

Testing Equality of Several Correlation Matrices

Prueba de Igualdad de Varias Matrices de Correlación

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Abstract

In this article we show that the Kullback's statistic for testing equality of several correlation matrices may be considered a modified likelihood ratio statistic when sampling from multivariate normal populations. We derive the asymptotic null distribution of L^* in series involving independent chi-square variables by expanding L^* in terms of other random variables and then inverting the expansion term by term. An example is also given to exhibit the procedure to be used when testing the equality of correlation matrices using the statistic L^* .

Key words: Asymptotic null distribution, Correlation matrix, Covariance matrix, Cumulants, Likelihood ratio test.

Resumen

En este artículo se muestra que el estadístico L^* de Kullback, para probar la igualdad de varias matrices de correlación, puede ser considerado como un estadístico modificado del test de razón de verosimilitud cuando se muestrean poblaciones normales multivariadas. Derivamos la distribución asintótica nula de L^* en series que involucran variables independientes chi-cuadrado, mediante la expansión de L^* en términos de otras variables aleatorias y luego invertir la expansión término a término. Se da también un ejemplo para mostrar el procedimiento a ser usado cuando se prueba igualdad de matrices de correlación mediante el estadístico L^* .

Palabras clave: distribución asintótica nula, matriz de correlación, matriz de covarianza, razón de verosimilitud.

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1. Introduction

The correlation matrix is one of the foundations of factor analysis and has found its way into such diverse areas as economics, medicine, physical science and political science. There is a fair amount of literature on testing properties of correlation matrices. Tests for certain structures in a correlation matrix have been proposed and studied by several authors, e.g, see Aitkin, Nelson, and Reinfurt (1968), Gleser (1968), Aitkin (1969), Modarres (1993), Kullback (1997) and Schott (2007). In a series of papers, Konishi (1978, 1979*a*, 1979*b*) has developed asymptotic expansions of correlation matrix and applied them to various problems of multivariate analysis. The exact distribution of the correlation matrix, when sampling from a multivariate Gaussian population, is derived in Ali, Fraser and Lee (1970) and Gupta and Nagar (2000).

If the covariance matrix of α -th population is given by Σ_α and Δ_α is a diagonal matrix of standard deviations for the population α , then $P_\alpha = \Delta_\alpha^{-1}\Sigma_\alpha\Delta_\alpha^{-1}$ is the correlation matrix for the population α . The null hypothesis that all k populations have the same correlation matrices may be stated as $H : P_1 = \dots = P_k$.

Let the vectors $\mathbf{x}_{\alpha 1}, \mathbf{x}_{\alpha 2}, \dots, \mathbf{x}_{\alpha N_\alpha}$ be a random sample of size $N_\alpha = n_\alpha + 1$ for $\alpha = 1, 2, \dots, k$ from k multivariate populations of dimensionality p . Further, we assume the independence of these k samples. Let $\bar{\mathbf{x}}_\alpha = \sum_{i=1}^{N_\alpha} \mathbf{x}_{\alpha i} / N_\alpha$, $A_\alpha = \sum_{i=1}^{N_\alpha} (\mathbf{x}_{\alpha i} - \bar{\mathbf{x}}_\alpha)(\mathbf{x}_{\alpha i} - \bar{\mathbf{x}}_\alpha)'$ and $S_\alpha = A_\alpha / N_\alpha$. Further, let D_α be a diagonal matrix of the square roots of the diagonal elements of S_α . The sample correlation matrix R_α is then defined by $R_\alpha = D_\alpha^{-1}S_\alpha D_\alpha^{-1}$. Let $n = \sum_{\alpha=1}^k n_\alpha$ and $\bar{R} = \sum_{\alpha=1}^k n_\alpha R_\alpha$.

Kullback (1967) derived the statistic $L^* = \sum_{\alpha=1}^k n_\alpha \ln\{\det(\bar{R}) / \det(R_\alpha)\}$ for testing the equality of k correlation matrices based on samples from multivariate populations. This statistic was later examined by Jennrich (1970) who observed that the statistic proposed by Kullback failed to have chi-square distribution ascribed to it. For further results on this topic the reader is referred to Browne (1978) and Modarres and Jernigan (1992).

Although the Kullback's statistic L^* is not equal to the modified likelihood ratio criterion, we here show that it may be considered an approximation of the modified likelihood ratio statistic when sampling from multivariate normal populations.

In Section 2, we show that Kullback's statistic can be viewed as an approximation of the modified likelihood ratio statistic based on samples from multivariate normal populations. Section 3 deals with some preliminary results and definitions which are used in subsequent sections. In sections 4 and 5, we obtain asymptotic null distribution of L^* by expanding L^* in terms of other random variables and then inverting the expansion term by term. Finally, in Section 6, an example is given to demonstrate the procedure to be used when testing the equality of correlation matrices using the statistic L^* . Some results on matrix algebra and distribution theory are given in the Appendix.

2. The Test Statistic

In this section, we give an approximation of the likelihood ratio test statistic λ for testing equality of correlation matrices of several multivariate Gaussian populations. The test statistic λ was derived and studied by Cole (1968*a*, 1968*b*) in two unpublished technical reports (see Browne 1978, Modarres and Jernigan 1992, 1993). However, these reports are scarcely available, and therefore the sake of completeness and for a better understanding it seems appropriate to first give a concise step-by-step derivation of the test statistic λ .

If the underlying populations follow multivariate normal distributions, then the likelihood function based on the k independent samples, when all parameters are unrestricted, is given by

$$\begin{aligned} L(\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k, \Sigma_1, \dots, \Sigma_k) &= \prod_{\alpha=1}^k \left[(2\pi)^{pN_\alpha/2} \det(\Sigma_\alpha)^{N_\alpha/2} \right]^{-1} \\ &\times \exp \left[-\frac{1}{2} \sum_{\alpha=1}^k \text{tr}(\Sigma_\alpha^{-1} A_\alpha) - \frac{1}{2} \sum_{\alpha=1}^k \text{tr} \left\{ \Sigma_\alpha^{-1} (\bar{\mathbf{x}}_\alpha - \boldsymbol{\mu}_\alpha) (\bar{\mathbf{x}}_\alpha - \boldsymbol{\mu}_\alpha)' \right\} \right] \end{aligned}$$

where for $\alpha = 1, \dots, k$ we have $\boldsymbol{\mu}_\alpha \in \mathbb{R}^p$ and $\Sigma_\alpha > 0$. It is well known that for any fixed value of Σ_α the likelihood function is maximized with respect to the $\boldsymbol{\mu}_\alpha$'s when $\hat{\boldsymbol{\mu}}_\alpha = \bar{\mathbf{x}}_\alpha$.

Let Δ_α be a diagonal matrix of standard deviations for the population α . Further, let $P_\alpha = \Delta_\alpha^{-1} \Sigma_\alpha \Delta_\alpha^{-1}$ be the population correlation matrix for the population α . The natural logarithm of the likelihood function, after evaluation at $\hat{\boldsymbol{\mu}}_\alpha = \bar{\mathbf{x}}_\alpha$, may then be written as

$$\begin{aligned} \ln[L(\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_k, \Delta_1 P_1 \Delta_1, \dots, \Delta_k P_k \Delta_k)] &= -\frac{1}{2} N p \ln(2\pi) - \frac{1}{2} \sum_{\alpha=1}^k N_\alpha \ln[\det(P_\alpha \Delta_\alpha^2)] - \frac{1}{2} \sum_{\alpha=1}^k \text{tr}(N_\alpha P_\alpha^{-1} G_\alpha R_\alpha G_\alpha) \end{aligned}$$

where $N = \sum_{\alpha=1}^k N_\alpha$ and $G_\alpha = \Delta_\alpha^{-1} D_\alpha$. Further, when the parameters are unrestricted, the likelihood function $L(\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_k, \Delta_1 P_1 \Delta_1, \dots, \Delta_k P_k \Delta_k)$ is maximized when $-\ln[\det(P_\alpha \Delta_\alpha^2)] - \text{tr}(P_\alpha^{-1} G_\alpha R_\alpha G_\alpha)$ is maximized for each α . This is true when

$$\begin{aligned} \ln[\det(P_\alpha \Delta_\alpha^2)] + \text{tr}(P_\alpha^{-1} G_\alpha R_\alpha G_\alpha) &= \ln[\det(\Delta_\alpha P_\alpha \Delta_\alpha)] + \text{tr}(\Delta_\alpha^{-1} P_\alpha^{-1} \Delta_\alpha^{-1} D_\alpha R_\alpha D_\alpha) \end{aligned}$$

is minimized for each α . This is achieved when $\Delta_\alpha P_\alpha \Delta_\alpha = D_\alpha R_\alpha D_\alpha$. From this it follows that the maximum value of $L(\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_k, \Delta_1 P_1 \Delta_1, \dots, \Delta_k P_k \Delta_k)$, when

the parameters are unrestricted, is given by

$$\begin{aligned} \ln[L(\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_k, D_1 R_1 D_1, \dots, D_k R_k D_k)] \\ = -\frac{1}{2} N p [\ln(2\pi) + 1] - \frac{1}{2} \sum_{\alpha=1}^k N_{\alpha} \ln[\det(R_{\alpha} D_{\alpha}^2)]. \end{aligned} \quad (1)$$

Let P be the common value of the population correlation matrices under the null hypothesis of equality of correlation matrices. The reduced parameter space for the covariance matrices is the set of all covariance matrices that may be written as $\Delta_{\alpha} P$ where P is a correlation matrix and Δ_{α} is a diagonal matrix with positive elements on the diagonal. The restricted log likelihood function is written as

$$\begin{aligned} \ln[L(\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_k, \Delta_1 P, \dots, \Delta_k P)] \\ = -\frac{1}{2} N p \ln(2\pi) - \frac{1}{2} \sum_{\alpha=1}^k N_{\alpha} \ln[\det(P \Delta_{\alpha}^2)] - \frac{1}{2} \sum_{\alpha=1}^k N_{\alpha} \operatorname{tr}(P^{-1} G_{\alpha} R_{\alpha} G_{\alpha}). \end{aligned}$$

Let $P^{-1} = (\rho^{ij})$. Since Δ_{α} is a diagonal matrix,

$$\ln[\det(\Delta_{\alpha}^2)] = 2 \ln[\det(\Delta_{\alpha})] = 2 \ln \left[\prod_{i=1}^p \sigma_{\alpha ii} \right] = 2 \sum_{i=1}^p \ln(\sigma_{\alpha ii})$$

Also, since $G_{\alpha} = \Delta_{\alpha}^{-1} D_{\alpha}$ is a diagonal matrix, we have

$$\operatorname{tr}(P^{-1} G_{\alpha} R_{\alpha} G_{\alpha}) = \sum_{i=1}^p \sum_{j=1}^p \rho^{ij} g_{\alpha j} r_{\alpha ij} g_{\alpha i}$$

Thus,

$$\begin{aligned} \ln[L(\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_k, \Delta_1 P, \dots, \Delta_k P)] \\ = -\frac{1}{2} N p \ln(2\pi) - \frac{1}{2} \sum_{\alpha=1}^k N_{\alpha} \sum_{i=1}^p \ln(\sigma_{\alpha ii}) - \frac{1}{2} \sum_{\alpha=1}^k N_{\alpha} \ln[\det(P)] \\ - \frac{1}{2} \sum_{\alpha=1}^k N_{\alpha} \sum_{i=1}^p \sum_{j=1}^p \rho^{ij} g_{\alpha j} r_{\alpha ij} g_{\alpha i} \end{aligned}$$

Since, $g_{\alpha i} = s_{\alpha ii} / \sigma_{\alpha ii}$, differentiation of $\ln[L(\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_k, \Delta_1 P, \dots, \Delta_k P)]$ with respect to $\sigma_{\alpha ii}$ yields

$$\frac{\partial}{\partial \sigma_{\alpha ii}} \ln[L(\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_k, \Delta_1 P, \dots, \Delta_k P)] = -\frac{N_{\alpha}}{2\sigma_{\alpha ii}} + \frac{N_{\alpha}}{2\sigma_{\alpha ii}} \sum_{j=1}^p g_{\alpha i} g_{\alpha j} \rho^{ij} r_{\alpha ij}$$

Further, setting this equal to zero gives $\sum_{j=1}^p g_{\alpha i} g_{\alpha j} \rho^{ij} r_{\alpha ij} - 1 = 0$. Differentiating $\ln[L(\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_k, \Delta_1 P, \dots, \Delta_k P)]$ with respect to the matrix P using Lemma 6, we obtain

$$\frac{\partial}{\partial P} \ln[L(\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_k, \Delta_1 P, \dots, \Delta_k P)] = -\frac{1}{2} N P^{-1} + \frac{1}{2} \sum_{\alpha=1}^k N_{\alpha} P^{-1} G_{\alpha} R_{\alpha} G_{\alpha} P^{-1}$$

Setting this equal to zero, multiplying by 2, pre and post multiplying by P and dividing by N gives $P = \sum_{\alpha=1}^k N_{\alpha} G_{\alpha} R_{\alpha} G_{\alpha} / N$ so that $\sum_{\alpha=1}^k N_{\alpha} g_{\alpha i}^2 / N = 1$.

The likelihood ratio test statistic λ for testing $H : P_1 = \dots = P_k$ is now derived as

$$\lambda = \prod_{\alpha=1}^k \frac{\det(R_{\alpha} D_{\alpha}^2)^{N_{\alpha}/2}}{\det(\widehat{P} \widehat{\Delta}_{\alpha}^2)^{N_{\alpha}/2}}$$

where \widehat{P} and $\widehat{\Delta}_{\alpha}^2$ are solutions of $\widehat{P} = \sum_{\alpha=1}^k N_{\alpha} \widehat{\Delta}_{\alpha}^{-1} S_{\alpha} \widehat{\Delta}_{\alpha}^{-1} / N$ and $\sum_{j=1}^p \rho^{ij} s_{\alpha ij} - 1 = 0$, $i = 1, \dots, p$, respectively.

To obtain an approximation of the likelihood ratio statistic we replace $\sigma_{\alpha ii}$ by its consistent estimator $\widehat{\sigma}_{\alpha ii}$. Then, it follows that $\widehat{g}_{\alpha ii} = s_{\alpha ii} / \widehat{\sigma}_{\alpha ii}$ and $\widehat{G}_{\alpha} = \text{diag}(\widehat{g}_{\alpha 1}, \dots, \widehat{g}_{\alpha p})$, and the estimator of P is given by $\widehat{P} = \sum_{\alpha=1}^k N_{\alpha} \widehat{G}_{\alpha} R_{\alpha} \widehat{G}_{\alpha} / N$. Thus, an approximation of the maximum of $\ln[L(\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_k, \Delta_1 P, \dots, \Delta_k P)]$ is given as

$$-\frac{1}{2} N p [\ln(2\pi) + 1] - \frac{1}{2} \sum_{\alpha=1}^k N_{\alpha} \ln[\det(\widehat{\Delta}_{\alpha})^2] - \frac{1}{2} N \ln[\det(\widehat{P})] \tag{2}$$

As the sample size goes to infinity, $s_{\alpha ii} / \widehat{\sigma}_{\alpha ii}$ converges in probability to 1 so that \widehat{G}_{α} converges in probability to I_p . This suggest further approximation of (2) as

$$-\frac{1}{2} N p [\ln(2\pi) + 1] - \frac{1}{2} \sum_{\alpha=1}^k N_{\alpha} \ln[\det(D_{\alpha})^2] - \frac{1}{2} N \ln \left[\det \left(\sum_{\alpha=1}^k \frac{N_{\alpha}}{N} R_{\alpha} \right) \right] \tag{3}$$

Now, using (1) and (3), the likelihood ratio statistic is approximated as

$$\tilde{\lambda} = \frac{\prod_{\alpha=1}^k \det(R_{\alpha})^{N_{\alpha}/2}}{\det(\sum_{\alpha=1}^k N_{\alpha} R_{\alpha} / N)^{N/2}} \tag{4}$$

Further, replacing N_{α} by n_{α} above, an approximated modified likelihood ratio statistic is derived as

$$M = \frac{\prod_{\alpha=1}^k \det(R_{\alpha})^{n_{\alpha}/2}}{\det(\sum_{\alpha=1}^k n_{\alpha} R_{\alpha} / n)^{n/2}} = \frac{\prod_{\alpha=1}^k \det(R_{\alpha})^{n_{\alpha}/2}}{\det(\bar{R})^{n/2}} \tag{5}$$

Since $-2 \ln M = \sum_{\alpha=1}^k n_{\alpha} \ln\{\det(\bar{R}) / \det(R_{\alpha})\} = L^*$, the statistic proposed by Kullback may be thought of as an approximated modified likelihood ratio statistic.

3. Preliminaries

Let the vectors $\mathbf{x}_{\alpha 1}, \dots, \mathbf{x}_{\alpha N_{\alpha}}$ be a random sample of size n_{α} for $\alpha = 1, \dots, k$ from k multivariate populations of dimensionality p and finite fourth moments. The characteristic function for the population α is given by $\phi_{\alpha}^*(\mathbf{t}) = E[\exp(\mathbf{t}' \mathbf{x})]$

where $\iota = \sqrt{-1}$ and $\mathbf{t} = (t_1, \dots, t_p)'$. The log characteristic function for population α may be written as

$$\ln[\phi_\alpha^*(\mathbf{t})] = \sum_{r_1 + \dots + r_p = 1}^{\infty} \kappa_\alpha^*(r_1, \dots, r_p) \prod_{j=1}^p \frac{(\iota t_j)^{r_j}}{r_j!}, \quad r_j \in \mathbf{I}^+ \quad (6)$$

where \mathbf{I}^+ is the set of non-negative integers. The cumulants of the distribution are the coefficients $\kappa_\alpha^*(r_1, \dots, r_p)$. If $r_1 + \dots + r_p = m$, then the associated cumulant is of order m . The relationship between the cumulants of a distribution and the characteristic function provide a convenient method for deriving the asymptotic distribution of statistic whose asymptotic expectations can be derived.

The cumulants of order m are functions of the moments of order m or lower. Thus if the m^{th} order moment is finite, so is the m^{th} order cumulant. Let $\mu_i = E(X_i)$, $\mu_{ij} = E(X_i X_j)$, $\mu_{ijk} = E(X_i X_j X_k)$, and $\mu_{ijkl} = E(X_i X_j X_k X_l)$ and κ_i , κ_{ij} , κ_{ijk} , and κ_{ijkl} be the corresponding cumulants. Then, Kaplan (1952) gives the following relationship:

$$\begin{aligned} \kappa_i &= \mu_i, \\ \kappa_{ij} &= \mu_{ij} - \mu_i \mu_j, \\ \kappa_{ijk} &= \mu_{ijk} - (\mu_i \mu_{jk} + \mu_j \mu_{ik} + \mu_k \mu_{ij}) + 2\mu_i \mu_j \mu_k, \\ \kappa_{ijkl} &= \mu_{ijkl} - \sum_4 \mu_i \mu_{jkl} - \sum_3 \mu_{ij} \mu_{kl} + 2 \sum_6 \mu_i \mu_j \mu_{kl} - 6\mu_i \mu_j \mu_k \mu_l \end{aligned}$$

where the summations are over the possible ways of grouping the subscripts, and the number of terms resulting is written over the summation sign.

Define the random matrix V_α as

$$V_\alpha = \sqrt{n_\alpha} \left(\frac{1}{n_\alpha} \Delta_\alpha^{-1} A_\alpha \Delta_\alpha^{-1} - P_\alpha \right) \quad (7)$$

Then, the random matrices $V_\alpha^{(0)}$, $V_\alpha^{(1)}$ and $V_\alpha^{(2)}$ are defined as

$$V_\alpha^{(0)} = \text{diag}(v_{\alpha 11}, v_{\alpha 22}, \dots, v_{\alpha pp}) \quad (8)$$

$$V_\alpha^{(1)} = V_\alpha - \frac{1}{2} V_\alpha^{(0)} P_\alpha - \frac{1}{2} P_\alpha V_\alpha^{(0)} \quad (9)$$

and

$$V_\alpha^{(2)} = \frac{1}{4} V_\alpha^{(0)} P_\alpha V_\alpha^{(0)} - \frac{1}{2} V_\alpha V_\alpha^{(0)} - \frac{1}{2} V_\alpha^{(0)} V_\alpha + \frac{3}{8} (V_\alpha^{(0)})^2 P_\alpha + \frac{3}{8} P_\alpha (V_\alpha^{(0)})^2 \quad (10)$$

Konishi (1979a, 1979b) has shown that

$$R_\alpha = P_\alpha + \frac{1}{\sqrt{n_\alpha}} V_\alpha^{(1)} + \frac{1}{n_\alpha} V_\alpha^{(2)} + O_p(n_\alpha^{-3/2})$$

The pooled estimate of the common correlation matrix is

$$\bar{R} = \sum_{\alpha=1}^k \omega_\alpha R_\alpha$$

so that

$$\bar{R} = \bar{P} + \frac{1}{\sqrt{n}}\bar{V}^{(1)} + \frac{1}{n}\bar{V}^{(2)} + O_p(n^{-3/2})$$

where $\omega_\alpha = n_\alpha/n$, $\bar{P} = \sum_{\alpha=1}^k \omega_\alpha P_\alpha$, $\bar{V}^{(1)} = \sum_{\alpha=1}^k \sqrt{\omega_\alpha} V_\alpha^{(1)}$ and $\bar{V}^{(2)} = \sum_{\alpha=1}^k V_\alpha^{(2)}$. The limiting distribution of $V_\alpha = \sqrt{n_\alpha} (\Delta_\alpha^{-1} A_\alpha \Delta_\alpha^{-1} / n_\alpha - P_\alpha)$ is normal with means 0 and covariances that depend on the fourth order cumulants of the parent population (Anderson 2003, p. 88).

Since Δ_α is a diagonal matrix of population standard deviations, $\Delta_\alpha^{-1} \mathbf{x}_{\alpha 1}, \dots, \Delta_\alpha^{-1} \mathbf{x}_{\alpha N_\alpha}$ may be thought of as N_α observations from a population with finite fourth order cumulants and characteristic function given by

$$\ln[\phi_\alpha(\mathbf{t})] = \sum_{r_1+\dots+r_p=1}^{\infty} \kappa_\alpha(r_1, \dots, r_p) \prod_{j=1}^p \frac{(t_j)^{r_j}}{r_j!}, \quad r_j \in \mathbf{I}^+ \tag{11}$$

where the standardized cumulants, $\kappa_\alpha(r_1, r_2, \dots, r_p)$, are derived from the expression (6) as

$$\kappa_\alpha(r_1, r_2, \dots, r_p) = \frac{\kappa_\alpha^*(r_1, r_2, \dots, r_p)}{\sigma_{\alpha 11} \chi_{r_1} \sigma_{\alpha 22} \chi_{r_2} \dots \sigma_{\alpha pp} \chi_{r_p}}$$

with $\chi_{r_j} = 1$ if $r_j = 0$, $\chi_{r_j} = 1/\sigma^{(\alpha)jj}$ if $r_j \neq 0$ and $\Sigma_\alpha^{-1} = (\sigma^{(\alpha)jj})$.

K -statistics are unbiased estimates of the cumulants of a distribution, and may be used to derive the moments of the statistics which are symmetric functions of the observations (Kendall and Stuart 1969). Kaplan (1952) gives a series of tensor formulae for computing the expectations of various functions of the k -statistics associated with a sample of size N from a multivariate population. For the definition of the k -statistics, let $N^{(r)} = N(N-1)\dots(N-r+1)$.

If $s_{i_1 i_2 \dots i_\ell}$ denotes the product $X_{i_1} X_{i_2} \dots X_{i_\ell}$ summed over the sample, the tensor formulae for the k -statistics may be shown to be as follows:

$$k_i = \frac{s_i}{N}, \quad k_{ij} = \frac{N s_{ij} - s_i s_j}{N^{(2)}}, \quad k_{ijk} = \frac{N^2 s_{ijk} - N \sum s_i s_j s_k + 2 s_i s_j s_k}{N^{(3)}}$$

$$k_{ijkl} = \frac{N(N+1)(N s_{ijkl} - \sum s_i s_j s_k s_l) - N^{(2)} \sum s_{ij} s_{kl} + 2N \sum s_i s_j s_k s_l - 6 s_i s_j s_k s_l}{N^{(4)}}$$

$$\begin{aligned} \kappa(ab, ij) &= E[(k_{ab} - \kappa_{ab})(k_{ij} - \kappa_{ij})] \\ &= \frac{\kappa_{abij}}{N} + \frac{\kappa_{ai} \kappa_{bj} + \kappa_{aj} \kappa_{bi}}{N-1} \end{aligned}$$

$$\begin{aligned} \kappa(ab, ij, pq) &= E[(k_{ab} - \kappa_{ab})(k_{ij} - \kappa_{ij})(k_{pq} - \kappa_{pq})] \\ &= \frac{\kappa_{abijpq}}{N^2} + \sum \frac{\kappa_{abip} \kappa_{jq}}{N(N-1)} + \sum \frac{(N-2) \kappa_{aip} \kappa_{bjq}}{N(N-1)^2} + \sum \frac{\kappa_{ai} \kappa_{bp} \kappa_{jq}}{(N-1)^2} \end{aligned}$$

The summations are over the possible ways of grouping the subscripts, and the number of terms resulting is written over the summation sign.

The matrix V_α is constructed from observations from the standardized distribution so that $v_{\alpha ij} = \sqrt{n_\alpha}(k_{\alpha ij} - \rho_{\alpha ij})$ where $k_{\alpha ij}$ is the related k -statistic for standardized population α . Kaplan's formulae may be applied to derive the following expressions for the expectations of elements of the matrices V_α (note that $\kappa_{\alpha ij} = \rho_{\alpha ij}$). We obtain

$$E(v_{\alpha ij}) = 0$$

$$E(v_{\alpha ij}v_{\alpha kl}) = \kappa_{\alpha ijkl} + \rho_{\alpha ik}\rho_{\alpha jl} + \rho_{\alpha il}\rho_{\alpha jk} + O(n_\alpha^{-1})$$

and

$$E(v_{\alpha ij}v_{\alpha kl}v_{\alpha ab}) = \frac{1}{\sqrt{n_\alpha}} \left[\kappa_{\alpha ijklab} + \sum^{12} \kappa_{\alpha ijka}\rho_{\alpha lb} + \sum^4 \kappa_{\alpha ika}\kappa_{\alpha jlb} + \sum^8 \rho_{\alpha ik}\rho_{\alpha ja}\rho_{\alpha lb} \right] + O(n_\alpha^{-3/2})$$

The random matrices $V_\alpha^{(0)}$, $V_\alpha^{(1)}$ and $V_\alpha^{(2)}$ are defined in (8), (9), and (10), respectively. The expectations associated with these random matrices are given as

$$E(v_{\alpha ij}^{(1)}) = 0$$

$$E(v_{\alpha ij}^{(2)}) = \frac{1}{4}\rho_{\alpha ij}\kappa_{\alpha iijj} - \frac{1}{2}(\kappa_{\alpha iijj} + \kappa_{\alpha ijjj}) + \frac{3}{8}\rho_{\alpha ij}(\kappa_{\alpha iiii} + \kappa_{\alpha jjjj}) + \frac{1}{2}(\rho_{\alpha ij}^3 - \rho_{\alpha ij}) + O(n_\alpha^{-1})$$

$$\begin{aligned} E(v_{\alpha ij}^{(1)}v_{\alpha kl}^{(1)}) &= \kappa_{\alpha ijkl} - \frac{1}{2}(\rho_{\alpha ij}\kappa_{\alpha iikl} + \rho_{\alpha ij}\kappa_{\alpha j jkl} + \rho_{\alpha kl}\kappa_{\alpha ijkk} + \rho_{\alpha kl}\kappa_{\alpha ijll}) \\ &\quad + \frac{1}{4}\rho_{\alpha ij}\rho_{\alpha kl}(\kappa_{\alpha iikk} + \kappa_{\alpha iill} + \kappa_{\alpha jjkk} + \kappa_{\alpha jjll}) \\ &\quad - (\rho_{\alpha kl}\rho_{\alpha ik}\rho_{\alpha jk} + \rho_{\alpha kl}\rho_{\alpha il}\rho_{\alpha jl} + \rho_{\alpha ij}\rho_{\alpha ik}\rho_{\alpha il} + \rho_{\alpha ij}\rho_{\alpha jk}\rho_{\alpha jl}) \\ &\quad + \frac{1}{2}\rho_{\alpha ij}\rho_{\alpha kl}(\rho_{\alpha ik}^2 + \rho_{\alpha il}^2 + \rho_{\alpha jk}^2 + \rho_{\alpha jl}^2) \\ &\quad + (\rho_{\alpha ik}\rho_{\alpha jl} + \rho_{\alpha il}\rho_{\alpha jk}) + O(n_\alpha^{-1}) \end{aligned} \tag{12}$$

and

$$E(v_{\alpha ij}^{(1)}v_{\alpha kl}^{(1)}v_{\alpha ab}^{(1)}) = \frac{1}{\sqrt{n_\alpha}} \left(t_{\alpha 1} - \frac{1}{2}t_{\alpha 2} + \frac{1}{4}t_{\alpha 3} - \frac{1}{8}t_{\alpha 4} \right) + O(n_\alpha^{-3/2})$$

where

$$t_{\alpha 1} = \kappa_{\alpha ijklab} + \sum^{12} \kappa_{\alpha ijka}\kappa_{\alpha lb} + \sum^4 \kappa_{\alpha ika}\kappa_{\alpha ilb} + \sum^8 \rho_{\alpha ik}\rho_{\alpha ja}\rho_{\alpha lb}$$

$$t_{\alpha 2} = \sum^3 \rho_{\alpha ij} \left[\kappa_{\alpha iiklab} + \kappa_{\alpha jjkla} + \sum^{12} (\kappa_{\alpha iika} + \kappa_{\alpha jjka}) + \sum^3 (\kappa_{\alpha ika} \kappa_{\alpha ilb} + \kappa_{\alpha jka} \kappa_{\alpha jlb}) + \sum^8 (\rho_{\alpha ik} \rho_{\alpha ia} \rho_{\alpha lb} + \rho_{\alpha jk} \rho_{\alpha ja} \rho_{\alpha lb}) \right]$$

$$t_{\alpha 3} = \sum^3 \rho_{\alpha ij} \rho_{\alpha kl} \left[\kappa_{\alpha iikkab} + \kappa_{\alpha iillab} + \kappa_{\alpha jjkkab} + \kappa_{\alpha jjllab} + \sum^{12} (\kappa_{\alpha iika} \rho_{\alpha kb} + \kappa_{\alpha iila} \rho_{\alpha lb} + \kappa_{\alpha jjka} \rho_{\alpha kb} + \kappa_{\alpha jjla} \rho_{\alpha lb}) + \sum^3 (\kappa_{\alpha ika} \kappa_{\alpha ikb} + \kappa_{\alpha ila} \kappa_{\alpha ilb} + \kappa_{\alpha jka} \kappa_{\alpha jkb} + \kappa_{\alpha jla} \kappa_{\alpha jlb}) + \sum^8 (\rho_{\alpha ik} \rho_{\alpha ia} \rho_{\alpha kb} + \rho_{\alpha il} \rho_{\alpha ia} \rho_{\alpha lb} + \rho_{\alpha jk} \rho_{\alpha ja} \rho_{\alpha kb} + \rho_{\alpha jl} \rho_{\alpha ja} \rho_{\alpha lb}) \right]$$

and

$$t_{\alpha 4} = \rho_{\alpha ij} \rho_{\alpha kl} \rho_{\alpha ab} \sum^8 \left[\kappa_{\alpha iikkaa} + \sum^{12} (\kappa_{\alpha iika} \rho_{\alpha ka}) + \sum^3 (\kappa_{\alpha ika} \kappa_{\alpha ikb}) + \sum^8 (\rho_{\alpha ik} \rho_{\alpha ia} \rho_{\alpha ka}) \right]$$

Lemma 1. *The diagonal elements of $V_{\alpha}^{(1)}$ are zero.*

Proof. Using (9) and the fact that $V_{\alpha}^{(0)}$ is a diagonal matrix, we have

$$v_{\alpha ij}^{(1)} = v_{\alpha ij} - \frac{1}{2} \rho_{\alpha ij} (v_{\alpha ii} + v_{\alpha jj})$$

The result follows by taking $j = i$ above and noting that diagonal elements of P_{α} are 1. □

Lemma 2. *The diagonal elements of $V_{\alpha}^{(2)}$ are zero.*

Proof. Using (10) and the fact that $V_{\alpha}^{(0)}$ is a diagonal matrix, we get

$$v_{\alpha ij}^{(2)} = \frac{1}{4} v_{\alpha ii}^{(0)} \rho_{\alpha ij} v_{\alpha jj}^{(0)} - \frac{1}{2} v_{\alpha ij} (v_{\alpha jj} + v_{\alpha ii}) + \frac{3}{8} \rho_{\alpha ij} (v_{\alpha ii}^2 + v_{\alpha jj}^2)$$

The result follows by substituting $j = i$ above and observing that $\rho_{\alpha ii} = 1$. □

4. Asymptotic Expansion of L^*

In order to derive the asymptotic distribution for L^* the statistic is first expanded in terms of other random variables (see Konishi and Sugiyama 1981).

The statistic L^* may be written as $L^* = ng(R_1, \dots, R_k)$ where $g(R_1, \dots, R_k) = \ln[\det(\bar{R})] - \sum_{\alpha=1}^k \omega_\alpha \ln[\det(R_\alpha)]$. Let

$$B_\alpha = \frac{1}{\sqrt{n_\alpha}} P_\alpha^{-1} V_\alpha^{(1)} + \frac{1}{n_\alpha} P_\alpha^{-1} V_\alpha^{(2)}$$

Since, P_α , $V_\alpha^{(1)}$ and $V_\alpha^{(2)}$ are all positive definite, so is B_α . This insures that the eigenvalues of B_α exist and are positive. Also, as n_α becomes large, the elements in B_α become small so that the characteristic roots may be assumed to be less than one. Using Lemma 5,

$$\begin{aligned} \ln[\det(R_\alpha)] &= \ln[\det(P_\alpha + P_\alpha B_\alpha)] + O_p(n_\alpha^{-3/2}) \\ &= \ln[\det(P_\alpha)] + \text{tr}(B_\alpha) - \frac{1}{2} \text{tr}(B_\alpha B_\alpha) + O_p(n_\alpha^{-3/2}) \end{aligned}$$

Now, $B_\alpha B_\alpha = n_\alpha^{-1} P_\alpha^{-1} V_\alpha^{(1)} P_\alpha^{-1} V_\alpha^{(1)} + O_p(n_\alpha^{-3/2})$ so that

$$\begin{aligned} \ln[\det(R_\alpha)] &= \ln[\det(P_\alpha)] + \frac{1}{\sqrt{n_\alpha}} \text{tr}(P_\alpha^{-1} V_\alpha^{(1)}) + \frac{1}{n_\alpha} \text{tr}(P_\alpha^{-1} V_\alpha^{(2)}) \\ &\quad - \frac{1}{2n_\alpha} \text{tr}(P_\alpha^{-1} V_\alpha^{(1)} P_\alpha^{-1} V_\alpha^{(1)}) + O_p(n_\alpha^{-3/2}) \end{aligned}$$

A similar expansion for $\ln[\det(\bar{R})]$ may be obtained by defining \bar{B} by

$$\bar{B} = \frac{1}{\sqrt{n}} \sum_{\alpha=1}^k \sqrt{\omega_\alpha} \bar{P}^{-1} V_\alpha^{(1)} + \frac{1}{n} \sum_{\alpha=1}^k \bar{P}^{-1} V_\alpha^{(2)}$$

Then

$$\begin{aligned} \ln[\det(\bar{R})] &= \ln[\det(\bar{P} + \bar{P}\bar{B})] + O_p(n^{-3/2}) \\ &= \ln[\det(\bar{P})] + \text{tr}(\bar{B}) - \frac{1}{2} \text{tr}(\bar{B}\bar{B}) + O_p(n^{-3/2}) \end{aligned}$$

Since $\bar{B}\bar{B} = n^{-1} \sum_{\alpha=1}^k \sum_{\beta=1}^k \sqrt{\omega_\alpha \omega_\beta} \bar{P}^{-1} V_\alpha^{(1)} \bar{P}^{-1} V_\beta^{(1)} + O_p(n^{-3/2})$,

$$\begin{aligned} \ln[\det(\bar{R})] &= \ln[\det(\bar{P})] + \frac{1}{\sqrt{n}} \sum_{\alpha=1}^k \sqrt{\omega_\alpha} \text{tr}(\bar{P}^{-1} V_\alpha^{(1)}) + \frac{1}{n} \sum_{\alpha=1}^k \text{tr}(\bar{P}^{-1} V_\alpha^{(2)}) \\ &\quad - \frac{1}{2n} \sum_{\alpha=1}^k \sum_{\beta=1}^k \sqrt{\omega_\alpha \omega_\beta} \text{tr}(\bar{P}^{-1} V_\alpha^{(1)} \bar{P}^{-1} V_\beta^{(1)}) + O_p(n^{-3/2}) \end{aligned}$$

Combining these expressions yields

$$\begin{aligned} g(R_1, \dots, R_k) &= \ln[\det(\bar{P})] - \sum_{\alpha=1}^k \omega_\alpha \ln[\det(P_\alpha)] + \frac{1}{\sqrt{n}} \sum_{\alpha=1}^k \sqrt{\omega_\alpha} \text{tr}(H_\alpha V_\alpha^{(1)}) \\ &\quad + \frac{1}{n} \sum_{\alpha=1}^k \text{tr}(H_\alpha V_\alpha^{(2)}) + \frac{1}{2} \sum_{\alpha=1}^k \frac{\omega_\alpha}{n_\alpha} \text{tr}(P_\alpha^{-1} V_\alpha^{(1)} P_\alpha^{-1} V_\alpha^{(1)}) \end{aligned}$$

$$-\frac{1}{2n} \sum_{\alpha=1}^k \sum_{\beta=1}^k \sqrt{\omega_\alpha \omega_\beta} \operatorname{tr}(\bar{P}^{-1} V_\alpha^{(1)} \bar{P}^{-1} V_\beta^{(1)}) + O_p(n^{-3/2})$$

where $H_\alpha = (h_{\alpha ij}) = (\bar{P}^{-1} - P_\alpha^{-1}) = H'_\alpha$. Let $G(R_1, \dots, R_k) = \sqrt{n}[g(R_1, \dots, R_k) - g(P_1, \dots, P_k)]$. Then, since $\sqrt{n}(\omega_\alpha/n_\alpha) = (\sqrt{n})^{-1}$, we obtain

$$\begin{aligned} G(R_1, \dots, R_k) &= \sum_{\alpha=1}^k \sqrt{\omega_\alpha} \operatorname{tr}(H_\alpha V_\alpha^{(1)}) + \frac{1}{\sqrt{n}} \sum_{\alpha=1}^k \operatorname{tr}(H_\alpha V_\alpha^{(2)}) \\ &\quad + \frac{1}{2\sqrt{n}} \sum_{\alpha=1}^k \operatorname{tr}(P_\alpha^{-1} V_\alpha^{(1)} P_\alpha^{-1} V_\alpha^{(1)}) \\ &\quad - \frac{1}{2\sqrt{n}} \sum_{\alpha=1}^k \sum_{\beta=1}^k \sqrt{\omega_\alpha \omega_\beta} \operatorname{tr}(\bar{P}^{-1} V_\alpha^{(1)} \bar{P}^{-1} V_\beta^{(1)}) + O_p(n^{-1}) \end{aligned} \quad (13)$$

Theorem 1. *The expression $G(R_1, \dots, R_k)$ may be written as*

$$\begin{aligned} G(R_1, \dots, R_k) &= \sum_{\alpha=1}^k \sum_{i < j} \sqrt{\omega_\alpha} \bar{h}_{\alpha ij} v_{\alpha ij}^{(1)} + \frac{1}{\sqrt{n}} \sum_{\alpha=1}^k \sum_{i < j} \bar{h}_{\alpha ij} v_{\alpha ij}^{(2)} \\ &\quad + \frac{1}{\sqrt{n}} \sum_{\alpha=1}^k \sum_{i < j} \sum_{k < \ell} q_\alpha(ij, k\ell) v_{\alpha ij}^{(1)} v_{\alpha k\ell}^{(1)} \\ &\quad - \frac{1}{\sqrt{n}} \sum_{\alpha=1}^k \sum_{\beta=1}^k \sum_{i < j} \sum_{k < \ell} \sqrt{\omega_\alpha \omega_\beta} q(ij, k\ell) v_{\alpha ij}^{(1)} v_{\beta k\ell}^{(1)} + O_p(n^{-1}) \end{aligned}$$

where $P_\alpha^{-1} = (\rho_\alpha^{ij})$, $\bar{P}^{-1} = (\bar{\rho}^{ij})$, $\bar{h}_{\alpha ij} = 2(\bar{\rho}^{ij} - \rho_\alpha^{ij})$, $q_\alpha(ij, k\ell) = \rho_\alpha^{i\ell} \rho_\alpha^{jk} + \rho_\alpha^{ik} \rho_\alpha^{j\ell}$ and $q(ij, k\ell) = \bar{\rho}^{i\ell} \bar{\rho}^{jk} + \bar{\rho}^{ik} \bar{\rho}^{j\ell}$.

Proof. Using results on matrix algebra, we have

$$\sum_{\alpha=1}^k \sqrt{\omega_\alpha} \operatorname{tr}(H_\alpha V_\alpha^{(1)}) = \sum_{\alpha=1}^k \sqrt{\omega_\alpha} \sum_{i=1}^p \sum_{j=1}^p h_{\alpha ji} v_{\alpha ij}^{(1)}$$

and since H_α is symmetric, application of Lemma 3 yields

$$\sum_{\alpha=1}^k \sqrt{\omega_\alpha} \operatorname{tr}(H_\alpha V_\alpha^{(1)}) = \sum_{\alpha=1}^k \sqrt{\omega_\alpha} \sum_{i < j} (h_{\alpha ji} + h_{\alpha ij}) v_{\alpha ij}^{(1)} = \sum_{\alpha=1}^k \sqrt{\omega_\alpha} \sum_{i < j} \bar{h}_{\alpha ij} v_{\alpha ij}^{(1)}$$

In an entirely similar manner,

$$\sum_{\alpha=1}^k \operatorname{tr}(H_\alpha V_\alpha^{(2)}) = \sum_{\alpha=1}^k \sum_{i < j} \bar{h}_{\alpha ij} v_{\alpha ij}^{(2)}$$

Using Lemma 4, results on matrix algebra and the symmetry of $V_\alpha^{(1)}$, we have

$$\begin{aligned} & \frac{1}{2} \sum_{\alpha=1}^k \text{tr}(P_\alpha^{-1} V_\alpha^{(1)} P_\alpha^{-1} V_\alpha^{(1)}) \\ &= \frac{1}{2} \sum_{\alpha=1}^k \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \sum_{\ell=1}^p \rho_\alpha^{i\ell} \rho_\alpha^{jk} v_{\alpha ij}^{(1)} v_{\alpha k\ell}^{(1)} \\ &= \frac{1}{2} \sum_{\alpha=1}^k \sum_{i < j} \sum_{k < \ell} (\rho_\alpha^{i\ell} \rho_\alpha^{jk} + \rho_\alpha^{j\ell} \rho_\alpha^{ik} + \rho_\alpha^{ik} \rho_\alpha^{j\ell} + \rho_\alpha^{i\ell} \rho_\alpha^{jk}) v_{\alpha ij}^{(1)} v_{\alpha k\ell}^{(1)} \\ &= \sum_{\alpha=1}^k \sum_{i < j} \sum_{k < \ell} q_\alpha(ij, k\ell) v_{\alpha ij}^{(1)} v_{\alpha k\ell}^{(1)} \end{aligned}$$

In a similar manner,

$$\begin{aligned} & \frac{1}{2} \sum_{\alpha=1}^k \sum_{\beta=1}^k \sqrt{\omega_\alpha \omega_\beta} \text{tr}(\bar{P}^{-1} V_\alpha^{(1)} \bar{P}^{-1} V_\beta^{(1)}) \\ &= \sum_{\alpha=1}^k \sum_{\beta=1}^k \sum_{i < j} \sum_{k < \ell} \sqrt{\omega_\alpha \omega_\beta} q(ij, k\ell) v_{\alpha ij}^{(1)} v_{\beta k\ell}^{(1)} \end{aligned}$$

Combining these expansions in (13) completes the proof. \square

Corollary 1. *In the special case $p = 2$, $G(R_1, \dots, R_k)$ may be written as*

$$\begin{aligned} G(R_1, \dots, R_k) &= 2 \sum_{\alpha=1}^k \sqrt{\omega_\alpha} \left(\frac{\rho_\alpha}{1 - \rho_\alpha^2} - \frac{\bar{\rho}}{1 - \bar{\rho}^2} \right) v_{\alpha 12}^{(1)} \\ &+ \frac{2}{\sqrt{n}} \sum_{\alpha=1}^k \left(\frac{\rho_\alpha}{1 - \rho_\alpha^2} - \frac{\bar{\rho}}{1 - \bar{\rho}^2} \right) v_{\alpha 12}^{(2)} + \frac{1}{\sqrt{n}} \sum_{\alpha=1}^k \frac{1 + \rho_\alpha^2}{(1 - \rho_\alpha^2)^2} (v_{\alpha 12}^{(1)})^2 \\ &- \frac{1}{\sqrt{n}} \sum_{\alpha=1}^k \sum_{\beta=1}^k \frac{1 + \bar{\rho}^2}{(1 - \bar{\rho}^2)^2} v_{\alpha 12}^{(1)} v_{\beta 12}^{(1)} + O_p(n^{-1}). \end{aligned}$$

Proof. For $p = 2$, $\sum_{i < j}$ consists of single term corresponding to $i = 1, j = 2$. Also, $P_\alpha = \begin{pmatrix} 1 & \rho_\alpha \\ \rho_\alpha & 1 \end{pmatrix}$ so that $P_\alpha^{-1} = (1 - \rho_\alpha^2)^{-1} \begin{pmatrix} 1 & -\rho_\alpha \\ -\rho_\alpha & 1 \end{pmatrix}$. Similarly, $\bar{P}^{-1} = (1 - \bar{\rho}^2)^{-1} \begin{pmatrix} 1 & -\bar{\rho} \\ -\bar{\rho} & 1 \end{pmatrix}$. Thus, the off diagonal element of H_α is given by $\rho_\alpha(1 - \rho_\alpha^2)^{-1} - \bar{\rho}(1 - \bar{\rho}^2)^{-1}$. Further, $q_\alpha(12, 12) = \rho_\alpha^{12} \rho_\alpha^{21} + \rho_\alpha^{11} \rho_\alpha^{22} = (1 + \rho_\alpha^2)/(1 - \rho_\alpha^2)^2$ and $q(12, 12) = (1 + \bar{\rho}^2)/(1 - \bar{\rho}^2)^2$. The result follows by using these values in the theorem. \square

5. Asymptotic Null Distribution of L^*

In this section we derive asymptotic distribution of the statistic L^* when the null hypothesis is true.

Define the $k \times k$ matrix W as $W = (w_{ij})$ where $w_{ii} = 1 - \omega_i$ and for $i \neq j$, $w_{ij} = -\sqrt{\omega_i \omega_j} = w_{ji}$, $1 \leq i, j, \leq k$. The matrix W has rank $k - 1$ and each of its non-zero eigenvalues is equal to 1.

Theorem 2. *Let the k correlation matrices R_1, \dots, R_k be based on independent samples of sizes N_1, \dots, N_k , respectively, with finite fourth order cumulants. Define the $kp(p - 1)/2 \times 1$ vector $\mathbf{v}^{(1)}$ by*

$$\mathbf{v}^{(1)} = (v_{1,1,2}^{(1)}, v_{1,1,3}^{(1)}, \dots, v_{1,p-1,p}^{(1)}, v_{2,1,2}^{(1)}, v_{2,1,3}^{(1)}, \dots, v_{2,p-1,p}^{(1)}, \dots, v_{k,1,2}^{(1)}, v_{k,1,3}^{(1)}, \dots, v_{k,p-1,p}^{(1)})'$$

where $V_\alpha^{(1)}$ is as defined in (9). Let $Q = (q(ij, k\ell))$ be the $p(p - 1)/2 \times p(p - 1)/2$ matrix of coefficients defined in Theorem 1.

Let T_α be the asymptotic dispersion matrix of $V_\alpha^{(1)}$ with entry $(ij, k\ell)$ equal to $E(v_{\alpha ij}^{(1)} v_{\alpha k\ell}^{(1)})$ given in (12). Then, the asymptotic dispersion matrix of $\mathbf{v}^{(1)}$ is

$$T^* = \begin{pmatrix} T_1 & 0 & \dots & 0 \\ 0 & T_2 & \dots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \dots & T_k \end{pmatrix}$$

Under the null hypothesis

$$L^* = \sum_{i=1}^{p(p-1)(k-1)/2} \lambda_i y_i + O_p(n^{-1/2})$$

where $y_1, \dots, y_{p(p-1)(k-1)/2}$ are independent, $y_i \sim \chi_{1}^2$, $1 \leq i \leq p(p-1)(k-1)/2$ and $\lambda_1, \dots, \lambda_{p(p-1)(k-1)/2}$ are the eigenvalues of $T^*(Q \otimes W)$. If the standardized fourth order cumulants of the populations are all equal, then $T_\alpha = T$ for $\alpha = 1, \dots, k$ and

$$L^* = \sum_{i=1}^{p(p-1)/2} \theta_i u_i + O_p(n^{-1/2}),$$

where $u_1, \dots, u_{p(p-1)/2}$ are independent, $u_i \sim \chi_{k-1}^2$ and $\theta_1, \dots, \theta_{p(p-1)/2}$ are the eigenvalues of TQ .

Proof. Under the null hypothesis we have $P_\alpha = \bar{P}$ for $\alpha = 1, \dots, k$ so that $g(P_1, \dots, P_k) = 0$, $h_{\alpha ij} = 0$ and $q_\alpha(ij, k\ell) = q(ij, k\ell) = \rho^{i\ell} \rho^{jk} + \rho^{ik} \rho^{j\ell}$ for all α . Since $g(R_1, \dots, R_k) = \ln[\det(\bar{R})] - \sum_{\alpha=1}^k \omega_\alpha \ln[\det(\bar{R}_\alpha)] = n^{-1} L^*$, using Theorem 1, one obtains

$$\begin{aligned} L^* &= ng(R_1, \dots, R_k) = n[g(R_1, \dots, R_k) - g(P_1, \dots, P_k)] \\ &= \sum_{\alpha=1}^k \sum_{i < j} \sum_{k < \ell} q(ij, k\ell) v_{\alpha ij}^{(1)} v_{\alpha k\ell}^{(1)} \\ &\quad - \sum_{\alpha=1}^k \sum_{\beta=1}^k \sum_{i < j} \sum_{k < \ell} \sqrt{\omega_\alpha \omega_\beta} q(ij, k\ell) v_{\alpha ij}^{(1)} v_{\beta k\ell}^{(1)} + O_p(n^{-1/2}) \end{aligned}$$

$$\begin{aligned} &= \sum_{\alpha=1}^k \sum_{\beta=1}^k w_{\alpha\beta} \sum_{i < j} \sum_{k < \ell} q(ij, k\ell) v_{\alpha ij}^{(1)} v_{\beta k\ell}^{(1)} + O_p(n^{-1/2}) \\ &= (\mathbf{v}^{(1)})'(Q \otimes W)\mathbf{v}^{(1)} + O_p(n^{-1/2}) \end{aligned}$$

Since Q is of rank $p(p-1)/2$ and W is of rank $k-1$, the matrix $Q \otimes W$ is of rank $p(p-1)(k-1)/2$. From (8) and (9) it is clear that elements of $V_{\alpha}^{(1)}$ are linear functions of elements of V_{α} and the limiting distribution of V_{α} is normal with means 0 and covariances that depend on the fourth order cumulants of the parent population. Therefore, $\mathbf{v}^{(1)}$ is asymptotically normal with means zero and dispersion matrix T^* . Thus, $L^* = \sum_{i=1}^{p(p-1)(k-1)/2} \lambda_i y_i + O_p(n^{-1/2})$.

If the standardized fourth order cumulants are the same for each underlying population, then $T^* = T \otimes I$. Further, $(T \otimes I)(Q \otimes W) = TQ \otimes W$ has as its eigenvalues $\theta_i \epsilon_j$, $i = 1, \dots, p(p-1)/2$, $j = 1, \dots, k$ where θ_i are the eigenvalues of TQ and ϵ_j are the eigenvalues of W . Since there are $p(p-1)/2$ non-zero eigenvalues of $(T \otimes I)(Q \otimes W)$ each occurring with multiplicity $k-1$, we have $L^* = \sum_{i=1}^{p(p-1)/2} \theta_i u_i + O_p(n^{-1/2})$. \square

Corollary 2. *Let the k sample correlation coefficients r_1, r_2, \dots, r_k be based on independent samples of sizes N_1, N_2, \dots, N_k from bivariate populations with finite fourth order cumulants. Let ρ be the hypothesized common correlation coefficient. Define the $k \times 1$ vector $\mathbf{v}^{(1)}$ by*

$$\mathbf{v}^{(1)} = (v_1^{(1)}, \dots, v_k^{(1)})'$$

where $v_{\alpha}^{(1)} = v_{\alpha 12} - \rho(v_{\alpha 11} + v_{\alpha 22})$ as defined in (9). Let

$$t_{\alpha} = (1 - \rho^2)^2 + \frac{1}{4}\rho^2(\kappa_{\alpha 1111} + \kappa_{\alpha 2222}) + \left(1 + \frac{1}{2}\rho^2\right) \kappa_{\alpha 1122} - \rho(\kappa_{\alpha 1113} + \kappa_{\alpha 1222})$$

and define $T^* = \text{diag}(t_1, \dots, t_k)$. Under the null hypothesis the statistic L^* is asymptotically expanded as

$$L^* = \frac{1 + \rho^2}{(1 - \rho^2)^2} \sum_{i=1}^{k-1} \lambda_i y_i + O_p(n^{-1/2})$$

where y_1, \dots, y_{k-1} are independent, $y_i \sim \chi_1^2$ and $\lambda_1, \dots, \lambda_{k-1}$ are the eigenvalues of T^*W . If the standardized fourth order cumulants are equal, then

$$t_{\alpha} = (1 - \rho^2)^2 + \frac{1}{4}\rho^2(\kappa_{1111} + \kappa_{2222}) + \left(1 + \frac{1}{2}\rho^2\right) \kappa_{1122} - \rho(\kappa_{1113} + \kappa_{1222})$$

for $\alpha = 1, 2, \dots, k$ and

$$\begin{aligned} L^* &= \left[(1 - \rho^2)^2 + \frac{1}{4}\rho^2(\kappa_{1111} + \kappa_{2222}) + \left(1 + \frac{1}{2}\rho^2\right) \kappa_{1122} \right. \\ &\quad \left. - \rho(\kappa_{1113} + \kappa_{1222}) \right] \frac{1 + \rho^2}{(1 - \rho^2)^2} \chi_{k-1}^2 + O_p(n^{-1/2}) \end{aligned}$$

Proof. As shown in Corollary 1, when $p = 2$, Q is a scalar. If ρ is the common correlation coefficient, then $Q = (1 + \rho^2)/(1 - \rho^2)^2$. The asymptotic variance of $v_{\alpha 12}^{(1)}$ is given in (12). Upon simplification,

$$E(v_{\alpha 12}^{(1)}v_{\alpha 12}^{(1)}) = t_{\alpha} = (1 - \rho^2)^2 + \frac{1}{4}\rho^2(\kappa_{\alpha 1111} + \kappa_{\alpha 2222}) + \left(1 + \frac{1}{2}\rho^2\right)\kappa_{\alpha 1122} - \rho(\kappa_{\alpha 1113} + \kappa_{\alpha 1222}) + O_p(n^{-1/2}) \quad (14)$$

so that T^* is the asymptotic covariance matrix of $\mathbf{v}^{(1)}$. Further, $T^*(Q \otimes W) = [(1 + \rho^2)/(1 - \rho^2)^2]T^*W$. Thus $L^* = [(1 + \rho^2)/(1 - \rho^2)^2] \sum_{i=1}^{k-1} \lambda_i y_i + O_p(n^{-1/2})$, where λ_i are the eigenvalues of T^*W . If the standardized fourth order cumulants are identical, $T = tI$, so that there is one eigenvalue of TQ with multiplicity k . This eigenvalue is merely $t(1 + \rho^2)/(1 - \rho^2)^2$ and the result follows immediately from Theorem 2. \square

Corollary 3. Let the k sample correlation coefficients r_1, r_2, \dots, r_k be based on independent samples of sizes N_1, N_2, \dots, N_k from bivariate populations which are elliptically contoured with a common kurtosis of 3κ and common correlation coefficient ρ . Then

$$L^* = [(1 - \rho^2)^2 + (1 + 2\rho^2)\kappa] \frac{1 + \rho^2}{(1 - \rho^2)^2} \chi_{k-1}^2 + O_p(n^{-1/2})$$

Proof. For elliptically contoured distributions (Muirhead 1982, Anderson 2003, Gupta and Varga 1993) the fourth order cumulants are such that $\kappa_{iiii} = 3\kappa_{iijj} = 3\kappa$ for $i \neq j$ and all other cumulants are zero (Waternaux 1984). Substituting this into the expression for t in Corollary 2 yields $t = (1 - \rho^2)^2 + (1 + 2\rho^2)\kappa$. The result then follows from Corollary 2. \square

Corollary 4. Let the k sample correlation coefficients r_1, \dots, r_k be based on independent samples of sizes N_1, \dots, N_k from bivariate normal populations with a common correlation coefficient ρ . Then

$$L^* = (1 + \rho^2)\chi_{k-1}^2 + O_p(n^{-1/2})$$

Proof. Normal distributions are special case of elliptically contoured distributions. The fourth order cumulants are all zero (Anderson 2003). The result follows by setting $\kappa = 0$ in Corollary 3. \square

6. An Example

This example is included to demonstrate the procedure to be used when testing the equality of correlation matrices by using the statistic L^* . The data represent random samples from three trivariate populations each with identical correlation matrix P given by

$$P = \begin{pmatrix} 1.0 & 0.3 & 0.2 \\ 0.3 & 1.0 & -0.3 \\ 0.2 & -0.3 & 1.0 \end{pmatrix}$$

Since the statistic L^* is an approximation of the modified likelihood ratio statistic for samples from multivariate normal populations, it is particularly suited to populations that are *near* normal. The contaminated normal model has been chosen to represent such a distribution.

Samples of size 25 from contaminated normal populations with mixing parameter $\epsilon = 0.1$ and $\sigma = 2$ were generated using the SAS system. These data are tabulated in Gupta, Johnson and Nagar (2012). The density of a contaminated normal model is given by

$$\phi_\epsilon(\mathbf{x}, \sigma, \Sigma) = (1 - \epsilon)\phi(\mathbf{x}, \Sigma) + \epsilon\phi(\mathbf{x}, \sigma\Sigma), \quad \sigma > 0, \quad 0 < \epsilon < 1$$

where $\phi(\mathbf{x}, \Sigma)$ is the density of a multivariate normal distribution with zero mean vector and covariance matrix Σ .

If the data were known to be from three normal populations all that would be required at this point would be the sample sizes and the matrix of corrected sums of squares and cross products. A key element, however, of the modified likelihood ratio procedure is that this assumption need not be made, but the fourth order cumulant must be estimated. To do this the k -statistics are calculated using Kaplan's formulae summarized in Section 3. The computations are made considerably easier by standardizing the data so that all of the first order sums are zero.

The computation using original (or standardized) data yields the following estimates of the individual correlation matrices:

$$R_1 = \begin{pmatrix} 1.0000 & 0.5105 & 0.3193 \\ 0.5105 & 1.0000 & -0.3485 \\ 0.3193 & -0.3485 & 1.0000 \end{pmatrix}, \quad \det(R_1) = 0.4024$$

$$R_2 = \begin{pmatrix} 1.0000 & 0.1758 & 0.2714 \\ 0.1758 & 1.0000 & -0.2688 \\ 0.2714 & -0.2688 & 1.0000 \end{pmatrix}, \quad \det(R_2) = 0.7975$$

$$R_3 = \begin{pmatrix} 1.0000 & 0.2457 & 0.3176 \\ 0.2457 & 1.0000 & -0.0331 \\ 0.3176 & -0.0331 & 1.0000 \end{pmatrix}, \quad \det(R_3) = 0.8325$$

Since each sample is of size 25, $\omega_i = 1/3$ for $i = 1, 2, 3$ and the pooled correlation matrix is merely the average of these three matrices:

$$\bar{R} = \begin{pmatrix} 1.0000 & 0.3107 & 0.3028 \\ 0.3107 & 1.0000 & -0.2168 \\ 0.3028 & -0.2168 & 1.0000 \end{pmatrix}, \quad \det(\bar{R}) = 0.7240$$

The value of the test statistic is now easily calculated as

$$\begin{aligned} L^* &= 72 \ln(0.7240) - 24[\ln(0.4024) + \ln(0.7975) + \ln(0.8325)] \\ &= 8.7473 \end{aligned}$$

The null hypothesis is to be rejected if the value of the test statistic is too large. The next step of the procedure is to estimate the coefficients in the linear combination of chi-square variables that make up the actual distribution under the null hypothesis. The most arduous part is the computation of the estimates of fourth order cumulants.

Since the data are standardized, the formula for the k -statistic for the four way product $x_i \times x_j \times x_k \times x_\ell$ simplifies to

$$k_{ijkl} = \frac{1}{N^{(4)}} [N^2(N+1)s_{ijkl} - N(N-1)(s_{ij}s_{kl} + s_{ik}s_{jl} + s_{il}s_{jk})]$$

where $N^{(4)} = N(N-1)(N-2)(N-3)$. Using this to estimate the cumulant corresponding to $x_1^2x_2^2$ yields $k_{1122} = 0.6670$. The computation for other fourth order cumulant are performed similarly. The resulting estimates are then pooled across population to yield an estimate of the common fourth order cumulants used in building the tau matrix (it is possible, of course, to drop the assumption of common fourth order cumulants and use the nine by nine matrix that would result if each separate tau matrix were joined in a block diagonal matrix). The estimates of the fourth order cumulants are summarized in the Table 1.

The pooled correlation matrix and these estimates are now used to build the estimated covariance matrix $V^{(1)}$. The entry corresponding to $v_{ij}^{(1)}v_{kl}^{(1)}$ is given by

$$\begin{aligned} k_{ijkl} - \frac{1}{2}(r_{ij}k_{iikl} + r_{ij}k_{jjkl} + r_{kl}k_{ijkk} + r_{kl}k_{ijll}) \\ + \frac{1}{4}r_{ij}r_{kl}(k_{iikk} + k_{iill} + k_{jjkk} + k_{jjll}) \\ - (r_{kl}r_{ik}r_{jk} + r_{kl}r_{il}r_{jl} + r_{ij}r_{ik}r_{il} + r_{ij}r_{jk}r_{jl}) \\ + \frac{1}{2}r_{ij}r_{kl}(r_{ik}^2 + r_{il}^2 + r_{jk}^2 + r_{jl}^2) + r_{ik}r_{jl} + r_{il}r_{jk} \end{aligned}$$

where r_{ij} is the pooled estimate of the correlation value and k_{ijkl} is the corresponding pooled fourth order cumulant. The entry corresponding to 12, 13 is given by $t_{12,13} = -0.3065$. Similar calculations yield the following covariance matrix corresponding to $(v_{\alpha 12}^{(1)}, v_{\alpha 13}^{(1)}, v_{\alpha 23}^{(1)})'$,

$$T = \begin{pmatrix} 1.0150 & -0.3065 & 0.1800 \\ -0.3065 & 0.7242 & 0.3974 \\ 0.1800 & 0.3974 & 0.8179 \end{pmatrix}$$

To complete the example, the inverse of the pooled correlation matrix is used to estimate the matrix Q . The entry corresponding to the element ij, kl is given by $r^{ik}r^{jl} + r^{il}r^{jk}$ where $R^{-1} = (r^{ij})$. These matrices are as follows:

$$\bar{R}^{-1} = \begin{pmatrix} 1.3163 & -0.5198 & -0.5113 \\ -0.5198 & 1.2546 & 0.4294 \\ -0.5113 & 0.4294 & 1.2479 \end{pmatrix}$$

TABLE 1: Estimated fourth order cumulants

Variables	Population 1	Population 2	Population 3	Pooled
1111	0.9077	0.1181	0.9355	0.6538
1112	0.7765	-0.0387	-0.0565	0.2271
1113	-0.3015	0.7008	0.0677	0.1105
1122	0.6670	0.3595	-0.3663	0.2201
1123	-0.3917	0.3519	-0.1333	-0.0574
1133	-0.1848	0.6608	-0.7475	-0.0905
1222	0.4896	-0.7128	-0.0178	-0.0803
1223	-0.3005	0.1637	-0.2243	-0.1204
1233	-0.0980	0.6343	-0.1394	0.1323
1333	-0.3430	0.3973	-0.0773	-0.0077
2222	-0.0787	-0.9989	0.8134	-0.0881
2223	-0.2543	0.0750	0.1887	0.0032
2233	0.3800	-0.1764	-0.5454	-0.1139
2333	-0.8386	0.8496	0.2869	0.0993
3333	0.9130	-0.9196	1.3068	0.4334

$$Q = \begin{pmatrix} 1.9217 & 0.8310 & -0.8647 \\ 0.8310 & 1.9041 & -0.8682 \\ -0.8647 & 0.8682 & 1.7500 \end{pmatrix}$$

Most eigenvalues extraction routines require that the matrix being analyzed be symmetric. Let A be the Cholesky decomposition of Q , that is $Q = A'A$ where A is an upper triangular matrix. Then the eigenvalues of TQ are the same as the eigenvalues of ATA' which is clearly symmetric. In this case

$$A = \begin{pmatrix} 1.3863 & 0.5995 & -0.6237 \\ 0 & 1.2429 & -0.3977 \\ 0 & 0 & 1.0967 \end{pmatrix}$$

$$ATA' = \begin{pmatrix} 1.4111 & -0.2877 & -0.0246 \\ -0.2877 & 0.8552 & 0.1849 \\ -0.0246 & 0.1849 & 0.9837 \end{pmatrix}$$

and the eigenvalue of this matrix are 1.55, 1.0473 and 0.6527. Using Theorem 2, the distribution of the statistic is estimated to be that of $Y = (1.55)\chi_2^2 + (1.0473)\chi_2^2 + (0.6527)\chi_2^2$ where each of the chi-square variate is independent. Using Lemma 7 the cumulative probability value associated with 8.7473 is obtained as 0.7665 so that the observed significance level is 0.2335. Thus, if the test is performed at the $\alpha = 0.1$ level of significance the conclusion reached is that there is insufficient evidence to reject the null hypothesis that the samples are from populations with identical correlation matrices.

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Appendix

Lemma 3. Let $V = (v_{ij})$ be a $p \times p$ symmetric matrix with zero on the diagonal and let $C = (c_{ij})$ be a $p \times p$ symmetric matrix. Then

$$\text{tr}(CV) = \sum_{i=1}^p \sum_{j=1}^p c_{ij}v_{ij} = 2 \sum_{i<j} c_{ij}v_{ij}$$

Proof. The proof is obtained by noting that $v_{jj} = 0$ and $c_{ij} = c_{ji}$. □

Lemma 4. Let $V_\alpha = (v_{\alpha ij})$ and $V_\beta = (v_{\beta ij})$ be $p \times p$ symmetric matrices with zero on the diagonal. Then

$$\sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p \sum_{\ell=1}^p c_{ijkl} v_{\alpha ij} v_{\beta k\ell} = \sum_{i<j} \sum_{k<\ell} (c_{ijkl} + c_{ijlk} + c_{jikl} + c_{jilk}) v_{\alpha ij} v_{\beta k\ell}.$$

Proof. Using Lemma 3, the sum may be written as

$$\sum_{i=1}^p \sum_{j=1}^p \sum_{k<\ell} (c_{ijkl} + c_{ijlk}) v_{\alpha ij} v_{\beta k\ell}$$

The proof is obtained by applying Lemma 3 second time. □

Lemma 5. Let A be a real symmetric matrix with eigenvalues that are less than one in absolute value, then

$$-\ln[\det(I - A)] = \text{tr}(A) + \frac{1}{2} \text{tr}(A^2) + \frac{1}{3} \text{tr}(A^3) + \dots$$

Proof. See Siotani, Hayakawa and Fujikoshi (1985). □

Lemma 6. Let R be a correlation of dimension p . Then

$$\frac{\partial}{\partial P} \ln[\det R] = R^{-1}$$

and

$$\frac{\partial}{\partial P} \text{tr}(R^{-1}B) = R^{-1}BR^{-1}$$

where B is a symmetric non-singular matrix of order p .

Proof. See Siotani, Hayakawa and Fujikoshi (1985). □

Lemma 7. Let Y_1, Y_2 and Y_3 be independent random variables, $Y_i \sim \chi_2^2$, $i = 1, 2, 3$. Define $Y = \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3$ where α_1, α_2 and α_3 are constants, $\alpha_1 > \alpha_2 > \alpha_3 > 0$. Then, the cumulative distribution function $F_Y(y)$ of Y is given by

$$F_Y(y) = \sum_{i=1}^3 C_i \left[1 - \exp\left(-\frac{y}{2\alpha_i}\right) \right], \quad y > 0,$$

where $C_1 = \alpha_1^2/(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_2)$, $C_2 = -\alpha_2^2/(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_2)$ and $C_3 = \alpha_3^2/(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_3)$

Proof. We get the desired result by inverting the moment generating function $M_Y(t) = \sum_{i=1}^3 C_i(1 - 2\alpha_i t)^{-1}$, $2\alpha_1 t < 1$. \square