

Estimating the Discounted Warranty Cost of a Minimally Repaired Coherent System

Estimación del costo de garantía descontado para un sistema coherente bajo reparo mínimo

NELFI GERTRUDIS GONZÁLEZ^{1,a}, VANDERLEI BUENO^{2,b}

¹ESCUELA DE ESTADÍSTICA, FACULTAD DE CIENCIAS, UNIVERSIDAD NACIONAL DE COLOMBIA, MEDELLÍN, COLOMBIA

²DEPARTAMENTO DE ESTATÍSTICA, INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, SÃO PAULO, BRASIL

Abstract

A martingale estimator for the expected discounted warranty cost process of a minimally repaired coherent system under its component level observation is proposed. Its asymptotic properties are also presented using the Martingale Central Limit Theorem.

Key words: Expected cost, martingale central limit theorem, reliability, repairable system, semimartingale, stochastic point process.

Resumen

En este trabajo modelamos los costos de garantía descontados para un sistema coherente reparado mínimamente a nivel de sus componentes y proponemos un estimador martingalas para el costo esperado para un período de garantía fijo, también probamos sus propiedades asintóticas mediante el Teorema del Limite Central para Martingalas.

Palabras clave: confiabilidad, costo esperado, proceso puntual estocástico, semi-martingalas, sistema reparable, teorema de límite central para martingalas.

1. Introduction

Warranties for durable consumer products are common in the market place. Its primary role is to offer a post sale remedy for consumers when a product fails

^aAssociate professor. E-mail: ngonzale@unal.edu.co

^bAssociate professor. E-mail: bueno@ime.usp.br

to fulfill its intended performance during the warranty period and generally, they also limit the manufacturer's liability for out-of-warranty product failure.

Manufacturers offer many types of warranties which have become an important promotional tool for their products. A discussion about various issues related to warranty policies can be found in Murthy (1990), Blischke & Murthy (1992*a*), Blischke & Murthy (1992*b*), Blischke & Murthy (1992*c*), Mitra & Patankar (1993), Blischke & Murthy (1994), Blischke & Murthy (1996).

Although warranties are used by manufacturers as a competitive strategy to boost their market share, profitability and image, they may cost a substantial amount of money and, from a manufacturer's perspective, the cost of a warranty program should be analyzed and estimated accurately.

A discounted warranty cost policy incorporates the time and provides an adequate measure for warranties because, in general, warranty costs can be treated as random cash flows in the future. Warranty issuers do not have to spend all the money at the stage of warranty planning, instead, they can allocate it along the life cycle of warranted products. Another reason why one should consider the time value is for the purpose of determining the warranty reserve, a fund set up specifically to meet future warranty claims. It is well known that the present value of warranty liabilities or rebates to be paid in the future are less than the face value and it is desirable to determine the warranty reserve according to the present value of the total warranty liability. Related issues to discounted warranty costs and warranty reserves have been studied by Mamer (1969), Mamer (1987), Patankar & Mitra (1995) and Thomas (1989), from both manufacturer and customer's perspectives for single-component products, either repairable or nonrepairable.

More recently, Jain & Maheshwari (2006) proposed a hybrid warranty model for renewing pro-rata warranties assuming constant failure rate and constant products maintenance and replacement costs. They derive the expected total discounted warranty costs for different lifetime distributions and determine the optimal number and optimal period for preventive maintenance after the expiry of the warranty; Jack & Murthy (2007) consider the costs for extended warranties offered after a base warranty and investigate optimal pricing strategies and optimal maintenance/replacement strategies; Hong-Zhong, Zhie-Jie, Yanfeng, Yu & Liping (2008) consider the cash flows of warranty reserve costs during the product lifecycle and estimate the expected warranty cost for repairable and non repairable products with both, the free replacement warranty and the pro-rata warranty policy. They also consider the case where the item has a heterogeneous usage intensity over the lifecycle and its usage is intermitent; Chattopadhyay & Rahman (2008) study lifetime warranties where the warranty coverage period depends on the lifetime of the product, they develop lifetime warranty policies and models for predicting failures and estimating costs; Jung, Park & Park (2010) consider optimal system maintenance policies during the post warranty period under the renewing warranty policy with maintenance costs dependent on life cycle.

In practice, most products are composed of several components. If warranties are offered for each component separately, then warranty models for single-component

products can be applied directly. However, sometimes warranty terms are defined upon an entire system. For such warranties, it is necessary to consider the system structure as well as the component level warranty service cost (Thomas 1989). Warranty analysis for multi-component systems based on the system structure has been addressed in a few papers: Ritchken (1986) provides an example of a two-component parallel system under a two-dimensional warranty; Chukova & Dimitrov (1996) derive the expected warranty cost for two-component series system and parallel system under a free-replacement warranty; Hussain & Murthy (1998) also discuss warranty cost estimation for parallel systems under the setting that uncertain quality of new products may be a concern for the design of warranty programs; Bai & Pham (2006) obtained the first two centered moments of the warranty cost of renewable full-services warranties for complex systems with series-parallel and parallel-series structures. A Markovian approach to the analysis of warranty cost for a three-component system can be found in Balachandran, Maschmeyer & Livingstone (1981); Ja, Kulkarni, Mitra & Patankar (2002) study the properties of the discounted warranty cost and total warranty program costs for non renewable warranty policy with non stationary processes.

There are many ways to model the impact of repair actions on system failure times. For complex systems, repair is often assumed to be minimal, which restores its failure rate. For a review about modeling failure and maintenance data from repairable systems, see Li & Shaked (2003) and Lindqvist (2006). For a generalization of minimal repair to heterogeneous populations, i.e., when the lifetime distribution is a mixture of distributions, see Finkelstein (2004). Nguyen & Murthy (1984) present a general warranty cost model for single-component repairable products considering as-good-as-new-repair, minimal repair and imperfect repair, but the value of time is not addressed. In Ja et al. (2002), several warranty reserve models for single-component products are derived for non stationary sale processes. Ja, Kulkarni, Mitra & Patankar (2001) analyze a warranty cost model on minimally repaired single-component systems with time dependent costs. Bai & Pham (2004) consider the free-repair warranty and the pro-rata warranty policies to derive some properties of a discounted warranty cost for a series system of repairable and independent components using a non homogeneous Poisson process. Recently, Duchesne & Marri (2009) consider, the same problem by analyzing some distributional properties (mean, variance, characteristic function) of the corresponding discounted warranty cost and using a general competing risk model to approach system reliability; Sheu & Yu (2005) propose a repair-replacement warranty policy which splits the warranty period into two intervals where only minimal repair can be undertaken and a middle interval in which no more than one replacement is allowed. Their model applies to products with bathtub failure rate considering random minimal repair costs. Other work about repair strategies, including imperfect and minimal repair, which consider their effects on warranty costs, can be found in Yun, Murthy & Jack (2008), Chien (2008), Yeo & Yuan (2009) and Samatli-Pac & Taner (2009).

For a series system with components which do not have common failures, system failures coincide with component failures and warranty models for single-component products can be applied directly. In this paper, we consider a dis-

counted warranty cost policy of a repairable coherent system under a minimal repair process on its component level. In this case, the system is set up as a series system with its components that survive to their critical levels, that is, the time from which a failure of a component would lead to system failure and, therefore, it is seen as a series system. We use the Martingale Central Limit Theorem to approximate the warranty cost distribution for a fixed warranty period of length w , and to estimate the warranty cost through the component failure/repair point processes.

In the Introduction of this paper we survey the recent developments in warranty models. In Section 2, we consider the dependent components lifetimes, as they appear in time through a filtration and use the martingale theory, a natural tool to consider the stochastic dependence and the increasing information in time. In Section 3, we consider independent copies of a coherent system, and its components, as given in Section 2 and develop a statistical model for the discounted warranty cost. Also, in this Section, we give an example. The paper is self contained but a mathematical basis of stochastic processes applied to reliability theory can be found in Aven & Jensen (1999). The extended proofs are in the Appendix.

2. The Warranty Discounted Cost Model of a Coherent System on its Component Level

We consider the vector $\mathbf{S} = (S_1, S_2, \dots, S_m)$ representing component lifetimes of a coherent system, with lifetime T , which are positive random variables in a complete probability space (Ω, \mathcal{F}, P) . The components can be dependent but simultaneous failures are ruled out, that is, for all i, j with $i \neq j$, $P(S_i = S_j) = 0$. We observe the system on its component level throughout a filtration, a family of sub σ -algebras of \mathcal{F} , $(\mathcal{F}_t)_{t \geq 0}$

$$\mathcal{F}_t = \sigma\{1_{\{T > s\}}, 1_{\{S_i > s\}} : s \leq t, 1 \leq i \leq m\},$$

which is increasing, right-continuous and complete. Clearly, the $S_i, 1 \leq i \leq m$ are (P, \mathcal{F}_t) -stopping time.

An extended and positive random variable τ is an (P, \mathcal{F}_t) -stopping time if, and only if, $\{\tau \leq t\} \in \mathcal{F}_t$, for all $t \geq 0$; an (P, \mathcal{F}_t) -stopping time τ is called predictable if an increasing sequence $(\tau_n)_{n \geq 0}$ of (P, \mathcal{F}_t) -stopping time, $\tau_n < \tau$, exists such that $\lim_{n \rightarrow \infty} \tau_n = \tau$; an (P, \mathcal{F}_t) -stopping time τ is totally inaccessible if $P(\tau = \sigma < \infty) = 0$ for all predictable (P, \mathcal{F}_t) -stopping time σ .

In what follows, to simplify the notation, we assume that relations such as $\subset, =, \leq, <, \neq$ between random variables and measurable sets, respectively, always hold “P-almost surely”, i.e., with probability one, which means that the term P -a.s., is suppressed.

2.1. Component Minimal Repair

For each i , $1 \leq i \leq m$, we consider the simple counting process $N_t^i = 1_{\{S_i \leq t\}}$, i.e., the counting process corresponding to the simple point process $(S_{i,n})_{n \geq 1}$ with $S_i = S_{i,1}$ and $S_{i,n} = \infty$, for $n \geq 2$. We use the Doob-Meyer decomposition

$$N_t^i = A_t^i + M_t^i, \quad M^i \in \mathcal{M}_0^2, \quad i = 1, \dots, m, \quad (1)$$

where \mathcal{M}_0^2 represents the class of mean zero and square integrable (P, \mathcal{F}_t) -martingales which are right-continuous with left-hand limits. A_t^i is a unique nondecreasing right continuous (P, \mathcal{F}_t) -predictable process with $A_0^i = 0$, called the (P, \mathcal{F}_t) -compensator of N_t^i .

We assume that the component lifetime S_i , $1 \leq i \leq m$ is a totally inaccessible (P, \mathcal{F}_t) -stopping time, which is a sufficient condition for the absolute continuity of A_t^i . It follows that

$$A_t^i = \int_0^t 1_{\{S_i > s\}} \lambda^i(s) ds < \infty, \quad i = 1, \dots, m, \quad (2)$$

where $\lambda^i(t)$ is the (P, \mathcal{F}_t) -failure rate of component i , a deterministic function of t .

Initially, consider the minimal repair process of component i . If we do a minimal repair at each failure of component i , the corresponding minimal repair counting process in $(0, t]$ is a non homogeneous Poisson process $\tilde{N}_t^i = \sum_{n=1}^{\infty} 1_{\{S_{i,n} \leq t\}}$, with Doob-Meyer decomposition given by,

$$\tilde{N}_t^i = \int_0^t \lambda^i(s) ds + \tilde{M}_t^i, \quad \tilde{M}^i \in \mathcal{M}_0^2, \quad (3)$$

and therefore the expected number of minimal repairs of component i is $E[\tilde{N}_t^i] = \int_0^t \lambda^i(s) ds$.

Let $H_i(t)$ be a deterministic, continuous (predictable) bounded and integrable function in $(0, t]$, corresponding to the minimal repair discounted cost of component i at time t , such that $\int_0^t H_i(s) \lambda^i(s) ds < \infty, 0 \leq t < \infty$.

The minimal repair cost process of component i is $\hat{B}_t^i = \sum_{j=1}^{\tilde{N}_t^i} H_i(S_{ij}) = \int_0^t H_i(s) d\tilde{N}_s^i$, where S_{ij} is the j -th minimal repair time of component i and $S_{i1} = S_i$.

Since $H_i(s)$ is predictable, the process $\int_0^t H_i(s) d\tilde{M}_s^i$ is a mean zero and square integrable (P, \mathcal{F}_t) -martingale and therefore, the (P, \mathcal{F}_t) -compensator of \hat{B}_t^i is B_t^i which is given by

$$B_t^i = \int_0^t H_i(s) \lambda^i(s) ds < \infty, \quad \forall 0 \leq t < \infty \quad (4)$$

Barlow and Proschan (1981) define the system lifetime T

$$T = \Phi(\mathbf{S}) = \min_{1 \leq j \leq k} \max_{i \in K_j} S_i$$

where $K_j, 1 \leq j \leq k$ are minimal cut sets, that is, a minimal set of components whose joint failure causes the system to fail. Aven & Jensen (1999) define the critical level of component i as the (P, \mathcal{F}_t) -stopping time $Y_i, 1 \leq i \leq m$ which describes the time when component i becomes critical for the system, i.e., the time from which the failure of component i leads to system failure. If either the system or component i fail before the latter becomes critical ($T \leq Y_i$ or $S_i \leq Y_i$) we assume that $Y_i = \infty$. Therefore, as in Aven & Jensen (1999) we can write

$$T = \min_{i: Y_i < \infty} S_i \quad (5)$$

Therefore, concerning the system minimal repairs at the component level, it is sufficient to consider the component minimal repairs after its critical levels. In what follows we consider the set $\mathcal{C}^i = \{\omega \in \Omega : S_i(\omega) > Y_i(\omega)\}$, where Y_i is the critical level of component i , and the minimal repair point process restricted to \mathcal{C}^i , that is, the process N_t^{i*} , defined as

$$N_t^{i*} = 1_{\mathcal{C}^i} N_t^i \quad (6)$$

which counts the failures of component i when it is critical, implying system failure.

Theorem 1. (González 2009) *The (P, \mathcal{F}_t) -compensator process of the indicator process $N_t^i = 1_{\{S_i \leq t\}}$ in \mathcal{C}^i is*

$$A_t^{i*} = \int_{Y_i}^t 1_{\{S_i > s\}} \lambda^i(s) ds = \int_0^t 1_{\{S_i > s\}} 1_{\{Y_i < s\}} \lambda^i(s) ds < \infty, \forall 0 \leq t < \infty \quad (7)$$

Note 1. Note that

$$E[N_t^i | S_i > Y_i] = E[A_t^{i*} | S_i > Y_i] = E\left[\int_{Y_i}^t 1_{\{S_i > s\}} \lambda^i(s) ds \mid S_i > Y_i\right] \quad (8)$$

From Theorem 1 the next Corollary follows easily.

Corollary 1. *Let \tilde{N}_t^i be the minimal repair counting process for the component i . Let $H_i(t)$ be a deterministic, continuous (predictable), bounded and integrable function in $[0, t]$, corresponding to the discounted warranty cost of component i at time t , such that $\int_0^t H_i(s) \lambda^i(s) ds < \infty, 0 \leq t < \infty$. In \mathcal{C}^i we have*

i. The (P, \mathcal{F}_t) -compensator of \tilde{N}_t^i is the process

$$\tilde{A}_t^{i*} = \int_{Y_i}^t \lambda^i(s) ds = \int_0^t 1_{\{Y_i < s\}} \lambda^i(s) ds < \infty, \quad \forall 0 \leq t < \infty \quad (9)$$

ii. The (P, \mathcal{F}_t) -compensator of the minimal repair cost process of component i ,

$$\widehat{B}_t^i = \sum_{j=1}^{\widetilde{N}_t^i} H_i(S_{ij}) \text{ is the process}$$

$$B_t^{i*} = \int_{Y_i}^t H_i(s) \lambda^i(s) ds = \int_0^t 1_{\{Y_i < s\}} H_i(s) \lambda^i(s) ds < \infty, \forall 0 \leq t < \infty \quad (10)$$

Note 2. For each $i = 1, \dots, m$ and $\omega \in \mathcal{C}^i$, the process $B_t^i(\omega)$ given in (4), is equal to the process $B_t^{i*}(\omega)$.

2.2. Coherent System Minimal Repair

Now we are going to define the minimal repair counting process and its corresponding coherent system cost process.

Let $N_t = 1_{\{T \leq t\}}$ be the system failure simple counting process and its (P, \mathcal{F}_t) -compensator process A_t , with decomposition

$$N_t = A_t + M_t, \quad M \in \mathcal{M}_0^2 \quad (11)$$

and

$$A_t = \int_0^t 1_{\{T > s\}} \lambda_s ds < \infty \quad (12)$$

where the process $(\lambda_t)_{t \geq 0}$ is the coherent system (P, \mathcal{F}_t) -failure rate process.

Since we do not have simultaneous failures the system will failure at time t when the first critical component for the system at t^- failures at t .

Under the above conditions Arjas (1981) proves that the (P, \mathcal{F}_t) -compensator of N_t is

$$A_t = \sum_{i=1}^m [A_{t \wedge T}^i - A_{Y_i}^i]^+ \quad (13)$$

and from (2) and (13) we get

$$A_t = \sum_{i=1}^m \int_0^t 1_{\{S_i > s\}} 1_{\{Y_i < s < T\}} \lambda^i(s) ds = \int_0^t 1_{\{T > s\}} \sum_{i=1}^m 1_{\{Y_i < s\}} \lambda^i(s) ds \quad (14)$$

From compensator unicity, it becomes clear that the (P, \mathcal{F}_t) -failure rate process of system is given by

$$\lambda_t = \sum_{i=1}^m 1_{\{Y_i < t\}} \lambda^i(t) \quad (15)$$

If the system is minimally repaired on its component level, its (P, \mathcal{F}_t) -failure rate process λ_t is restored at its condition immediately before failure and therefore

the critical component that fails at the system failure time is minimally repaired. Therefore, the number of minimal repairs of the system on its component level, is

$\tilde{N}_t = \sum_{n=1}^{\infty} 1_{\{T_n \leq t\}}$, with Doob-Meyer decomposition given by

$$\tilde{N}_t = \int_0^t \lambda_s ds + \tilde{M}_t = \sum_{i=1}^m \int_0^t 1_{\{Y_i < s\}} \lambda^i(s) ds + \tilde{M}_t. \quad \tilde{M} \in \mathcal{M}_0^2$$

Definition 1. For a fixed $\omega \in \Omega$ let $\mathcal{C}^\Phi(\omega) = \{i \in \{1, \dots, m\} : S_i(\omega) > Y_i(\omega)\}$ be the set of components surviving its corresponding critical levels. For each $i = 1, \dots, m$, let C^i be the indicator variable

$$C^i(\omega) = \begin{cases} 1 & \text{if } i \in \mathcal{C}^\Phi(\omega) \\ 0 & \text{otherwise} \end{cases} \quad (16)$$

Then, the minimal repair counting process of the coherent system is

$$\tilde{N}_t(\omega) = \sum_{i \in \mathcal{C}^\Phi(\omega)} \tilde{N}_t^i(\omega) = \sum_{i=1}^m C^i(\omega) \tilde{N}_t^i(\omega) \quad (17)$$

with corresponding cost process

$$\hat{B}_t(\omega) = \sum_{i \in \mathcal{C}^\Phi(\omega)} \hat{B}_t^i(\omega) = \sum_{i=1}^m C^i(\omega) \hat{B}_t^i(\omega) \quad (18)$$

Note 3. Note that $C^i(\omega) = 1 \Leftrightarrow \omega \in \mathcal{C}^i$ and in each realization $\omega \in \Omega$, the indicator variables $C^i(\omega), i = 1, \dots, m$, are constant in $[0, t]$. Therefore, if $C^i(\omega) = 0$, then $\hat{B}_s^i = 0, \forall 0 \leq s \leq t$. It means that in each realization of the system repair/failure process, we only observe the repair/cost processes of components which fail after their corresponding critical levels. Therefore, in each realization, the repair/cost process for the system with structure Φ is equivalent to the repair/cost process for a series system of components which are critical for the initial system in such realization.

2.3. Martingale Estimator of the Warranty Cost

In the following results and definitions, for each realization w , the minimal repair costs of a coherent system is the sum of the minimal repair costs of its critical components in a given realization ω .

Suppose

$$\sum_{i=1}^m \int_0^t H_i(s) \lambda^i(s) ds < \infty, \quad \forall 0 \leq t < \infty \quad (19)$$

For a fixed $\omega \in \Omega$, let $B_t(\omega)$ be the process

$$B_t(\omega) = \sum_{i \in \mathcal{C}^\Phi(\omega)} B_t^i(\omega) = \sum_{i=1}^m C^i(\omega) B_t^i(\omega) \quad (20)$$

Following Karr (1986), for each $i = 1, \dots, m$, $i \in \mathcal{C}^i$, the (P, \mathcal{F}_t) -martingale estimator for the process B_t^i , is the process

$$\widehat{B}_t^i(\omega) = \int_0^t H_i(s) d\widetilde{N}_s^i(\omega), \text{ in } \mathcal{C}^i \quad (21)$$

respectively.

Definition 2. For each $\omega \in \Omega$, the process $\widehat{B}_t(\omega)$ given in (18) is the (P, \mathcal{F}_t) -martingale estimator for the process B_t given in (20).

Proposition 1. Let $H_i(t), i = 1, \dots, m$, be a bounded and continuous functions in $[0, t]$, such that

$$\sum_{i=1}^m \int_0^t H_i^2(s) \lambda^i(s) ds < \infty, \quad \forall 0 \leq t < \infty \quad (22)$$

Then, for each realization ω and each $i \in \mathcal{C}^\Phi(\omega)$, the processes $(\widehat{B}^i - B^i)_{t \geq 0}$, are orthogonal, mean zero, and square integrable (P, \mathcal{F}_t) martingales with predictable variation processes $(\langle \widehat{B}^i - B^i \rangle)_{t \geq 0}$ given by

$$\langle \widehat{B}^i - B^{i*} \rangle_t = \int_{Y_i}^t H_i^2(s) \lambda^i(s) ds = \int_0^t H_i^2(s) 1_{\{Y_i < s\}} \lambda^i(s) ds \quad (23)$$

respectively.

Proof. Note that $\forall i \in \mathcal{C}^\Phi(\omega)$ we have $\omega \in \mathcal{C}^i$. Therefore, from Corollary 1, the (P, \mathcal{F}_t) -compensator of $\widehat{B}_t^i = \sum_{j=1}^{\widetilde{N}_t^i} H_i(S_{ij}) = \int_0^t H_i(s) d\widetilde{N}_s^i$ is the process $B_t^{i*} = \int_{Y_i}^t H_i(s) \lambda^i(s) ds = \int_0^t H_i(s) 1_{\{Y_i < s\}} \lambda^i(s) ds$ which represents B_t^i in \mathcal{C}^i (see Note 2).

So, for all $i \in \mathcal{C}^\Phi(\omega)$, the predictable variation process of the martingale $(\widehat{B}_t^i - B_t^i)$ is the predictable variation process of the martingale $(\widehat{B}_t^i - B_t^{i*})$,

$$\langle \widehat{B}^i - B^{i*} \rangle_t = \int_0^t H_i^2(s) d\langle \widetilde{M}^{i*} \rangle_s = \int_0^t H_i^2(s) 1_{\{Y_i < s\}} \lambda^i(s) ds$$

Otherwise, since $P(S_i = S_j) = 0$ P-a.s. for all i, j with $i \neq j$, the processes N_t^i and N_t^j do not have simultaneous jumps and so are \widetilde{N}_t^i and \widetilde{N}_t^j . Then, for all $i, j \in \mathcal{C}^\Phi(\omega)$, the (P, \mathcal{F}_t) -martingales \widetilde{M}_t^{*i} and \widetilde{M}_t^{*j} are orthogonal and square integrable, so that for $i \neq j$, the martingales $(\widehat{B}_t^i - B_t^{i*})$ and $(\widehat{B}_t^j - B_t^{j*})$ are also

orthogonal and square integrable. It follows that, for all $i \in \mathcal{C}^\Phi(\omega)$, the predictable covariation process

$$\langle \widehat{B}^i - B^i, \widehat{B}^j - B^j \rangle_t = \langle \widehat{B}^i - B^{i*}, \widehat{B}^j - B^{j*} \rangle_t = 0$$

that is, for all $i \in \mathcal{C}^\Phi(\omega)$, $(\widehat{B}^i - B^i)(\widehat{B}^j - B^j)$ is a mean zero (P, \mathcal{F}_t) -martingale. \square

Corollary 2. Let $H_i(t)$, $i, 1 \leq i \leq m$ be bounded and continuous functions in $[0, t]$ satisfying the condition in (22), and the processes $(\widehat{B}_t)_{t \geq 0}$, $(B_t)_{t \geq 0}$ as were given in (18) and (20), respectively. Then, for a realization $\omega \in \Omega$ and the corresponding set $\mathcal{C}^\Phi(\omega)$, the process $(\widehat{B} - B)_{t \geq 0}$ is a mean zero and square integrable (P, \mathcal{F}_t) -martingale with predictable variation process $(\langle \widehat{B} - B \rangle)_{t \geq 0}$ given by

$$\langle \widehat{B} - B \rangle_t = \sum_{i \in \mathcal{C}^\Phi(\omega)} \int_{Y_i}^t H_i^2(s) \lambda^i(s) ds = \sum_{i=1}^m C^i(\omega) \int_0^t H_i^2(s) 1_{\{Y_i < s\}} \lambda^i(s) ds \quad (24)$$

Proof. For all $i \in \mathcal{C}^\Phi(\omega)$ and from Proposition 1, the processes $(\widehat{B}^i - B^i)_{t \geq 0} = (\widehat{B}^i - B^{i*})_{t \geq 0}$, $1 \leq i \leq m$, are orthogonal, mean zero, and square integrable (P, \mathcal{F}_t) -martingales with predictable variation processes given by $\langle \widehat{B}^i - B^{i*} \rangle_t = \int_0^t H_i^2(s) 1_{\{Y_i < s\}} \lambda^i(s) ds$, respectively. Therefore,

$$\begin{aligned} \widehat{B}_t(\omega) - B_t(\omega) &= \sum_{i \in \mathcal{C}^\Phi(\omega)} (\widehat{B}_t^i(\omega) - B_t^i(\omega)) \\ &= \sum_{i \in \mathcal{C}^\Phi(\omega)} \int_0^t H_i(s) \widetilde{M}_s^{i*}(\omega) \in \mathcal{M}_0^2 \end{aligned} \quad (25)$$

and $\langle \widehat{B}^i - B^i, \widehat{B}^j - B^j \rangle = 0, \forall i \neq j$. From (23) we have

$$\langle \widehat{B} - B \rangle_t = \sum_{i \in \mathcal{C}^\Phi(\omega)} \langle \widehat{B}^i - B^{i*} \rangle_t = \sum_{i=1}^m C^i(\omega) \int_0^t H_i^2(s) 1_{\{Y_i < s\}} \lambda^i(s) ds$$

\square

Note 4. From (24) we have that the expected value of the predictable variation process of the system warranty cost process is

$$E[\langle \widehat{B} - B \rangle_t] = \sum_{i=1}^m P(S_i > Y_i) E \left[\int_{Y_i}^t H_i^2(s) \lambda^i(s) ds \mid S_i > Y_i \right] \quad (26)$$

3. Statistical Model

3.1. Preliminary

We intend to estimate the expected minimal repair cost $E[\widehat{B}_t]$, over the interval $[0, t]$. First, we need asymptotic results for the estimator of each component expected warranty costs.

From Definitions 1, 2 and Corollary 2 we have

$$\begin{aligned}
 E[\widehat{B}_t] &= E\left[\sum_{i \in \mathcal{C}^\Phi(\omega)} B_t^i\right] = \sum_{i=1}^m P(S_i > Y_i) E\left[\int_{Y_i}^t H_i(s) \lambda^i(s) ds \mid S_i > Y_i\right] \\
 &= E[B_t]
 \end{aligned}
 \tag{27}$$

where $P(S_i > Y_i) E[\int_{Y_i}^t H_i(s) \lambda^i(s) ds \mid S_i > Y_i]$ corresponds to the system minimal repairs expected cost related to the component i .

We consider the sequences $(\widehat{B}_t^{(j)}, C^{i(j)}, i = 1, \dots, m)_{t \geq 0}, 1 \leq j \leq n$, of n independent and identically distributed copies of the m -variate process $(\widehat{B}_t^i, C^i, i = 1, \dots, m)_{t \geq 0}$.

For $j = 1, \dots, n$ let $\mathcal{C}^{\Phi(j)} = \{i \in \{1, \dots, m\} : S_i^{(j)} > Y_i^{(j)}\}$ be the set of critical components for the j -th observed system, where $S_i^{(j)}$ is the first failure time of component i and $Y_i^{(j)}$ its critical level. Then, the minimal repairs expected cost for the j -th system is

$$\widehat{B}_t^{(j)} = \sum_{i \in \mathcal{C}^{\Phi(j)}} \widehat{B}_t^{i(j)} = \sum_{i=1}^m C^{i(j)} \widehat{B}_t^{i(j)}
 \tag{28}$$

and its compensator process is (from Corollary 2)

$$B_t^{(j)} = \sum_{i \in \mathcal{C}^{\Phi(j)}} B_t^{i(j)} = \sum_{i=1}^m C^{i(j)} \int_{Y_i^{(j)}}^t H_i(s) \lambda^i(s) ds
 \tag{29}$$

For n copies we consider the mean processes

$$\overline{\widehat{B}}_t^{(n)} = \frac{1}{n} \sum_{j=1}^n \widehat{B}_t^{(j)} = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^m C^{i(j)} \int_0^t H_i(s) d\widetilde{N}_s^{i(j)}
 \tag{30}$$

$$\overline{B}_t^{(n)} = \frac{1}{n} \sum_{j=1}^n B_t^{(j)} = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^m C^{i(j)} \int_{Y_i^{(j)}}^t H_i(s) \lambda^i(s) ds
 \tag{31}$$

Let

$$\overline{\widehat{B}}_t^{i(n)} = \frac{1}{n} \sum_{j=1}^n C^{i(j)} \widehat{B}_t^{i(j)} \quad \text{and} \quad \overline{B}_t^{i(n)} = \frac{1}{n} \sum_{j=1}^n C^{i(j)} B_t^{i(j)}
 \tag{32}$$

Then, from (30) and (31), we also have

$$\overline{\widehat{B}}_t^{(n)} = \sum_{i=1}^m \overline{\widehat{B}}_t^{i(n)} \quad \text{and} \quad \overline{B}_t^{(n)} = \sum_{i=1}^m \overline{B}_t^{i(n)}
 \tag{33}$$

For each $i = 1, \dots, m$ we propose $\overline{\widehat{B}}_t^{i(n)}$ as the estimator for the system minimal repairs expected cost related to the component i .

Theorem 2. For each $i = 1, \dots, m$ let $B^{i*}(t)$ be

$$B^{i*}(t) = P(S_i > Y_i) E \left[\int_{Y_i}^t H_i(s) \lambda^i(s) ds \mid S_i > Y_i \right] \quad (34)$$

Then, under conditions in Proposition 1, $\overline{B}_t^{i(n)}$ is a consistent and unbiased estimator for the minimal repairs expected cost related to component i , $B^{i*}(t)$.

Proof. See Appendix A. □

3.2. The Central Limit Theorem

In what follows we prove that the m -variate error process of the proposed estimators, $(\overline{B}_t^{i(n)} - B^{i*}(t), i = 1, \dots, m)$, conveniently standardized, satisfies the Martingale Central Limit Theorem, as in Karr (1986).

Theorem 3. (Karr 1986, Theorem 5.11). For fixed m and for each n , $n \geq 1$, let $(M_t^{i(n)}, i = 1, \dots, m)$ be a sequence of orthogonal, mean zero and square integrable martingales with jumps, at time t , $\Delta M_t^{i(n)} = M_t^{i(n)} - M_t^{i(n)-}$, where $M_t^{i(n)-} = \lim_{h \downarrow 0} M_{t-h}^{i(n)}$. For each $i, i = 1, \dots, m$ let $V_i(t)$ be a continuous and non decreasing function with $V_i(0) = 0$. If

(a) $\forall t \geq 0$ and $i = 1, \dots, m$

$$\langle M^{i(n)} \rangle_t \xrightarrow[n \rightarrow \infty]{\mathcal{D}} V_i(t) \quad (35)$$

(b) There is a sequence $(c_n)_{n \geq 1}$, such that $c_n \xrightarrow[n \rightarrow \infty]{} 0$ and

$$P(\sup_{s \leq t} |\Delta M_s^{i(n)}| \leq c_n) \xrightarrow[n \rightarrow \infty]{} 1 \quad (36)$$

Then exist an m -variate Gaussian continuous process, $\mathbf{M} = (M^i, i = 1, \dots, m)$ where M^i is a martingale with

$$\langle M^i, M^k \rangle_t = 1_{\{i=k\}} V_i(t) \quad (37)$$

such that $\mathbf{M}^{(n)} = (M^{1(n)}, \dots, M^{m(n)}) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathbf{M} = (M^1, \dots, M^m)$ in $D[0, t]^m$

Note 5. In the above theorem the conditions (a) and (b) are sufficient to prove the convergence of the finite-dimensional distributions and tightness of the sequence $\mathbf{M}^{(n)}$ in the m -dimensional space $D[0, t]^m$ of the right-continuous functions with left limits, in $[0, t]$ (Karr 1986).

Corollary 3. Suppose that for each $i, i = 1, \dots, m$, $\int_0^t H_i^2(s)\lambda^i(s)ds < \infty$ and let $V_i^*(t)$ be the function

$$V_i^*(t) = P(S_i > Y_i)E\left[\int_{Y_i}^t H_i^2(s)\lambda^i(s)ds \mid S_i > Y_i\right] \tag{38}$$

Let $\overline{\mathbf{B}}_t^{(n)} = \left(\overline{B}_t^{1(n)}, \dots, \overline{B}_t^{m(n)}\right)$ and $\overline{\mathbf{B}}_t^{(n)} = \left(\overline{B}_t^{1(n)}, \dots, \overline{B}_t^{m(n)}\right)$ be m -variate processes. Then, the process $\mathbf{M}^{(n)} = \sqrt{n}(\overline{\mathbf{B}}_t^{(n)} - \overline{\mathbf{B}}_t^{(n)}) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathbf{M}$ in $D[0, t]^m$, where \mathbf{M} is an m -variate Gaussian continuous process with martingales components.

Proof. We establish the conditions (a) and (b) of Theorem 3. Denote $M_t^{i(n)} = \sqrt{n} \left(\overline{B}_t^{i(n)} - \overline{B}_t^{i(n)}\right)$, $i = 1, \dots, m$. As $P(S_i = S_j) = 0$ for all i, j with $i \neq j$, from Proposition 1

$$\left(\overline{B}_t^{i(n)} - \overline{B}_t^{i(n)}\right) = \frac{1}{n} \sum_{j=1}^n C^{i(j)} \left(\widehat{B}_t^{i(j)} - B_t^{i(j)}\right), \quad 1 \leq i \leq m,$$

are orthogonal, mean zero and square integrable (P, \mathcal{F}_t) -martingales, for each $n \geq 1$.

Therefore, for all $n \geq 1$ and $i \neq j$, $\langle M^{i(n)}, M^{j(n)} \rangle_t = 0$, from the Strong Law of Large Numbers and (60), for all $i, 1 \leq i \leq m$

$$\begin{aligned} \langle M^{i(n)} \rangle_t &= n \sum_{j=1}^n C^{i(j)} \left[\frac{1}{n^2} \int_{Y_i^{(j)}}^t H_i^2(s)\lambda^i(s)ds \right] = \frac{1}{n} \sum_{j=1}^n C^{i(j)} \left[\int_{Y_i^{(j)}}^t H_i^2(s)\lambda^i(s)ds \right] \\ &\xrightarrow[n \rightarrow \infty]{} P(S_i > Y_i)E\left[\int_{Y_i}^t H_i^2(s)\lambda^i(s)ds \mid S_i > Y_i\right] = V_i^*(t) < \infty \end{aligned} \tag{39}$$

and we have $\langle M^{i(n)} \rangle_t \xrightarrow[n \rightarrow \infty]{\mathcal{D}} V_i^*(t)$, for all $t \geq 0$.

Furthermore, the jumps of $M^{i(n)}$ arise only from $\sqrt{n}\overline{B}_t^{i(n)}$ and they are of size $\Delta M_t^{i(n)} = \frac{H_i(t)}{\sqrt{n}}$. By hypothesis, $H_i(t)$ is continuous and bounded in $[0, t]$, say by a constant $\Gamma < \infty$. Taking $c_n = \Gamma n^{-\frac{1}{4}}$, the condition (b) of Theorem 3 is satisfied. Therefore, $\mathbf{M}^{(n)} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathbf{M}$, where \mathbf{M} is an m -variate Gaussian continuous process, $\mathbf{M} = (M^i, i = 1, \dots, m)$, with martingale components $M^i, i = 1, \dots, m$ such that $\langle M^i, M^k \rangle_t = 1_{\{i=k\}}V_i^*(t)$. \square

Proposition 2. Let $\mathbf{Z}_t^{(n)}$ be the m -variate process $\mathbf{Z}_t^{(n)} = \left(Z_t^{1(n)}, \dots, Z_t^{m(n)}\right)$ where $Z_t^{i(n)} = \sqrt{n}(\overline{B}_t^{i(n)} - B^{i*}(t))$, $i = 1, \dots, m$ and suppose that for all $i, 1 \leq i \leq m$ and $t \geq 0$,

$$\sigma^{2i*}(t) = \text{Var}[C^i B_t^i] = E\left[C^i \left(\int_{Y_i}^t H_i(s)\lambda^i(s)ds\right)^2\right] - (B^{i*}(t))^2 < \infty. \tag{40}$$

Then $\mathbf{Z}_t^{(n)} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathbf{Z}_t$, where \mathbf{Z}_t is an m -variate Normal random vector with mean zero and covariance matrix $\Sigma(t)$ such that $\Sigma_{ij}(t) = 1_{\{i=j\}}\sigma^{2i^*}(t)$.

Proof. See Appendix B. □

Theorem 4. Let $\boldsymbol{\mu}(t) = (B^{1^*}(t), \dots, B^{m^*}(t))$, $\delta^{2i^*}(t) = \text{Var}[C^i \widehat{B}_t^i] < \infty$, $i = 1, \dots, m$ and suppose that the conditions of Corollary 3 and Proposition 2 holds. Then, the process $\mathbf{E}_t^{(n)} = \sqrt{n}(\overline{\mathbf{B}}_t^{(n)} - \boldsymbol{\mu}(t)) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathbf{W}_t$ in $D[0, t]^m$, where $\mathbf{W}_t = (W_t^1, \dots, W_t^m)$ is an m -variate Gaussian process with mean zero and covariance matrix $\mathbf{U}(t)$ with $U_{ij}(t) = 1_{\{i=j\}}\delta^{2i^*}(t)$.

Proof. See Appendix C. □

Note 6. In order to apply Theorem 4 we must estimate for fixed t , the variances $\delta^{2i^*}(t)$, $i = 1, \dots, m$, which can be done through the sample estimator of the variance.

For $t \geq 0$ we consider n independent and identically distributed copies of the m -variate process $((\widehat{B}_t^i, C^i), i = 1, \dots, m)$, with covariance matrix given by

$$\mathbf{U}(t) = \begin{bmatrix} \delta^{21^*}(t) & 0 & 0 & \dots & 0 \\ 0 & \delta^{22^*}(t) & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & \delta^{2m^*}(t) \end{bmatrix} \quad (41)$$

We propose as estimator of $\mathbf{U}(t)$ to the sample covariance matrix, $\mathbf{S}^{(n)}(t)$, where $S_{ij}^{(n)}(t) = 1_{\{i=j\}}S_t^{2i(n)}$, that is

$$\mathbf{S}^{(n)}(t) = \begin{bmatrix} S_t^{21(n)} & 0 & 0 & \dots & 0 \\ 0 & S_t^{22(n)} & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & S_t^{2m(n)} \end{bmatrix} \quad (42)$$

with

$$S_t^{2i(n)} = \left(\frac{n}{n-1}\right) \left[\frac{1}{n} \sum_{j=1}^n (C^{i(j)} \widehat{B}_t^{i(j)} - B^{i^*}(t))^2 - (\overline{\widehat{B}}_t^{i(n)} - B^{i^*}(t))^2 \right] \quad (43)$$

Therefore, for each i , $1 \leq i \leq m$ and fixed $t \geq 0$, we calculate the corresponding sample estimator of variance, $S_t^{2i(n)}$, which satisfies the properties enunciated in the following proposition.

Proposition 3. For each i , $1 \leq i \leq m$, $S_t^{2i(n)}$ is an unbiased and uniformly consistent estimator for $\delta^{2i^*}(t)$ and therefore, $\mathbf{S}^{(n)}(t)$ and $\sum_{i=1}^m S_t^{2i(n)}$ are unbiased and uniformly consistent estimator for $\mathbf{U}(t)$ and $\sum_{i=1}^m \delta^{2i^*}(t)$, respectively.

Proof. For each $i, 1 \leq i \leq m$ and $t \geq 0$ we have $E[C^i \widehat{B}_t^i] = E[\overline{B}_t^{i(n)}] = B^{i*}(t)$ and $\delta^{2i*}(t) = E[(C^i \widehat{B}_t^i - B^{i*}(t))^2]$. As the copies are independent and identically distributed from (43) we get

$$E[S_t^{2i(n)}] = \left(\frac{n}{n-1}\right) \left[\delta^{2i*}(t) - \frac{1}{n}\delta^{2i*}(t)\right] = \delta^{2i*}(t), \text{ and therefore,}$$

$$E[\mathbf{S}^{(n)}(t)] = \mathbf{U}(t) \quad \text{and} \quad E\left[\sum_{i=1}^m S_t^{2i(n)}\right] = \sum_{i=1}^m \delta^{2i*}(t) \tag{44}$$

Also, we apply the Strong Law of Large Number to obtain, for all $t \geq 0$,

$$\frac{1}{n} \sum_{j=1}^n (C^{i(j)} \widehat{B}_t^{i(j)} - B^{i*}(t))^2 \xrightarrow[n \uparrow \infty]{} \delta^{2i*}(t)$$

From the Strong Law of Large Number and the Continuous Mapping Theorem (See, Billingsley 1968), we have,

$$\left(\overline{B}_t^{i(n)} - B^{i*}(t)\right)^2 \xrightarrow[n \uparrow \infty]{} 0$$

and $\left(\frac{n}{n-1}\right) \xrightarrow[n \uparrow \infty]{} 1$. From the above results and (43), for all $i, 1 \leq i \leq m$ we conclude

$$S_t^{2i(n)} \xrightarrow[n \uparrow \infty]{} \delta^{2i*}(t), \quad \forall t \geq 0$$

Then

$$S_s^{2i(n)} \xrightarrow[n \uparrow \infty]{} \delta^{2i*}(s), \quad \forall s \leq t, \quad \sup_{s \leq t} |S_s^{2i(n)} - \delta^{2i*}(s)| \xrightarrow[n \uparrow \infty]{} 0$$

and therefore,

$$\sup_{s \leq t} (S_s^{2i(n)} - \delta^{2i*}(s))^2 \xrightarrow[n \uparrow \infty]{} 0$$

It follows from the above results that

$$E\left[\sup_{s \leq t} (S_s^{2i(n)} - \delta^{2i*}(s))^2\right] \xrightarrow[n \uparrow \infty]{} 0 \tag{45}$$

This result gives the uniformly consistence of the estimators $S_t^{2i(n)}$ and $\sum_{i=1}^m S_t^{2i(n)}$, which also warranties the consistence of the estimator $\mathbf{S}^{(n)}(t)$ given in (42). \square

3.3. Estimation of the Expected Warranty Cost for a Fixed Warranty Period of Length w

From (27) and (34), the expected warranty cost for a fixed period of length w is $B^*(w) = E[\widehat{B}_w] = \sum_{i=1}^m B^{i*}(w) = E[B_w]$. In this section we obtain a $(1 - \alpha)100\%$ confidence interval from results in Section 2.3 to Section 3.2.

Let $\mathbf{1}_m = (1, 1, \dots, 1)$ be the m -dimensional unit vector and $(\mathbf{A})^t$ the transpose of corresponding vector or matrix \mathbf{A} . From (33) and Theorem 2 an estimator of $B^*(w)$ is $\widehat{B}^*(w) = \overline{\widehat{B}}_w^{(n)}$, which can be write as

$$\overline{\widehat{B}}_w^{(n)} = \sum_{i=1}^m \overline{\widehat{B}}_w^{i(n)} = \mathbf{1}_m (\overline{\widehat{\mathbf{B}}}_w^{(n)})^t = \overline{\widehat{\mathbf{B}}}_w^{(n)} (\mathbf{1}_m)^t \quad (46)$$

where $\overline{\widehat{\mathbf{B}}}_w^{(n)} = (\overline{\widehat{B}}_w^{1(n)}, \dots, \overline{\widehat{B}}_w^{m(n)})$. Also, we can write the expected warranty cost $B^*(w)$ as

$$B^*(w) = \sum_{i=1}^m B^{i*}(w) = \mathbf{1}_m (\boldsymbol{\mu}(w))^t = \boldsymbol{\mu}(w) (\mathbf{1}_m)^t \quad (47)$$

with $\boldsymbol{\mu}(w) = (B^{1*}(w), \dots, B^{m*}(w))$ as defined in Theorem 4.

Theorem 5.

- i. $\overline{\widehat{B}}_w^{(n)}$ is a consistent and unbiased estimator for $B^*(w)$.
- ii. A consistent and unbiased estimator for $\text{Var}[\overline{\widehat{B}}_w^{(n)}]$ is $\widehat{\text{Var}}[\overline{\widehat{B}}_w^{(n)}] = \frac{1}{n} \sum_{i=1}^m S_w^{2i(n)}$.
- iii. An approximate $(1 - \alpha)100\%$ confidence interval for $B^*(w)$, is

$$\overline{\widehat{B}}_w^{(n)} \pm Z_{1-\alpha/2} \sqrt{\frac{1}{n} \sum_{i=1}^m S_w^{2i(n)}} \quad (48)$$

where Z_γ is the γ -quantile of the standard normal distribution.

Proof.

i. For each i , $1 \leq i \leq m$, from Theorem 2, $\overline{\widehat{B}}_w^{i(n)}$ is a consistent and unbiased estimator for $B^{i*}(w)$. Then, $\overline{\widehat{B}}_w^{(n)} = \sum_{i=1}^m \overline{\widehat{B}}_w^{i(n)}$ is a consistent and unbiased estimator for $B^*(w)$.

ii. Since for $i \neq j$ the processes \widehat{B}_w^i and \widehat{B}_w^j do not have simultaneous jumps, we have

$$\text{Var}[\overline{\widehat{B}}_w^{(n)}] = \frac{1}{n} \sum_{i=1}^m \delta^{2i*}(w) = \frac{1}{n} \mathbf{1}_m \mathbf{U}(w) (\mathbf{1}_m)^t = \frac{1}{n} \text{Var}[(\widehat{B}_w^1, \dots, \widehat{B}_w^m) (\mathbf{1}_m)^t]$$

Therefore, from Proposition 3, (42) and (44), an unbiased and consistent estimator for $\text{Var}[\overline{\widehat{B}}_w^{(n)}]$ is

$$\widehat{\text{Var}}[\overline{\widehat{B}}_w^{(n)}] = \frac{1}{n} \sum_{i=1}^m S_w^{2i(n)} = \frac{1}{n} \mathbf{1}_m \mathbf{S}^{(n)}(w) (\mathbf{1}_m)^t \quad (49)$$

iii. As a consequence of Theorem 4, and the Crámer-Wold procedure (See, Fleming & Harrington 1991, Lemma 5.2.1) we have

$$\mathbf{1}_m(\mathbf{E}_w^{(n)})^t = \mathbf{E}_w^{(n)}(\mathbf{1}_m)^t = \sum_{i=1}^m E_w^{i(n)} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathbf{1}_m(\mathbf{W}_w)^t = \mathbf{W}_w(\mathbf{1}_m)^t \sim N(0, \mathbf{1}_m \mathbf{U}(w)(\mathbf{1}_m)^t) = N\left(0, \sum_{i=1}^m \delta^{2i^*}(w)\right)$$

then

$$\frac{\sum_{i=1}^m E_w^{i(n)}}{\sqrt{\sum_{i=1}^m \delta^{2i^*}(w)}} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} N(0, 1) \tag{50}$$

From Proposition 3 and the Slutsky Theorem,

$$\begin{aligned} \frac{\sum_{i=1}^m E_w^{i(n)}}{\sqrt{\sum_{i=1}^m S_w^{2i(n)}}} &= \frac{\sqrt{n} \sum_{i=1}^m (\overline{B}_w^{i(n)} - B^{i^*}(w))}{\sqrt{\sum_{i=1}^m S_w^{2i(n)}}} \\ &= \frac{\sqrt{n}(\overline{B}_w^{(n)} - B^*(w))}{\sqrt{\sum_{i=1}^m S_w^{2i(n)}}} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} N(0, 1) \end{aligned} \tag{51}$$

From the last equation we get

$$\lim_{n \rightarrow \infty} P \left\{ \frac{|\sqrt{n}(\overline{B}_w^{(n)} - B^*(w))|}{\sqrt{\sum_{i=1}^m S_w^{2i(n)}}} \leq Z_{1-\alpha/2} \right\} \geq P \{ |Z| \leq Z_{1-\alpha/2} \} = 1 - \alpha$$

and a $(1 - \alpha)100\%$ approximate pointwise confidence interval for $B^*(w)$, is

$$\overline{B}_w^{(n)} \pm Z_{1-\alpha/2} \sqrt{\frac{1}{n} \sum_{i=1}^m S_w^{2i(n)}}$$

□

The confidence interval for $B^*(w)$ given in (48) can have negative values and it is not acceptable. We propose to build a confidence interval through a convenient bijective transformation $g(x)$ such that $\frac{d}{dx}g(x)|_{x=B^*(w)} \neq 0$, which does not contain negative values. Conveniently, we consider $g(x) = \log x$, $x > 0$ with $\frac{d}{dx}g(x) = 1/x$, $x > 0$.

Corollary 4. Suppose that for a fixed $w > 0$, $B^*(w) > 0$ and $\overline{B}_w^{(n)} > 0$. Then

$$\overline{B}_w^{(n)} \times \exp \left\{ \pm Z_{1-\alpha/2} \sqrt{\frac{\sum_{i=1}^m S_w^{2i(n)}}{n[\overline{B}_w^{(n)}]^2}} \right\} \quad (52)$$

is an approximate $(1 - \alpha)100\%$ confidence interval for $B^*(w)$.

Proof. Using the Delta Method (See, Lehmann 1999, Section 2.5) and formula (50) we get

$$\sqrt{n}[\log(\overline{B}_w^{(n)}) - \log(B^*(w))] \xrightarrow[n \rightarrow \infty]{\mathcal{D}} N\left(0, [B^*(w)]^{-2} \sum_{i=1}^m \delta^{2i^*}(w)\right) \quad (53)$$

From literal i in Theorem 5, Proposition 3, and the Continuous Mapping Theorem (Billingsley 1968),

$$\frac{\overline{B}_w^{(n)}}{\sqrt{\sum_{i=1}^m S_w^{2i(n)}}} \xrightarrow[n \rightarrow \infty]{} \frac{B^*(w)}{\sqrt{\sum_{i=1}^m \delta^{2i^*}(w)}}, \quad (54)$$

Therefore, for fixed w , from (53), (54) and using Slutsky Theorem, we have

$$\frac{\sqrt{n}\overline{B}_w^{(n)}}{\sqrt{\sum_{i=1}^m S_w^{2i(n)}}} [\log(\overline{B}_w^{(n)}) - \log(B^*(w))] \xrightarrow[n \rightarrow \infty]{\mathcal{D}} N(0, 1) \quad (55)$$

From the last equation, an approximate $(1 - \alpha)100\%$ confidence interval for $\log(B^*(w))$ is

$$\log(\overline{B}_w^{(n)}) \pm Z_{1-\alpha/2} \sqrt{\frac{\sum_{i=1}^m S_w^{2i(n)}}{n[\overline{B}_w^{(n)}]^2}} \quad (56)$$

from which, applying the inverse transformation, that is, $\exp(x)$, we obtain (52). \square

3.4. Example

To illustrate the results we simulated the minimal repair warranty cost process for a parallel system of three independent components with lifetimes $S_i \sim \text{Weibull}(\theta_i, \beta_i)$, $i = 1, 2, 3$, respectively, where θ_i is the scale parameter and β_i is the shape parameter, that is, with survival function $\overline{F}^i(t) = \exp[-(t/\theta_i)^{\beta_i}]$ and hazard rate function $\lambda^i(t) = (\beta_i/\theta_i^{\beta_i})t^{\beta_i-1}$, $t > 0$.

We use two possible cost functions: the first one is $H_i(t) = C_i e^{-\delta t}$ and the second one is $H_i(t) = C_i \left(1 - \frac{t}{w}\right) e^{-\delta t}$, $0 \leq t \leq w$, with $\delta = 1$ in both cases. Clearly they are bounded and continuous functions in $[0, t]$. The parameter values are indicated in Table 1, $w = 5$ is the fixed warranty period and the sample sizes are $n = 30, 50, 100, 500, 1000, 2000, 5000, 10000$.

The critical levels of the components for the system under minimal repair are

$$Y_i = \begin{cases} \max_{j \neq i} S_j & \text{if } \max_{j \neq i} S_j < S_i, \\ \infty & \text{if } \max_{j \neq i} S_j \geq S_i, \end{cases} \quad i = 1, 2, 3 \tag{57}$$

Therefore, if the component failure times are observed in order S_2, S_3, S_1 , then $T = \max\{S_1, S_2, S_3\} = \min_{\{Y_i < \infty\}} S_i = S_1$, and, in this case, component 1, is the only one critical for the system. Therefore, after the second component failure time, S_3 , the system is reduced to component 1, which is minimally repaired in each observed failure over the warranty period.

TABLE 1: Parameter values.

i	θ_i	β_i	C_i
1	1	1.5	3
2	1	1.5	3
3	2	2.0	5

The simulation results considering the cost function as $H_i(t) = C_i e^{-\delta t}$ are:

In Table 2, the limits correspond to the confidence interval defined in (52), with confidence level of $\alpha = 0.05$. In Figure 1, we show the 95% approximate pointwise confidence intervals for sample size of $n = 100$ and $w \in (0, 5]$.

TABLE 2: Estimations for some sample sizes, $H_i(t) = C_i e^{-\delta t}$, $w = 5$, $\alpha = 0.05$.

n	$\widehat{B}^*(w)$	$\sum_{i=1}^3 S_w^{2i(n)}$	$\sum_{i=1}^3 S_w^{2i(n)} / n$	Lower limit	Upper limit
30	1.90	2.30	0.07654	1.430	2.529
50	1.89	2.96	0.05921	1.471	2.435
100	1.85	2.95	0.02954	1.543	2.221
500	1.83	2.84	0.00568	1.685	1.980
1000	1.81	2.76	0.00276	1.712	1.918
2000	1.85	2.84	0.00142	1.780	1.928
5000	1.84	2.83	0.00057	1.794	1.887
10000	1.86	2.90	0.00029	1.825	1.891

Table 3, presents the theoretical values for the expected cost for a warranty period of length $w = 5$, where $E[B_w^i] = \int_0^w H_i(s) \lambda^i(s) ds$ (that is, when the component i is minimally repaired at each observed failure) and $E[B_w^i | S_i > Y_i] = E\left[\int_{Y_i}^w H_i(s) \lambda^i(s) ds \mid S_i > Y_i\right]$. Based on these results, we can conclude that for the considered system, the estimated values are closer to the expected values for sample sizes greater than $n = 50$.

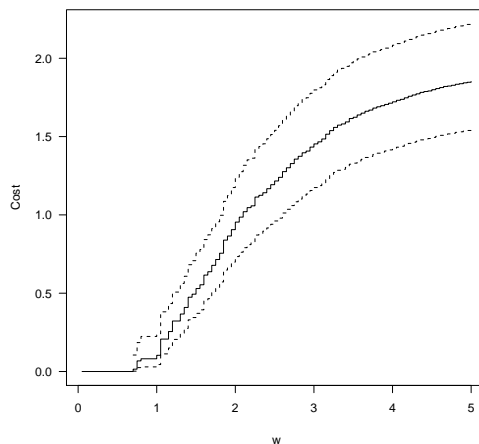


FIGURE 1: 95% Approximate pointwise confidence intervals using limits in (52) with simulated samples and $H_i(t) = C_i e^{-\delta t}$.

TABLE 3: Expected costs, $H_i(t) = C_i e^{-\delta t}$, $w = 5$.

i	$E[B_w^i]$	$E[B_w^i S_i > Y_i]$	$P(S_i > Y_i)$	$P(S_i > Y_i)E[B_w^i S_i > Y_i]$
1	3.9138	2.14648	0.1620753	0.348
2	3.9138	2.14648	0.1620753	0.348
3	2.3989	1.70345	0.6758494	1.151
System cost				1.847

The following results correspond to the Monte Carlo simulations in which we got the mean cost for $w = 5$ and 1000 samples of size $n = 100$ and $n = 200$, respectively. Table 4, presents several statistics and the Shapiro Wilk normality test. In Figure 2, we show the histograms of mean costs.

TABLE 4: Statistics of Monte Carlo simulation, $H_i(t) = C_i e^{-\delta t}$, $w = 5$.

n	\bar{X}_n	S_n^2	\bar{X}_n	$P_{2.5}$	P_{25}	P_{75}	$P_{97.5}$	S.Wilk	P-value
100	1.848	0.01574	1.848	1.612	1.761	1.936	2.091	0.9990	0.8576
200	1.843	0.00851	1.841	1.666	1.782	1.905	2.036	0.9987	0.7200

From results in Tables 2 to 4, and Figures 1 and 2, we observe that the mean cost is approximately 1.85 for a warranty period of length $w = 5$. Also, the sample variance and the 2.5th and 97.5th sample percentiles for the mean costs from samples of size $n = 100$ showed in Table 4. They are close to the corresponding values in Table 2 for $\sum_{i=1}^3 S_w^{2i(n)}/n$ and the confidence limits, respectively, and it becomes clear that, in this case, the normal approximation is already achieved with samples of size 100.

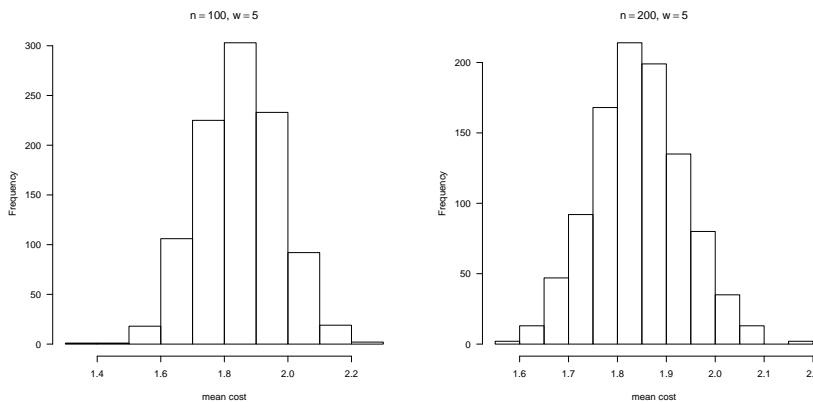


FIGURE 2: Histograms of mean costs, $H_i(t) = C_i e^{-\delta t}$, $n = 100, 200$.

The results related to the minimal repair costs with functions given by $H_i(t) = C_i (1 - \frac{t}{w}) e^{-\delta t}$ are showed in Tables 5, 6 and 7 and Figures 3 and 4. The conclusions are similar to the previous case.

TABLE 5: Estimations for different sample sizes, $H_i(t) = C_i (1 - \frac{t}{w}) e^{-\delta t}$, $w = 5$, $\alpha = 0.05$.

n	$\hat{B}^*(w)$	$\sum_{i=1}^3 S_w^{2i(n)}$	$\sum_{i=1}^3 S_w^{2i(n)}/n$	Lower limit	Upper limit
30	1.16	1.35	0.04516	0.811	1.662
50	1.10	1.27	0.02544	0.827	1.460
100	1.04	1.23	0.01232	0.847	1.286
500	1.08	1.30	0.00259	0.985	1.185
1000	1.02	1.22	0.00122	0.953	1.090
2000	1.03	1.25	0.00063	0.978	1.076
5000	1.04	1.27	0.00025	1.007	1.069
10000	1.04	1.26	0.00013	1.014	1.058

TABLE 6: Expected costs, $H_i(t) = C_i (1 - \frac{t}{w}) e^{-\delta t}$, $w = 5$.

i	$E[B_w^i]$	$E[B_w^i S_i > Y_i]$	$P(S_i > Y_i)$	$P(S_i > Y_i)E[B_w^i S_i > Y_i]$
1	2.8076	1.26053	0.1620753	0.204
2	2.8076	1.26053	0.1620753	0.204
3	1.5236	0.93862	0.6758494	0.634
System cost				1.043

TABLE 7: Statistics of Monte Carlo simulation, $H_i(t) = C_i (1 - \frac{t}{w}) e^{-\delta t}$, $w = 5$.

n	\hat{X}_n	S_n^2	\tilde{X}_n	$P_{2.5}$	P_{25}	P_{75}	$P_{97.5}$	S.Wilk	P-value
100	1.038	0.00925	1.037	0.846	0.973	1.100	1.227	0.9987	0.6595
200	1.042	0.00418	1.040	0.917	0.996	1.085	1.171	0.9985	0.5533

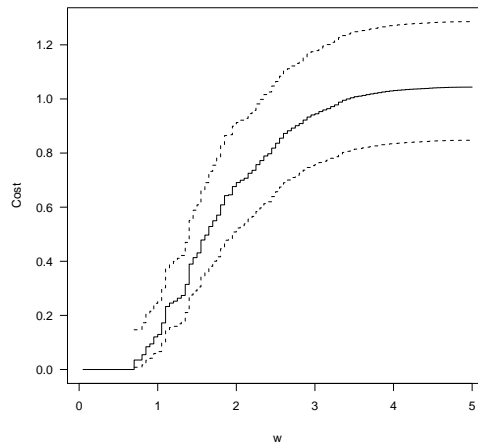


FIGURE 3: 95% Approximate pointwise confidence intervals using limits in (52) with simulated samples and $H_i(t) = C_i \left(1 - \frac{t}{w}\right) e^{-\delta t}$.

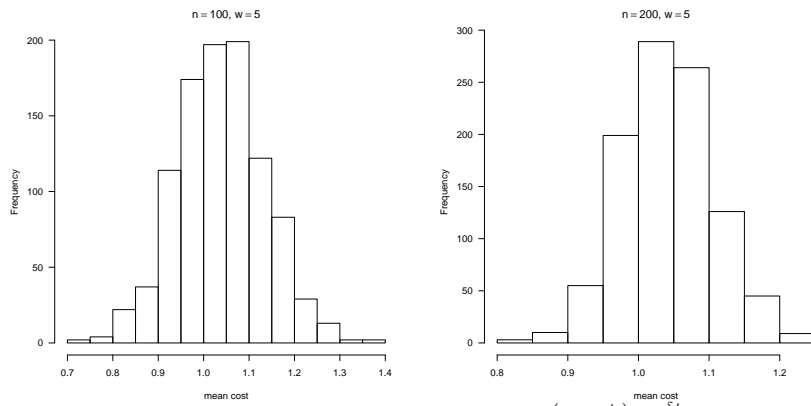


FIGURE 4: Histograms of mean costs, $H_i(t) = C_i \left(1 - \frac{t}{w}\right) e^{-\delta t}$, $n = 100, 200$.

4. Conclusions

A martingale estimator for the expected discounted warranty cost process of a minimally repaired coherent system under its component level observation was proposed. Its asymptotic properties were also presented using the Martingale Central Limit Theorem.

Acknowledgements

Dr. Bueno was partially supported by FAPESP process No. 2010/52227-0, Brazil and Mrs. González was partially supported by CNPq process No. 140903/2007-0, Brazil.

[Recibido: octubre de 2010 — Aceptado: agosto de 2011]

References

- Arjas, E. (1981), 'The failure and hazard processes in multivariate reliability systems', *Mathematics of Operations Research* **6**(4), 551–562.
- Aven, T. & Jensen, U. (1999), *Stochastic Models in Reliability*, Springer-Verlag, Inc., New York.
- Bai, J. & Pham, H. (2004), 'Discounted warranty cost of minimally repaired series systems', *IEEE Transactions on Reliability* **53**, 37–42.
- Bai, J. & Pham, H. (2006), 'Cost analysis on renewable full-service warranties for multi-component systems', *European Journal of Operational Research* **168**, 492–508.
- Balachandran, K. R., Maschmeyer, R. & Livingstone, J. (1981), 'Product warranty period: A Markovian approach to estimation and analysis of repair and replacement costs', *The Accounting Review* **1**, 115–124.
- Billingsley, P. (1968), *Convergence of Probability Measures*, John Wiley and Sons, Inc., New York.
- Blischke, W. R. & Murthy, D. N. P. (1992a), 'Product warranty management-1: A taxonomy for warranty policies', *European Journal of Operational Research* **62**, 127–148.
- Blischke, W. R. & Murthy, D. N. P. (1992b), 'Product warranty management-2: An integrated framework for study', *European Journal of Operational Research* **62**, 261–281.
- Blischke, W. R. & Murthy, D. N. P. (1992c), 'Product warranty management-3: A review of mathematical models', *European Journal of Operational Research* **63**, 1–34.
- Blischke, W. R. & Murthy, D. N. P. (1994), *Warranty Cost Analysis*, Marcel Dekker, Inc., New York.
- Blischke, W. R. & Murthy, D. N. P. (1996), *Product Warranty Handbook*, Marcel Dekker, Inc., New York.

- Chattopadhyay, G. & Rahman, A. (2008), 'Development of lifetime warranty policies and models for estimating costs', *Reliability Engineering and System Safety* **93**, 522–529.
- Chien, Y. H. (2008), 'A general age-replacement model with minimal repair under renewing free-replacement warranty', *European Journal of Operational Research* **186**, 1046–1058.
- Chukova, S. & Dimitrov, B. (1996), Warranty analysis for complex systems, in W. R. Blischke & D. N. P. Murthy, eds, 'Product Warranty Handbook', Chukova-Dimitrov, New York.
- Duchesne, T. & Marri, F. (2009), 'General distributional properties of discounted warranty costs with risk adjustment under minimal repair', *IEEE Transactions on Reliability* **58**, 143–151.
- Finkelstein, M. S. (2004), 'Minimal repair in heterogeneous populations', *Journal of Applied Probability* **41**, 281–286.
- Fleming, T. R. & Harrington, D. P. (1991), *Counting Processes and Survival Analysis*, John Wiley and Sons, Inc., New York.
- González, N. G. (2009), Processos de burn-in e de garantia em sistemas coerentes sob o modelo de tempo de vida geral, Tese De Doutorado, Universidade de Sao Paulo, Instituto de Matemática e Estatística.
- Hong-Zhong, H., Zhie-Jie, L., Yanfeng, L., Yu, L. & Liping, H. (2008), 'A warranty cost model with intermittent and heterogeneous usage', *Eksploatacja I Niezawodnosc-Maintenance and Reliability* **40**, 9–15.
- Hussain, A. Z. M. O. & Murthy, D. N. P. (1998), 'Warranty and redundancy design with uncertain quality', *IIE Transactions* **30**, 1191–1199.
- Ja, S. S., Kulkarni, V., Mitra, A. & Patankar, G. (2001), 'A non renewable minimal-repair warranty policy with time-dependent costs', *IEEE Transactions on Reliability* **50**, 346–352.
- Ja, S. S., Kulkarni, V., Mitra, A. & Patankar, G. (2002), 'Warranty reserves for nonstationary sales processes', *Naval Research Logistics* **49**, 499–513.
- Jack, N. & Murthy, D. N. P. (2007), 'A flexible extended warranty an related optimal strategies', *Journal of the Operational Research Society* **58**, 1612–1620.
- Jain, M. & Maheshwari, S. (2006), 'Discounted costs for repairable units under hybrid warranty', *Applied Mathematics and Computation* **173**, 887–901.
- Jung, K. M., Park, M. & Park, D. H. (2010), 'System maintenance cost depend on life cycle under renewing warranty policy', *Reliability Engineering and System Safety* **95**, 816–821.

- Karr, A. F. (1986), *Point Processes and their Statistical Inference*, Marcel Dekker, Inc., New York.
- Lehmann, E. L. (1999), *Elements of Large-Sample Theory*, Springer-Verlag, Inc., New York.
- Li, H. & Shaked, M. (2003), 'Imperfect repair models with preventive maintenance', *Journal of Applied Probability* **40**, 1043–1059.
- Lindqvist, B. H. (2006), 'On the statistical modeling and analysis of repairable systems', *Statistical Science* **21**, 532–551.
- Lipster, R. S. & Shiryaev, A. N. (2001), *Statistics of Random Processes I. General Theory*, 2 edn, Springer-Verlag, Inc., New York.
- Mamer, J. W. (1987), 'Discounted and per unit costs of product warranty', *Management Science* **33**, 916–930.
- Mamer, W. (1969), 'Determination of warranty reserves', *Management Science* **15**, 542–549.
- Mitra, A. & Patankar, J. G. (1993), 'Market share and warranty costs for renewable warranty programs', *International Journal of Production Economics* **20**, 111–123.
- Murthy, D. N. P. (1990), 'Optimal reliability choice in product design', *Engineering Optimization* **15**, 280–294.
- Nguyen, D. G. & Murthy, D. N. P. (1984), 'Cost analysis of warranty policies', *Naval Research Logistics Quarterly* **31**, 525–541.
- Patankar, J. & Mitra, A. (1995), 'Effect of warranty execution on warranty reserve costs', *Management Science* **41**, 395–400.
- Ritchken, P. H. (1986), 'Optimal replacement policies for irreparable warranty item', *IEEE Transactions on Reliability* **35**, 621–624.
- Samatliy-Pac, G. & Taner, M. R. (2009), 'The role of repair strategy in warranty cost minimization: An investigation via quasi-renewal processes', *European Journal of Operational Research* **197**, 632–641.
- Sheu, S. H. & Yu, S. L. (2005), 'Warranty strategy accounts for bathtub failure rate and random minimal repair cost', *An International Journal Computers & Mathematics with Applications* **49**, 1233–1242.
- Thomas, M. U. (1989), 'A prediction model of manufacturer warranty reserves', *Management Science* **35**, 1515–1519.
- Yeo, W. M. & Yuan, X. M. (2009), 'Optimal warranty policies for systems with imperfect repair', *European Journal of Operational Research* **199**, 187–197.
- Yun, W. Y., Murthy, D. N. P. & Jack, N. (2008), 'Warranty servicing with imperfect repair', *International Journal of Production Economics* **111**, 159–169.

Appendix A. Proof of Theorem 2

If $i \in \mathcal{C}^\Phi(\omega)$, from Proposition 1 and the martingale property we have

$$E[\widehat{B}_t^i | S_i > Y_i] = E\left[\int_0^t H_i(s) d\widetilde{N}_s^i \middle| S_i > Y_i\right] = E\left[\int_{Y_i}^t H_i(s) \lambda^i(s) ds \middle| S_i > Y_i\right]$$

Since the sequences $(\widehat{B}_t^{i(j)}, C^{i(j)}, 1 \leq i \leq m), 1 \leq j \leq n$, are independent and identically distributed copies of the m -variate process $(\widehat{B}_t^i, C^i, 1 \leq i \leq m)$, from (32) we have

$$E[\widetilde{B}_t^{i(n)}] = \frac{1}{n} \sum_{j=1}^n P(S_i > Y_i) E\left[\int_{Y_i}^t H_i(s) \lambda^i(s) ds \middle| S_i > Y_i\right]$$

and therefore, $E[\widetilde{B}_t^{i(n)}] = \frac{1}{n} \sum_{j=1}^n B^{i*}(t) = B^{i*}(t) = E[\widehat{B}_t^{i(n)}]$.

To set the consistency of proposed estimator we have to prove that

$$E[\sup_{s \leq t} (\widetilde{B}_s^{i(n)} - B^{i*}(s))^2] \xrightarrow{n \uparrow \infty} 0 \quad (58)$$

First, from (32) and Proposition 1 and for fixed n we have

$$\widetilde{B}_t^{i(n)} - \overline{B}_t^{i(n)} = \frac{1}{n} \sum_{j=1}^n C^{i(j)} (\widehat{B}_t^{i(j)} - B_t^{i(j)}) = \frac{1}{n} \sum_{j=1}^n C^{i(j)} (\widehat{B}_t^{i(j)} - B_t^{i*(j)}) \quad (59)$$

is a mean zero and square integrable (P, \mathcal{F}_t) -martingale. Furthermore, from the independence conditions and (23) we have

$$\langle \widetilde{B}^{i(n)} - \widehat{B}^{i(n)} \rangle_t = \frac{1}{n} \times \left[\frac{1}{n} \sum_{j=1}^n C^{i(j)} \int_{Y_i^{(j)}}^t H_i^2(s) \lambda^i(s) ds \right] \quad (60)$$

By hypothesis, for each $i = 1, \dots, m$

$$E\left[C^i \int_{Y_i}^t H_i^2(s) \lambda^i(s) ds\right] = P(S_i > Y_i) E\left[\int_{Y_i}^t H_i^2(s) \lambda^i(s) ds \middle| S_i > Y_i\right] < \infty \quad (61)$$

and, therefore, using the Strong Law of Large Numbers we have

$$\frac{1}{n} \sum_{j=1}^n C^{i(j)} \int_{Y_i^{(j)}}^t H_i^2 \lambda^i(s) ds \xrightarrow{n \uparrow \infty} P(S_i > Y_i) E\left[\int_{Y_i}^t H_i^2(s) \lambda^i(s) ds \middle| S_i > Y_i\right] \quad (62)$$

Using (60) and (62) we conclude that

$$\langle \widetilde{B}^{i(n)} - \overline{B}^{i(n)} \rangle_t \xrightarrow{n \rightarrow \infty} 0 \times P(S_i > Y_i) E\left[\int_{Y_i}^t H_i^2(s) \lambda^i(s) ds \middle| S_i > Y_i\right] = 0 \quad (63)$$

Furthermore, we have (Lipster & Shiryaev 2001, Theorem 2.4)

$$E[\sup_{s \leq t} (\widehat{B}_s^{i(n)} - \widehat{B}_s^{i(n)})^2] \leq 4E[(\overline{B}_t^{i(n)} - \overline{B}_t^{i(n)})^2] = 4E[\langle \overline{B}^{i(n)} - \overline{B}^{i(n)} \rangle_t] \quad (64)$$

where the last equality is because $(\overline{B}_t^{i(n)} - \overline{B}_t^{i(n)})$ is a mean zero and square integrable (P, \mathcal{F}_t) -martingale. From (63) and (64), we have

$$E[\sup_{s \leq t} (\overline{B}_s^{i(n)} - \overline{B}_s^{i(n)})^2] \xrightarrow{n \uparrow \infty} 0 \quad (65)$$

Also, from the Strong Law of Large Numbers and continuity in t , we get

$$(\overline{B}_s^{i(n)} - B^{i*}(s)) \xrightarrow{n \rightarrow \infty} 0, \quad \forall s \leq t \text{ and therefore, } \sup_{s \leq t} |\overline{B}_s^{i(n)} - B^{i*}(s)| \xrightarrow{n \rightarrow \infty} 0$$

then, we conclude

$$\sup_{s \leq t} (\overline{B}_s^{i(n)} - B^{i*}(s))^2 \xrightarrow{n \rightarrow \infty} 0$$

and

$$E[\sup_{s \leq t} (\overline{B}_s^{i(n)} - B^{i*}(s))^2] \xrightarrow{n \rightarrow \infty} 0 \quad (66)$$

Furthermore, we have

$$E[\sup_{s \leq t} (\widehat{B}_s^{i(n)} - B^{i*}(s))^2] \leq E[\sup_{s \leq t} (\overline{B}_s^{i(n)} - \overline{B}_s^{i(n)})^2] + E[\sup_{s \leq t} (\overline{B}_s^{i(n)} - B^{i*}(s))^2]$$

and taking limits in the above inequality, from (65) and (66) we get

$$\lim_{n \rightarrow \infty} E[\sup_{s \leq t} (\widehat{B}_s^{i(n)} - B^{i*}(s))^2] = 0 \quad (67)$$

and (58) is proved.

Appendix B. Proof of Proposition 2

First, as the sequences $(\widehat{B}_t^{i(j)}, C^{i(j)}, 1 \leq j \leq n)$ are independent and identically distributed copies of (\widehat{B}_t^i, C^i) , we have that, for all $t \geq 0$ and $i = 1, \dots, m$,

$$E[\overline{B}_t^{i(n)}] = \frac{1}{n} \sum_{j=1}^n P(S_i > Y_i) E \left[\int_{Y_i}^t H_i(s) \lambda^i(s) ds \mid S_i > Y_i \right] = B^{i*}(t)$$

and $\overline{B}_t^{i(n)}$ is an unbiased estimator for $B^{i*}(t)$.

Furthermore, from the Strong Law of Large Numbers, $\overline{B}_t^{i(n)}$ converges almost surely to $B^{i*}(t)$.

Otherwise, as $(\mathbf{Z}_t^{(n)})_{n \geq 1}$ is a sequence of independent and identically distributed random vectors and the components do not have simultaneous failures, the processes $\overline{B}_t^{i(n)}$ and $\overline{B}_t^{j(n)}$ are uncorrelated in $[0, t]$, for all $i, j, i \neq j$, all n and

$$COV[Z_t^{i(n)}, Z_t^{j(n)}] = COV[\sqrt{n} \overline{B}_t^{i(n)}, \sqrt{n} \overline{B}_t^{j(n)}] = COV[C^i B_t^i, C^j B_t^j] = 0 \quad (68)$$

Consequently

$$\text{Var}[Z_t^{i(n)}] = \text{Var}[\sqrt{n} \overline{B}_t^{i(n)}] = \text{Var}[C^i B_t^i] = \sigma^{2i^*}(t) \quad (69)$$

Therefore, applying the Central Limit Theorem for a sequence of independent and identically distributed random vectors with mean $\boldsymbol{\mu}(t) = (B^{1^*}(t), \dots, B^{m^*}(t))$ and finite covariance matrix $\boldsymbol{\Sigma}(t)$, where $\Sigma_{ij}(t) = 1_{\{i=j\}} \sigma^{2i^*}(t)$, we obtain that $\mathbf{Z}_t^{(n)} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathbf{Z}_t$, where \mathbf{Z}_t is an m -variate Normal random vector with mean zero and covariance matrix $\boldsymbol{\Sigma}(t)$. In what follows we prove the convergence of the finite-dimensional distributions of the process $\mathbf{Z}^{(n)}$. For that we consider:

- (a) Since $\forall t \geq 0, n \geq 1, i \neq j$, $COV[Z_t^{i(n)}, Z_t^{j(n)}] = COV[C^i B_t^i, C^j B_t^j] = 0$, we have $\forall t_k \leq t_l, t_k, t_l \in [0, t]$, $COV[Z_{t_k}^{i(n)}, Z_{t_l}^{j(n)}] = COV[C^i B_{t_k}^i, C^j B_{t_l}^j] = 0$;
- (b) From the above, we can prove the convergence of the finite-dimensional distributions of the process $\mathbf{Z}^{(n)}$ using the Crámer-Wold procedure: proving the convergence for each component $Z^{i(n)}$, for all $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq t$, we prove that $\forall a_{il}$ arbitrary constants,

$$\sum_{i=1}^m \sum_{l=1}^{k-1} a_{il} (Z_{t_{i+1}}^{i(n)} - Z_{t_i}^{i(n)}) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \sum_{i=1}^m \sum_{l=1}^{k-1} a_{il} (Z_{t_{i+1}}^i - Z_{t_i}^i) \quad (70)$$

which, using the Cramer-Wold procedure, is equivalent to

$$(\mathbf{Z}_{t_1}^{(n)}, \mathbf{Z}_{t_2}^{(n)}, \dots, \mathbf{Z}_{t_k}^{(n)}) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} (\mathbf{Z}_{t_1}, \mathbf{Z}_{t_2}, \dots, \mathbf{Z}_{t_k})$$

Now, for each $i, 1 \leq i \leq m$ and $t_1 \leq t_2 \in [0, t]$, consider n independent and identically distributed copies of $(C^i B_{t_1}^i, C^i B_{t_2}^i)$. Then, for each n and $i = 1, \dots, m$ we get the random vector $(Z_{t_1}^{i(n)}, Z_{t_2}^{i(n)})$. Therefore

$$E(Z_{t_1}^{i(n)}, Z_{t_2}^{i(n)}) = (0, 0), \quad \forall n \geq 1, i = 1, \dots, m.$$

Furthermore, since the copies $C^{i(j)} B_{t_1}^{i(j)} B_{t_2}^{i(j)}$, $j = 1, \dots, n$ are independent and identically distributed and as, for independent copies j and k , the random variables $C^{i(j)} B_{t_1}^{i(j)}$ and $C^{i(k)} B_{t_2}^{i(k)}$ are also independent, we have

$$COV[Z_{t_1}^{i(n)}, Z_{t_2}^{i(n)}] = E[C^i B_{t_1}^i B_{t_2}^i] - B^{i^*}(t_1) B^{i^*}(t_2) = \sigma^{i^*}(t_1, t_2) < \infty. \quad (71)$$

From (69), $\text{Var}[Z_{t_1}^{i(n)}] = \sigma^{2i^*}(t_1)$ and $\text{Var}[Z_{t_2}^{i(n)}] = \sigma^{2i^*}(t_2)$. Then, from the Central Limit Theorem for a sequence of independent and identically distributed random vectors, with finite mean vector and finite covariance matrix, we have

$$(Z_{t_1}^{i(n)}, Z_{t_2}^{i(n)}) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} (Z_{t_1}^i, Z_{t_2}^i), \forall t_1 \leq t_2 \in [0, t] \tag{72}$$

where $(Z_{t_1}^i, Z_{t_2}^i)$ is a bivariate normal vector with mean zero and covariance matrix $\Sigma^i(t_1, t_2)$,

$$\Sigma^i(t_1, t_2) = \begin{bmatrix} \sigma^{2i^*}(t_1) & \sigma^{i^*}(t_1, t_2) \\ \sigma^{i^*}(t_1, t_2) & \sigma^{2i^*}(t_2) \end{bmatrix} \tag{73}$$

Using an induction argument we can generalize the above result for all partition $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq t$, of the interval $[0, t]$ and we get for all $i, 1 \leq i \leq m$,

$$(Z_{t_1}^{i(n)}, Z_{t_2}^{i(n)}, \dots, Z_{t_k}^{i(n)}) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} (Z_{t_1}^i, Z_{t_2}^i, \dots, Z_{t_k}^i)$$

where $(Z_{t_1}^i, Z_{t_2}^i, \dots, Z_{t_k}^i)$ is a k -variate Normal vector with mean zero and finite covariance matrix.

Finally, we analyze Stone's tightness condition in $D[0, t]^m$ (Fleming & Harrington 1991), that is: If for each $i, 1 \leq i \leq m$ and for all $\epsilon > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \uparrow \infty} P \left\{ \sup_{\substack{|s-u| < \delta \\ 0 \leq s, u \leq t}} |Z_s^{i(n)} - Z_u^{i(n)}| > \epsilon \right\} = 0 \tag{74}$$

Since $Z_s^{i(n)}$ is continuous and monotone in $[0, t]$, we have

$$\begin{aligned} &P \left\{ \sup_{\substack{|s-u| < \delta \\ 0 \leq s, u \leq t}} |Z_s^{i(n)} - Z_u^{i(n)}| \leq \epsilon \right\} \\ &\leq P \left\{ |Z_s^{i(n)} - Z_u^{i(n)}| \leq \epsilon, \text{ for } s \text{ and } u \text{ fixed: } 0 \leq s, u \leq t, |s - u| < \delta \right\} \end{aligned} \tag{75}$$

From (72) and (73), for all $0 \leq s \leq u$

$$(Z_s^{i(n)} - Z_u^{i(n)}) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} N(0, \gamma^2(s, u)), \gamma^2(s, u) = \sigma^{2i^*}(s) + \sigma^{2i^*}(u) - 2\sigma^{i^*}(s, u). \tag{76}$$

Finally, from (69) and (71) it is clear that $\lim_{\delta \downarrow 0} \gamma^2(s, u) = 0, |s - u| < \delta, 0 \leq s, u \leq t$. Then, from (75) we have

$$\begin{aligned} \lim_{\delta \downarrow 0} \lim_{n \rightarrow \infty} P \left\{ \sup_{\substack{|s-u| < \delta \\ 0 \leq s, u \leq t}} |Z_s^{i(n)} - Z_u^{i(n)}| \leq \epsilon \right\} &\leq \lim_{\delta \downarrow 0} 2\Phi \left(\frac{\epsilon}{\sqrt{\gamma^2(s, u)}} \right) - 1 \\ &= 2\Phi(\infty) - 1 = 1 \end{aligned} \quad \square$$

Appendix C. Proof of Theorem 4

From Theorem 2 we have $E[\mathbf{E}_t^{(n)}] = 0$ for all $n \geq 1$ and $t \geq 0$.

Let $\mathbf{M}_t^{(n)} = \sqrt{n}(\overline{\mathbf{B}}_t^{(n)} - \overline{\mathbf{B}}_t^{(n)})$ and $\mathbf{Z}_t^{(n)} = \sqrt{n}(\overline{\mathbf{B}}_t^{(n)} - \boldsymbol{\mu}(t))$. Note that

$$\mathbf{E}_t^{(n)} = \sqrt{n}(\overline{\mathbf{B}}_t^{(n)} - \boldsymbol{\mu}(t)) = \mathbf{M}_t^{(n)} + \mathbf{Z}_t^{(n)} \quad (77)$$

Now, for all $t \geq 0$ and $1 \leq i \leq m$ we are going to calculate the asymptotic variance for the processes $E_t^{i(n)} = \sqrt{n}(\overline{B}_t^{i(n)} - B^{i*}(t)) = M_t^{i(n)} + Z_t^{i(n)}$. For fixed t ,

$$\text{Var}[E_t^{i(n)}] = \text{Var}[M_t^{i(n)}] + \text{Var}[Z_t^{i(n)}] + 2 \text{COV}[M_t^{i(n)}, Z_t^{i(n)}] \quad (78)$$

Since the copies are independent and identically distributed, from Corollary 3 and for all $t \geq 0$, we have that $\text{Var}[M_t^{i(n)}]$ corresponds to

$$E[\langle M^{i(n)} \rangle_t] = E\left[\frac{1}{n} \sum_{j=1}^n C^{i(j)} \int_{Y_i^{(j)}}^t H_i^2(s) \lambda^i(s) ds\right] = E[C^i(\widehat{B}_t^i - B_t^i)^2] = V_i^*(t); \quad (79)$$

and $\text{Var}[Z_t^{i(n)}]$ is given by (69).

In order to calculate $\text{COV}[M_t^{i(n)}, Z_t^{i(n)}]$, we use the covariance definition, the martingale property, the fact that for independent copies j and l , $C^{i(j)}\widehat{B}_t^{i(j)}$ and $C^{i(l)}\widehat{B}_t^{i(l)}$ are also independent, concluding

$$\text{COV}[M_t^{i(n)}, Z_t^{i(n)}] = E[C^i \widehat{B}_t^i B_t^i] - E[C^i (B_t^i)^2] \quad (80)$$

Therefore, from (69), (79) and (80), we obtain in (78) that, for all $n \geq 1$ and $t \geq 0$

$$\text{Var}[E_t^{i(n)}] = E[C^i (\widehat{B}_t^i)^2] - (B^{i*}(t))^2$$

In addition, we have $E[C^i \widehat{B}_t^i] = B^{i*}(t)$ and then,

$$\text{Var}[E_t^{i(n)}] = \text{Var}[C^i \widehat{B}_t^i] = \delta^{2i*}(t) \quad (81)$$

We also calculate $\text{COV}[E_t^{i(n)}, E_t^{j(n)}]$ for $n \geq 1$ and $i \neq j$, and since processes \widehat{B}_t^i and \widehat{B}_t^j do not have simultaneous jumps, we obtain,

$$\text{COV}[E_t^{i(n)}, E_t^{j(n)}] = \text{COV}[C^i \widehat{B}_t^i, C^j \widehat{B}_t^j] = 0 \quad (82)$$

From results (81) and (82) we conclude that the asymptotic covariance for the process $\mathbf{E}_t^{(n)}$ is $\mathbf{U}(t)$ where $U_{ij}(t) = 1_{\{i=j\}} \delta^{2i*}(t)$. Next, we set the asymptotic normality of $\mathbf{E}_t^{(n)}$ by considering the results from its asymptotic covariance structure and the convergence in distribution of the processes $\mathbf{M}_t^{(n)}$ (Corollary 3) and $\mathbf{Z}_t^{(n)}$ (Proposition 2):

As the processes $\mathbf{M}^{(n)}$ and $\mathbf{Z}^{(n)}$ satisfy the tightness condition in $D[0, t]^m$ and their finite-dimensional distributions converge to Gaussian continuous processes, such that $\forall t_k, t_l \in [0, t], COV[E_{t_k}^{i(n)}, E_{t_l}^{j(n)}] = 0$, the process $\mathbf{E}^{(n)} = \mathbf{M}^{(n)} + \mathbf{Z}^{(n)}$ also satisfies the tightness condition.

Also, its finite-dimensional distributions converge to Gaussian continuous processes and for all partition $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq t$, we can prove that,

$$\begin{aligned} &\forall a_{il} \text{ arbitrary constants,} \\ &\sum_{i=1}^m \sum_{l=1}^{k-1} a_{il} (E_{t_{l+1}}^{i(n)} - E_{t_l}^{i(n)}) = \sum_{i=1}^m \sum_{l=1}^{k-1} a_{il} (M_{t_{l+1}}^{i(n)} - M_{t_l}^{i(n)}) + \sum_{i=1}^m \sum_{l=1}^{k-1} a_{il} (Z_{t_{l+1}}^{i(n)} - Z_{t_l}^{i(n)}) \\ &\hspace{15em} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \\ &\sum_{i=1}^m \sum_{l=1}^{k-1} a_{il} (M_{t_{l+1}}^i - M_{t_l}^i) + \sum_{i=1}^m \sum_{l=1}^{k-1} a_{il} (Z_{t_{l+1}}^i - Z_{t_l}^i) = \sum_{i=1}^m \sum_{l=1}^{k-1} a_{il} (W_{t_{l+1}}^i - W_{t_l}^i) \quad \square \end{aligned}$$

which, using the Cramer-Wold procedure, is equivalent to

$$\begin{aligned} (\mathbf{E}_{t_1}^{(n)}, \mathbf{E}_{t_2}^{(n)}, \dots, \mathbf{E}_{t_k}^{(n)}) &= (\mathbf{M}_{t_1}^{(n)}, \mathbf{M}_{t_2}^{(n)}, \dots, \mathbf{M}_{t_k}^{(n)}) + (\mathbf{Z}_{t_1}^{(n)}, \mathbf{Z}_{t_2}^{(n)}, \dots, \mathbf{Z}_{t_k}^{(n)}) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \\ &(\mathbf{M}_{t_1}, \mathbf{M}_{t_2}, \dots, \mathbf{M}_{t_k}) + (\mathbf{Z}_{t_1}, \mathbf{Z}_{t_2}, \dots, \mathbf{Z}_{t_k}) = (\mathbf{W}_{t_1}, \mathbf{W}_{t_2}, \dots, \mathbf{W}_{t_k}) \end{aligned}$$