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NATURAL MICROSTRUCTURES ASSOCIATED WITH SINGULARITY FREE GRADIENT FIELDS IN THREE-SPACE AND QUANTIZATION

Abstract.

Any singularity free vector field X defined on an open set in a three-dimensional Euclidean space with curl $X = 0$ admits a complex line bundle F^a with a fibre-wise defined symplectic structure, a principal bundle $\mathcal{P}^a$ and a Heisenberg group bundle G^a. For the non-vanishing constant vector field X the geometry of $\mathcal{P}^a$ defines for each frequency a Schrödinger representation of any fibre of the Heisenberg group bundle and in turn a quantization procedure for homogeneous quadratic polynomials on the real line.

1. Introduction

In [2] we described microstructures on a deformable medium by a principal bundle on the body manifold. The microstructure at a point of the body manifold is encoded by the fibre over it, i.e. the collection of all internal variables at the point. The structure group expresses the internal symmetries.

In these notes we will show that each singularity free gradient field defined on an open set of the Euclidean space hides a natural microstructure. The structure group is $U(1)$.

If the vector field X is a gradient field with a nowhere vanishing principal part a, say, then there are natural bundles over O such as a complex line bundle F^a with a fibre-wise defined symplectic form ω^a, a Heisenberg group bundle G^a and a four-dimensional principal bundle $\mathcal{P}^a$ with structure group $U(1)$. (Fibres over O are indicated by a lower index x.) For any $x \in O$ the fibre F^a_x is the orthogonal complement of $a(x)$ formed in E and encodes internal variables at x. It is, moreover, identified as a coadjoint orbit of G^a_x. The principal bundle $\mathcal{P}^a$, a subbundle of the fibre bundle F^a, is equipped with a natural connection form α^a, encoding the vector field in terms of the geometry of the local level surfaces: The field X can be reconstructed from α^a. The collection of all internal variables provides all tangent vectors to all locally given level surfaces. The curvature Ω^a of α^a describes the geometry of the level surfaces of the gradient field in terms of ω^a and the Gaussian curvature.

There is a natural link between this sort of microstructure and quantum mechanics. To demonstrate the mechanism we have in mind, the principal part a of the vector field X is assumed to be constant (for simplicity only). Thus the integral curves, i.e. the field lines, are straight lines. Fixing some $x \in O$ and a solution curve β passing through $x \in O$, we consider the collection of all geodesics on the restriction of the principal bundle $\mathcal{P}^a$ to β. Each of these geodesics with the same speed is called a periodic lift of β and passes through a common initial point $v_x \in \mathcal{P}^a_x$, say. If the periodic lifts rotate in time, circular polarized waves are established. Hence the integral curve β is accompanied by circular polarized waves on $\mathcal{P}^a$ of arbitrarily given frequencies. This collection of periodic lifts of β defines unitary representations ρ_ν of the Heisenberg group G_x^a, the Schrödinger representations (cf. [11] and [13]). The frequencies of the polarized waves correspond to the equivalence classes of ρ_ν due to the theorem of Stone-von Neumann.

The automorphism group of G_x^a is the symplectic group $Sp(F_x^a)$ of the symplectic complex line F_x^a. Therefore, the representation ρ_1 of G_x^a yields a projective representation of $Sp(F_x^a)$, due to the theorem of Stone-von Neumann again. This projective representation is resolved to a unitary representation W of the metaplectic group $Mp(F_x^a)$ in the usual way. Its infinitesimal representation dW of the Lie algebra $mp(F_x^a)$ of $Mp(F_x^a)$ yields the quantization procedure for all homogeneous quadratic polynomials defined on the real line. Of course, this is in analogy to the quantization procedure emanating from the quadratic approximation in optics.

2. The complex line bundle associated with a singularity free gradient field in Euclidean space

Let O be an open subset not containing the zero vector 0 in a three-dimensional oriented $\mathbb{R}$-vector space E with scalar product $< \ , \ >$. The orientation on the Euclidean space E shall be represented by the Euclidean volume form μ_E.

Our setting relies on a smooth, singularity free vector field $X : O \longrightarrow O \times E$ with principal part $a : O \longrightarrow E$, say. We shall frequently identify X with its principal part.

Moreover, let $\mathbb{H} := \mathbb{R} \cdot e \oplus E$ be the skew field of quaternions where e is the multiplicative unit element. The scalar product $< \ , \ >$ and the orientation on E extend to all of $\mathbb{H}$ such that $e \in \mathbb{H}$ is a unit vector and the above splitting of $\mathbb{H}$ is orthogonal. The unit sphere S^3, i.e. $Spin(E)$, is naturally isomorphic to $SU(2)$ and covers $SO(E)$ twice (cf. [8] and [9]).

Given any $x \in O$, the orthogonal complement F_x^a of $a(x) \in E$ is a complex line as can be seen from the following: Let $\mathbb{C}_x^a \subset \mathbb{H}$ be the orthogonal complement of F_x^a. Hence the field of quaternions $\mathbb{H}$ splits orthogonally into

$$\mathbb{H} = \mathbb{C}_x^a \oplus F_x^a. \tag{1}$$

As it is easily observed,

$$\mathbb{C}_x^a = \mathbb{R} \cdot e \oplus \mathbb{R} \cdot \frac{a(x)}{|a(x)|}$$

is a commutative subfield of $\mathbb{H}$ naturally isomorphic to $\mathbb{C}$ due to

$$\left(\frac{a(x)}{|a(x)|} \right)^2 = -e \qquad \forall\, x \in O,$$

where $| \cdot |$ denotes the norm defined by $< \ , \ >$. This isomorphism shall be called

$$j_x^a : \mathbb{C} \longrightarrow \mathbb{C}_x^a \ ;$$

it maps 1 to e and i to $\frac{a(x)}{|a(x)|}$. The multiplicative group on the unit circle of $\mathbb{C}_x^a$ is denoted by $U_x^a(1)$. It is a subgroup of $SU(2) \subset \mathbb{H}$ and hence a group of spins. Obviously $a(x)$ generates the Lie algebra of $U_x^a(1)$.

F_x^a is a $\mathbb{C}_x^a$-linear space under the (right) multiplication of $\mathbb{H}$ and hence a $\mathbb{C}$-linear space, a complex line. Moreover, $\mathbb{H}$ is the Clifford algebra of F_x^a equipped with $- < \ , \ >$ (cf. [9]).

The topological subspace $F^a := \bigcup_{x \in O} \{x\} \times F_x^a$ of $O \times E$ is a $\mathbb{C}$-vector subbundle of $O \times E$, if curl $X = 0$, as can easily be seen. In this case F^a is a complex line bundle (cf. [15]), the complex line bundle associated with X. Let $\mathrm{pr}^a : F^a \longrightarrow O$ be its projection. Accordingly there is a bundle of fields $\mathbb{C}^a \longrightarrow O$ with fibre $\mathbb{C}^a_x$ at each $x \in O$. Clearly,

$$O \times \mathbb{H} = \mathbb{C}^a \times F^a$$

as vector bundles over O. Of course, the bundle $F^a \longrightarrow O$ can be regarded as the pull-back of TS^2 via the Gauss map assigning $\frac{a(x)}{|a(x)|}$ to any $x \in O$.

We, therefore, assume that curl $X = 0$ from now on. Due to this assumption there is a locally given real-valued function V, a potential of a, such that $a = \mathrm{grad}\ V$. Each (locally given) level surface S of V obviously satisfies $TS = F^a|_S$. Here $F^a|_S = \bigcup_{x\in S}\{x\} \times F^a_x$. Each fibre F^a_x of F^a is oriented by its Euclidean volume form $i_{\frac{a(x)}{|a(x)|}}\mu_E := \mu_E\left(\frac{a(x)}{|a(x)|}, \ldots, \ldots\right)$. For any level surface the scalar product yields a Riemannian metric g_S on S given by

$$g_S(x; v_x, w_x) := < v_x, w_x > \qquad \forall\, x \in O \quad \text{and} \quad \forall\, v_x, w_x \in T_x S.$$

For any vector field Y on S, any $x \in O$ and any $v_x \in T_x S$, the covariant derivative ∇^S of Levi-Cività determined by g_S satisfies

$$\nabla^S_{v_x} Y(x) \quad = \quad dY(x; v_x) + < Y(x), W^a_x(v_x) > .$$

Here $W^a_x : T_x S \longrightarrow T_x S$ is the Weingarten map of S assigning to each $w_x \in T_x S$ the vector $d\frac{a}{|a|}(x; w_x)$, the differential of $\frac{a}{|a|}$ at x evaluated at w_x. The Riemannian curvature R of ∇^S at any x is expressed by the well-known equation of Gauss as

$$\begin{aligned} R(x; v_x, w_x.u_x, y_x) \quad &= \quad < W^a_x(w_x), u_x > \cdot < W^a_x(v_x), y_x > \\ &\quad - < W^a_x(v_x), u_x > \cdot < W^a_x(w_x), y_x > \end{aligned} \tag{2}$$

for any choice of the vectors $v_x, w_x, u_x, y_x \in T_x S$.

A simple but fundamental observation in our setting is that each fibre $F^a_x \subset F^a$ carries a natural symplectic structure ω^a defined by

$$\omega^a(x; h, k) := < h \times a(x), k > = < h \cdot a(x), k > \qquad \forall\, h, k \in F^a_x,$$

where $\times$ is the cross product, here being identical with the product in $\mathbb{H}$. In the context of F^a_x as a complex line we may write

$$\omega^a(x; h_0, h_1) = |a(x)| \cdot < h_0 \cdot i, h_1 > .$$

This is due to the fact that h and $a(x)$ are perpendicular elements in E. The bundle F^a is fibre-wise oriented by $-\omega^a$. In fact ω^a extends on all of E by setting

$$\omega^a(x; y, z) :=< y \times a(x), z >$$

for all $y, z \in E$; it is not a symplectic structure on O, of course. Let $\kappa(x) := \det W^a_x$ for all $x \in S$, the Gaussian curvature of S. Provided v_x, w_x is an orthonormal basis of $T_x S$, the relation between the Riemannian curvature R and ω is given by

$$R(x; v_x, w_x.u_x, y_x) = \frac{\kappa(x)}{|a(x)|} \cdot \omega^a(x; u_x, y_x)$$

for every $x \in S$ and $u_x, y_x \in T_x S = F^a_x$.

3. The natural principal bundle $\mathcal{P}^a$ associated with X

We recall that the singularity free vector field X on O has the form $X = (\mathrm{id}\,, a)$. Let $\mathcal{P}^a_x \subset F^a_x$ be the circle centred at zero with radius $|a(x)|^{-\frac{1}{2}}$ for any $x \in O$. Then

$$\mathcal{P}^a := \bigcup_{x\in O} \{x\} \times \mathcal{P}^a_x$$

equipped with the topology induced by F^a is a four-dimensional fibre-wise oriented submanifold of F^a. It inherits its smooth fibre-wise orientation from F^a. Moreover, $\mathcal{P}^a$ is a $U(1)$-principal bundle. $U(1)$ acts from the right on the fibre $\mathcal{P}^a_x$ of $\mathcal{P}^a$ via $j^a_x|_{U(1)} : U(1) \longrightarrow U^a_x(1)$ for any $x \in O$. This operation is fibre-wise orientation preserving. The reason for choosing the radius of $\mathcal{P}^a_x$ to be $|a(x)|^{-\frac{1}{2}}$ will be made apparent below.

Both F^a and $\mathcal{P}^a$ encode collections of internal variables over O and both are constructed out of X, of course. Clearly, the vector bundle F^a is associated with $\mathcal{P}^a$.

The vector field X can be reconstructed out of the smooth, fibre-wise oriented principal bundle $\mathcal{P}^a$ as follows: For each $x \in O$ the fibre $\mathcal{P}^a_x$ is a circle in F^a_x centred at zero. The orientation of this circle yields an orientation of the orthogonal complement of F^a_x formed in E, the direction of the field at x. Hence $|a(x)|$ is determined by the radius of the circle $\mathcal{P}^a_x$. Therefore, the vector field X admits a characteristic geometric object, namely the smooth, fibre-wise oriented principal bundle $\mathcal{P}^a$ on which all properties of X can be reformulated in geometric terms. Vice versa, all geometric properties of $\mathcal{P}^a$ reflect characteristics of a. The fibre-wise orientation can be implemented in a more elegant way by introducing a connection form, α^a, say, which is in fact much more powerful. This will be our next task. Since $\mathcal{P}^a \subset O \times E$, any tangent vector $\xi \in T_{v_x}\mathcal{P}^a$ can be represented as a quadruple

$$\xi = (x, v_x, h, \zeta_{v_x}) \in O \times E \times E \times E$$

for $x \in O$, $v_x \in \mathcal{P}^a_x$ and $h, \zeta_{v_x} \in E \subset \mathbb{H}$ with the following restrictions, expressing the fact that ξ is tangent to $\mathcal{P}^a$:
Given a curve $\sigma = (\sigma_1, \sigma_2)$ on $\mathcal{P}^a$ with $\sigma_1(s) \in O$ and $\sigma_2(s) \in \mathcal{P}^a_{\sigma_1(s)}$ for all s, then

$$<\sigma_2(s), a(\sigma_1(s))> = 0 \quad \text{and} \quad |\sigma_2(s)|^2 = \frac{1}{|a(\sigma_1(s))|} \qquad \forall s.$$

Each $\zeta \in T_{v_x}\mathcal{P}^a$ given by $\zeta = \dot{\sigma}_2\,(0)$ is expressed as

$$\zeta = r_1 \cdot \frac{a(x)}{|a(x)|} + r_2 \cdot \frac{v_x}{|v_x|} + r \cdot \frac{v_x \times a(x)}{|v_x| \cdot |a(x)|}$$

with

$$r_1 = -<W^a_x(v_x), h> \quad , \quad r_2 = -\frac{|v_x|}{2} \cdot d\ln|a|(x;h)$$

and a free parameter $r \in \mathbb{R}$. The Weingarten map W^a_x is of the form

$$da(x;k) = |a(x)| \cdot W^a_x(k) + a(x) \cdot d\ln|a|(x;k) \qquad \forall\, x \in O\ ,\ \forall\, k \in E,$$

where we set $W^a_x(a(x)) = 0$ for all $x \in O$. With these preparations we define the one-form

$$\alpha^a : T\mathcal{P}^a \longrightarrow \mathbb{R}$$

for each $\xi \in T\mathcal{P}^a$ with $\xi = (x, v_x, h, \zeta)$ to be

$$\alpha^a(v_x, \xi) := < v_x \times a(x), \zeta > . \tag{3}$$

One easily shows that α^a is a connection form (cf. [10] and for the field theoretic aspect [1]). To match the requirement of a connection form in this metric setting, the size of the radius of $\mathcal{P}^a_x$ is crucial for any $x \in O$. The negative of the connection form on $\mathcal{P}^a$ is in accordance with the smooth fibre-wise orientation, of course.

Thus the principal bundle $\mathcal{P}^a$ together with the connection form α^a characterizes the vector field X, and vice versa. To determine the curvature Ω^a which is defined to be the exterior covariant derivative of α^a, the horizontal bundles in $T\mathcal{P}^a$ will be characterized. Given $v_x \in \mathcal{P}^a$, the horizontal subspace $Hor_{v_x} \subset T\mathcal{P}^a$ is defined by

$$Hor_{v_x} := ker\ \alpha^a(v_x; \ldots).$$

A vector $\xi_{v_x} \in Hor_{v_x}$, being orthogonal to $v_x \times a(x)$, has the form $(x, v_x, h, \zeta^{hor}) \in O \times E \times E \times E$ where h varies in O and ζ^{hor} satisfies

$$\zeta^{hor} = - < W^a_x(v_x), h > \cdot \frac{a(x)}{|a(x)|} - \frac{|v_x|}{2} \cdot d \ln |a|(x; h) \cdot \frac{v_x}{|v_x|}.$$

Since $T\mathrm{pr}^a : Hor_{v_x} \longrightarrow T_x O$ is an isomorphism for any $v_x \in \mathcal{P}^a$, dim $Hor_{v_x} = 3$ for all $v_x \in \mathcal{P}^a$ and for all $x \in O$. The collection $Hor \subset T\mathcal{P}^a$ of all horizontal subspaces in the tangent bundle $T\mathcal{P}^a$ inherits a vector bundle structure $T\mathcal{P}^a$.
The exterior covariant derivative $d^{hor}\alpha^a$ is defined by

$$d^{hor}\alpha^a(v_x, \xi_0, \xi_1) := d\alpha^a(v_x; \xi_0^{hor}, \xi_1^{hor})$$

for every $\xi_0, \xi_1 \in T_{v_x}\mathcal{P}^a$, $v_x \in \mathcal{P}^a_x$ and $x \in O$.
The curvature $\Omega^a := d^{hor}\alpha^a$ of α^a is sensitive in particular to the geometry of the (locally given) level surfaces, as is easily verified by using equation (2):

PROPOSITION 1. *Let X be a smooth, singularity free vector field on O with principal part a. The curvature Ω^a of the connection form α^a is*

$$\Omega^a = \frac{\kappa}{|a|} \cdot \omega^a$$

where $\kappa : O \longrightarrow \mathbb{R}$ is the leaf-wise defined Gaussian curvature on the foliation of O given by the collection of all level surfaces of the locally determined potential V. The curvature Ω^a vanishes along field lines of X.

The fact that the curvature Ω^a vanishes along field lines plays a crucial role in our set-up. It will allow us to establish (on a simple model) the relation between the transmission of internal variables along field lines of X and the quantization of homogeneous quadratic polynomials on the real line.

4. Two examples

If we consider specific vector fields in these notes, we will concentrate on the two types presented in more detail in this section. At first let us regard a constant vector field X on $O \subset E\backslash\{0\}$ with a principal part having the non-zero value $a \in E$ for all $x \in O$. Obviously the principal bundle $\mathcal{P}^a$ is trivial, i.e.

$$\mathcal{P}^a \cong O \times U^a(1).$$

Since an integral curve β of X is a straight line segment parametrized by

$$\beta(t) = t \cdot a + x_0 \quad \text{with} \quad \beta(t_0) = x_0,$$

the restriction $\mathcal{P}^a|_{im\ \beta}$ of $\mathcal{P}^a$ to the image $im\ \beta$ is a cylinder with radius $|a|^{-\frac{1}{2}}$.

As the second type of example of a principal bundle $\mathcal{P}^a$ associated with a singularity free vector field let us consider a central symmetric field $X = \text{grad}\ V_{sol}$ on $E\backslash\{0\}$ with the only singularity at the origin. The potential V_{sol} is given by

$$V_{sol}(x) := -\frac{\bar{m}}{|x|} \qquad \forall\, x \in O$$

where $\bar{m}$ is a positive real. This potential governs planetary motions and hence grad V_{sol} is called the solar field here. The principal part a of the gradient field is

$$\text{grad}\ V_{sol}(x) = -\frac{\bar{m}}{|x|^2} \cdot \frac{x}{|x|} \qquad \forall\, x \in E\backslash\{0\}. \tag{4}$$

For reasons of simplicity we illustrate from a longitudinal point of view the principal bundle $\mathcal{P}^a$ associated with the gradient field. An integral curve β passing through x at the time $t_0 = 1$ is of the form

$$\beta(t) = -\bar{m} \cdot (3 \cdot t - 2)^{\frac{1}{3}} \cdot x \qquad \text{for} \quad \frac{2}{3} < t < \infty. \tag{5}$$

Hence the (trivial) principal bundle $\mathcal{P}^a|_{im\ \beta}$ is a cone. The radius r of a circle $\mathcal{P}^a_x$ with $x \in im\ \beta$ is $r = \frac{|x|}{\sqrt{\bar{m}}}$ for all $x \in O$ (cf. [12]).

5. Heisenberg group bundles associated with the singularity free vector field and curves and the solar field

Associated with the $(2+1)$-splitting of the Euclidean space E caused by the vector field X there is a natural Heisenberg group bundle G^a with ω^a as symplectic form. The bundle G^a allows us to reconstruct X as well. Heisenberg groups play a central role in signal theory (cf. [13], [14]). We essentially restrict us to the two types of examples presented in the previous section.

Given $x \in O$, the vector $a(x) \neq 0$ determines F^a_x with the symplectic structure $\omega^{a(x)}$ and $\mathbb{C}^a_x$ which decompose $\mathbb{H}$ according to (1).

The submanifold $G^a_x := |a(x)|^{-\frac{1}{2}} \cdot e \cdot U^a_x(1) \oplus F^a_x$ of $\mathbb{H}$ carries the Heisenberg group structure the (non-commutative) multiplication of which is defined by

$$(z_1 + h_1) \cdot (z_2 + h_2) := |a(x)|^{-\frac{1}{2}} \cdot z_1 \cdot z_2 \cdot e^{\frac{1}{2} \cdot \omega^a(x; h_1, h_2) \cdot \frac{a}{|a|}} + h_1 + h_2 \tag{6}$$

for any two $z_1, z_2 \in |a(x)|^{-\frac{1}{2}} \cdot e \cdot U^a_x(1)$ and any pair $h_1, h_2 \in F^a_x$ (cf. [12]). The (commutative) multiplication in the centre $|a(x)|^{\frac{1}{2}} \cdot e \cdot U^a_x(1)$ of G^a_x is given by adding angles. The reason the centre has radius $|a(x)|^{-\frac{1}{2}}$ is the length scale on $\mathcal{P}^a_x$ for any $x \in O$. The group bundle

$\cup_{x\in O}\{x\}\times|a(x)|^{-\frac{1}{2}}\cdot e\cdot U_x^a(1)$, which is the collection of all centres, is associated with $\mathcal{P}^a$ and forms a natural torus bundle together with $\mathcal{P}^a$. The collection

$$G^a := \bigcup_{x\in O}\{x\}\times G_x^a$$

can be made into a group bundle which is associated with the principal bundle $\mathcal{P}^a$, too. Clearly $F^a\subset G^a$ as fibre bundles. In the cases of a constant vector field and the solar field the Heisenberg group bundle along field lines is trivial.

In particular, a in (6) takes the values $|a(x)|^{-\frac{1}{2}}=|a|^{-\frac{1}{2}}$ and $|a(x)|^{-\frac{1}{2}}=\frac{|x|}{m}$ for all $x\in O$ in the cases of the constant vector field respectively the solar field.

The Lie algebra $\mathcal{G}_x^a$ of G_x^a is

$$\mathcal{G}_x^a := \mathbb{R}\cdot\frac{a}{|a|}\oplus F_x^a$$

together with the operation

$$\left[\vartheta_1\cdot\frac{a}{|a|}+h_1,\vartheta_2\cdot\frac{a}{|a|}+h_2\right]:=\omega^a(x;h_1,h_2)\cdot\frac{a}{|a|}$$

for any $\vartheta_1,\vartheta_2\in\mathbb{R}$ and any $h_1,h_2\in F_x^a$. The exponential map $\exp_{G_x^a}:\mathcal{G}_x^a\longrightarrow G_x^a$ is surjective. Obviously, X can be reconstructed from both G^a and $\mathcal{G}^a$. The coadjoint orbit of Ad^{a^*} passing through $<\vartheta\cdot\frac{a}{|a|}+h_1,..>\in\mathcal{G}_x^{a*}$ with $\vartheta\neq 0$ is $\vartheta\cdot\frac{a}{|a|}\oplus F_x^a$.

In this context we will study the solar field next (cf. [12]). At first let us see how it emanates from Keppler's laws of circular planetary motion. Suppose σ is a closed planetary orbit in $E\setminus\{0\}$ defined on all of $\mathbb{R}$; it lies in a plane $F^{b'}$, say, with $b'\in E\setminus\{0\}$, due to Keppler's second law. Let σ be a circle of radius r. It is generated by a one-parameter group φ in $SO(F^b)$ with generator b, say, yielding

$$\varphi(t)=e^{t\cdot b}\qquad\forall t\in\mathbb{R}.$$

Hence

$$\ddot{\varphi}=b^2\cdot\varphi=-|b|^2\cdot\varphi.$$

This generator, a skew linear map in $so(F^b)$, is identified with a vector in E in the obvious way. The invariant norms on $so(F^b)$ are positive real multiples of the trace norm, and hence on $so(F^b)$ the generator has a norm

$$||b||^2=-G'^2\cdot\ \mathrm{tr}\, b^2=G'^2\cdot|b|^2$$

for some positive real number G' and a fixed constant $||b||$.

The time of revolution $T:=\frac{2\pi}{|b|}$ is determined by Keppler's third law which states

$$T^2=r^3\cdot\ \text{const.}\tag{7}$$

Therefore $\ddot{\varsigma}$ of $\varsigma:=\varphi\cdot x_0$ with $|x_0|=r$ has the form

$$\ddot{\varsigma}=-\frac{||b||^2}{G'^2}\cdot\varsigma=-\frac{G\cdot m}{|\varsigma|^2}\cdot\frac{\varsigma}{|\varsigma|}$$

with $G'^2=G^{-1}\cdot r^3$ and $m:=||b||^2$ as solar mass. This is the reason why X with principal part grad V_{sol} here is called the solar field. Newton's field of gravitation includes the mass of the planet, which is not involved here.

Next let us point out a consequence of the comparison of the cone $\mathcal{P}^a|_\beta$ embedded into $\mathcal{G}^a_x$ for a fixed $x \in im\ \beta$, but shifted forward such that its vertex is in $0 \in E$, with the cone C_M of a Minkowski metric g^a_M on $\mathcal{G}^a_x$. The metric g^a_M relies on the following observation: Up to the choice of a positive constant c, there is a natural Minkowski metric on $\mathbb{H}$ inherited from squaring any quaternion $k = \lambda \cdot e + u$ with $\lambda \in \mathbb{R}$ and $u \in E$ since the e-component $(k^2)_e$ of k^2 is

$$-(k^2)_e = (|u|^2 - \lambda^2) \cdot e = (b^2 \cdot k^2)_e$$

with $b \in S^2$. Introducing the positive constant c, the Minkowski metric g^a_M on $\mathcal{G}^a_x$ mentioned above is pulled back to $\mathcal{G}^a_x$ by the right multiplication with $\frac{a}{|a|}$ and reads

$$g^a_M(h_1, h_2) := < u_1, u_2 > -c \cdot \lambda_1 \cdot \lambda_2$$

for any $h_r \in F^{\frac{a}{|a|}}$ represented by $h_r = \lambda_r \cdot \frac{a}{|a|} + u_r$ for $r = 1, 2$. The respective interior angles φ^a and φ_{C_M} which the meridians on $\mathcal{P}|_{im\ \beta}$ and C_M form with the axis $\mathbb{R} \cdot \frac{x}{|x|}$ satisfy

$$\tan \varphi^a = \bar{m}^{-\frac{1}{2}} \quad \text{and} \quad \tan \varphi_{C_M} = \frac{1}{c},$$

and

$$m \cdot c^2 = G^{-1} \cdot \cot^2 \varphi^a \cdot \cot^2 \varphi_{C_M},$$

provided $m := \frac{\bar{m}}{G}$. This is a geometric basis to derive within our setting $E = m \cdot c^2$ from special relativity (cf. [12]).

Now we will study planetary motions in terms of Heisenberg algebras. In particular we will deduce Keppler's laws from the solar field by means of a holographic principle (we will make this terminology precise below). To this end we first describe natural Heisenberg algebras associated with each time derivative of a smooth injective curve σ in O defined on an interval $I \subset \mathbb{R}$. For any $t \in I$ the n-th derivative $\sigma^{(n)}(t)$, assumed to be different from zero, defines a Heisenberg algebra bundle $\mathcal{G}^{(n)}$ for $n = 0, 1 \ldots$ with fibre

$$\mathcal{G}^{(n)}_{\sigma(t)} := \mathbb{R} \cdot \sigma^{(n)}(t) \oplus F^{(n)}_{\sigma(t)}$$

where $F^{(n)}_{\sigma(t)} := \sigma^{(n)}(t)^\perp$ (formed in E) with the symplectic structure $\omega^{(n)}$ defined by

$$\omega^{(n)}(\sigma(t); h_1, h_2) \quad = \quad < h_1 \times \sigma^{(n)}(t), h_2 > \qquad \forall h_1, h_2 \in F^{(n)}_{\sigma(t)}.$$

Here $F^{(n)}$ is the complex line bundle along $im\ \sigma$ for which $F^{(n)}_{\sigma(t)} := \sigma^{(n)}(t)^\perp$ for each t. The two-forms $\omega^{(n)}$ are extended to all of O by letting h_1 and h_2 vary also in $\mathbb{R} \cdot \frac{\sigma^{(n)}(t)}{|\sigma^{(n)}(t)|}$ for all $t \in I$. The Heisenberg algebra $\mathcal{G}^{(n)}_{\sigma(t)}$ is naturally isomorphic to $\mathcal{G}^{(n)}_{\sigma(t_0)}$ for a given $t_0 \in I$, any t and any n for which $\sigma^{(n)}(t) \neq 0$.

As a subbundle of $F^{(n)}$ we construct $\mathcal{P}^{(n)} \subset F^{(n)}$ which constitutes of the circles $\mathcal{P}^{(n)}_{\sigma(t)} \subset F^{(n)}_{\sigma(t)}$ with radius $|\sigma^{(n)}(t)|^{-\frac{1}{2}}$. On $F^{(n)}$ the curve σ admits an analogue $\alpha^{(n)}$ of the one-form α^a described in (3), determined by

$$\alpha^{(n)}(\sigma(t); h) \ = < \sigma(t) \times \sigma^{(n)}(t), h > \qquad \forall\, h \in F^{(n)}_{\sigma(t)}$$

for any t. Since the Heisenberg algebra bundle evolves from $\mathcal{G}_0^{(n)}$ we may ask how $\alpha^{(n)}$ evolves along σ, in particular for $\alpha^{(1)}$. The evolution of $\alpha^{(n)}$ can be expressed in terms of $\dot{\alpha}^{(n)}$ defined by

$$\begin{aligned}\dot{\alpha}^{(n)}(\sigma(t);h) &:= \frac{d}{dt}\alpha^{(n)}(\sigma(t);h) - \alpha^{(n)}(\dot{\sigma}(t),h)\\ &= <\sigma(t)\times\sigma^{(n+1)}(t),h> \qquad \forall\, h\in F^{(n)}_{\sigma(t)}.\end{aligned}$$

A slightly more informative form for $\dot{\alpha}^{(1)}$ is

$$\dot{\alpha}^{(1)}(\sigma(t);h) = \omega^{(2)}(\sigma(t);\sigma(t),h) \qquad \forall\, h\in F^{(1)}_{\sigma(t)}.$$

Thus the evolution of $\alpha^{(1)}$ along σ is governed by the Heisenberg algebras $\mathcal{G}^{(2)}$, yielding in particular

$$\alpha^{(1)} = \text{ const.} \quad \text{iff} \quad \sigma\times\ddot{\sigma}=0, \quad \text{meaning} \quad i_\sigma\omega^{(2)}=0.$$

Hence $\alpha^{(1)}$ = const. is the analogue of Keppler's second law. In this case the quaternion $b := \sigma\times\dot{\sigma}$ is constant and hence σ is in the plane $F\subset E$ perpendicular to b. Thus $\mathbb{R}\cdot b\times F^b$ is a Heisenberg algebra with

$$\omega^b(h_1,h_2) :=< h_1\times b, h_2 > \qquad \forall\, h_1,h_2\in F^b$$

as symplectic form on F^a. Hence the planetary motion can be described in only one Heisenberg algebra, namely in $\mathcal{G}^b$, which is caused by the angular momentum b, of course. We have $\ddot{\sigma} = f\cdot\sigma$ for some smooth real-valued function f defined along a planetary motion σ, implying $\omega^{(2)} = f\cdot\frac{|\sigma|^2}{\bar{m}}\cdot\omega^a$. In case σ is a circle, f is identical with the constant map with value $\frac{\bar{m}}{|\sigma|^2}$, due to the third Kepplerian law (cf. equation (7)). This motivates us to set

$$\mathcal{G}^{(2)}_{\sigma(t)} = \mathcal{G}^a_{\sigma(t)} \qquad \forall t \tag{8}$$

along any closed planetary motion σ which hence implies $\omega^{(2)} = \omega^a$ along σ. In turn one obtains

$$\ddot{\sigma}(t) = \text{grad}\ V_{sol}(\sigma(t)) \qquad \forall\, t, \tag{9}$$

a well-known equation from Newton implying Keppler's laws. Equation (9) is derived from a holographic principle in the sense that equation (8) states that the oriented circle of $\mathcal{P}^2_{\sigma(t)}$ matches the oriented circle of $\mathcal{P}^a_{\sigma(t)}$ at any t.

6. Horizontal and periodic lifts of β

Since, in general, $\Omega^a\neq 0$, the horizontal distribution in $T\mathcal{P}^a$ does not need to be integrable along level surfaces. However, Ω^a vanishes along field lines and thus the horizontal distribution is integrable along these curves. Let us look at $\mathcal{P}^a|_\beta$ where β is a field line of the singularity free vector field X.

A horizontal lift of $\dot{\beta}$ is a curve $\dot{\beta}^{hor}$ in $Hor_\beta = \ker\alpha^a$ which satisfies $T\text{pr}^a\dot{\beta}^{hor} = \dot{\beta}$ and obeys an initial condition in $T\mathcal{P}^a|_\beta$. Hence there is a unique curve β^{hor} passing through $v_{\beta(t_0)}\in\mathcal{P}^a_{\beta(t_0)}$, say, called horizontal lift of β. In the case of a constant vector field or in the case of the solar field this is nothing else but a meridian of the cylinder respectively the cone $\mathcal{P}^a|_\beta$ containing $v_{\beta(t_0)}$. Let $\beta(t_0) = x$ for a fixed $x \in O$.

Obviously, a horizontal lift is a geodesic on $\mathcal{P}^a|_\beta$ equipped with the metric g_{Hor_β}, say, induced by the scalar product $< \, , \, >$ on E.

At first let a be a non-vanishing constant. A curve γ on $\mathcal{P}^a|_\beta$ here is called a periodic lift of β through v_x iff it is of the form

$$\gamma(s) = \beta^{hor}(s) \cdot e^{p\cdot s\cdot \frac{a}{|a|}} \in \mathcal{P}^a_{\beta(s)} \qquad \forall s$$

where p is a fixed real.

Clearly, γ is a horizontal lift through v_x iff $\gamma = \beta^{hor}$, i.e. iff $p = 0$. In fact any periodic lift γ of β is a geodesic on $\mathcal{P}^a|_\beta$. Hence $\ddot{\gamma}$ is perpendicular to $\mathcal{P}^a|_\beta$. Due to the $U(1)$-symmetry of $\mathcal{P}^a|_\beta$, a geodesic σ on $\mathcal{P}^a|_\beta$ is of the form

$$\sigma(s) = \beta^{hor}(\theta \cdot s) \cdot e^{p\cdot\theta\cdot s\cdot \frac{a}{|a|}} \qquad \forall s$$

as it is easily verified. Here p and θ denote reals. θ determines the speed of the geodesic. Thus σ and β have accordant speeds if $\theta = 1$ (which will be assumed from now on), as can be easily seen from

$$\dot{\gamma}(0) = p \cdot v_x \cdot \frac{a}{|a|} + \dot{\beta}^{hor}(0)$$

for $t_0 = 0$. The real number p determines the spatial frequency of the periodic lift γ due to $\frac{2\cdot\pi}{T} = \frac{p}{|v_x|}$. The spatial frequency of γ counts the number of revolutions around $\mathcal{P}^a|_\beta$ per unit time and is determined by the F^a_x-component p of the initial velocity due to the $U(1)$-symmetry of the cylinder $\mathcal{P}^a|_\beta$. We refer to p as a momentum.

For the solar field $X(x) = \left(x, -\frac{x}{|x|^3}\right)$ with $x \in O$, let $|x_0| = 1$ and let a parametrization of the body of revolution $\mathcal{P}^a|_\beta$ be given in Clairaut coordinates via the map $\mathbf{x} : \mathcal{U} \to E$ defined by

$$\mathbf{x}(u, v) := -\bar{m} \cdot (3v - 2)^{\frac{1}{3}} \cdot r\left(e^{u\cdot\frac{a}{|a|}}\right) \cdot \left(v_x + \frac{a}{|a|}\right)$$

on an open set $\mathcal{U} \subset \mathbb{R}^2$. Here r is the representation of $U^{\frac{a}{|a|}}(1)$ onto $SO\left(F^{\frac{a}{|a|}}\right)$ for any $x \in O$. Then a geodesic γ on $\mathcal{P}^a|_\beta$ takes the form

$$\gamma(s) = \mathbf{x}(u(s), v(s)) \quad = \quad -\bar{m} \cdot (3v(s) - 2)^{\frac{1}{3}} \cdot r\left(e^{u(s)\cdot\frac{a}{|a|}}\right) \cdot \left(v_x + \frac{a}{|a|}\right)$$

where the functions u and v are determined by

$$u(s) \quad = \quad \sqrt{2} \cdot \arctan\left(\frac{s}{\sqrt{2}\,d} + \frac{c_1}{2\,d}\right) + c_2 \tag{10}$$

$$\text{and} \quad v(s) \quad = \quad \pm\frac{1}{3}\left(\left(\frac{1}{\sqrt{2}}s + c_1\right)^2 + d^2\right)^{\frac{3}{2}} + \frac{2}{3} \tag{11}$$

(cf. [12]) with s in an open interval $I \subset \mathbb{R}$ containing 1. Here c_1 and c_2 are integration constants determining the initial conditions. Since we are concerned with a forward movement along the channel $\mathbb{R} \cdot \frac{a}{|a|}$, only the positive sign in (11) is of interest. The constant d fixes the slope of the geodesic via

$$\cos\vartheta = \frac{d}{\sqrt{\left(\frac{1}{\sqrt{2}}s + c_1\right)^2 + d^2}}$$

where ϑ is the constant angle between the geodesic γ, called periodic lift, again, and the parallels given in Clairaut coordinates. This means that d vanishes precisely for a meridian. A periodic lift γ is a horizontal lift of β iff γ is a meridian. Thus the parametrization of a meridian as a horizontal lift β^{hor} of an integral curve β parametrized as in (5) has the form

$$\beta^{hor}(t) = -\bar{m} \cdot (3t-2)^{\frac{1}{3}} \cdot v_x$$

with $\beta^{hor}(1) = -\bar{m} \cdot v_x$ as well as $\beta(1) = -\bar{m} \cdot x$ for $\frac{2}{3} \leq t < 1$ and any initial $v_x \in \mathcal{P}^a_{\beta(1)}$.

For the constant vector field from above, any periodic lift γ of β through v_x is uniquely determined by the $U^a(1)$-valued map

$$s \mapsto e^{p \cdot s \cdot \frac{a}{|a|}},$$

while for the solar field a periodic lift is characterized by

$$s \mapsto e^{u(s) \cdot \frac{a}{|a|}}$$

with $u(s)$ as in (10). These two maps here are called an elementary periodic function respectively an elementary Clairaut map. Therefore, we can state:

PROPOSITION 2. *Let $x = \beta(0)$. Under the hypothesis that a is a non-zero constant, there is a one-to-one correspondence between all elementary periodic $U^a(1)$-valued functions and all periodic lifts of β passing through a given $v_x \in \mathcal{P}^a_x$. In case X is the solar field there is a one-to-one correspondence between all periodic lifts passing through a given $v_x \in \mathcal{P}^a_x$ and all elementary Clairaut maps.*

An internal variable can be interpreted as a piece of information. Thus the fibres F^a_x and $\mathcal{P}^a_x$ can be regarded as a collection of pieces of information at x. The periodic lifts of β on $\mathcal{P}^a|_\beta$ describe the evolution of information of $\mathcal{P}^a|_\beta$ along β. This evolution can be further realized by a circular polarized wave: Let the lift rotate with frequency $\nu \neq 0$. Then a point $w(s;t)$, say, on this rotating lift is described by

$$w(s;t) = |v_x| \cdot \frac{\beta^{hor}_{v_x}(s)}{|\beta^{hor}_{v_x}(s)|} \cdot e^{2\pi\nu(t - p\cdot s)\cdot\frac{a}{|a|}} \qquad \forall s,t \in \mathbb{R}, \quad s \neq 0 \tag{12}$$

a circular polarized wave on the cylinder with $\frac{1}{|p|}$ as speed of the phase and $|v_x|$ as amplitude. w travels along $\mathbb{R} \cdot \frac{a}{|a|}$, the channel of information. Clearly, $\mathcal{P}^a|_{im\ \beta}$ is in $O \times E$ and not in E. However, w could be coupled to the space E and could be a wave in E traveling along β, e.g. as an electric or magnetic field. More types of waves can be obtained by using the complex line bundle F^a instead of the principal bundle $\mathcal{P}^a$, of course.

7. Representation of the Heisenberg group associated with periodic lifts of β on $\mathcal{P}^a|_\beta$ of a constant vector field

Let $a \neq 0$ be constant on O and $x \in im\ \beta$ a fixed vector. There is a unique periodic lift γ of β passing through $v_x = \gamma(0)$ with prescribed velocity $\dot{\gamma}(0)$. At first we will associate with $\dot{\gamma}(0)$ a well-defined unitary linear operator on a Hilbert space as follows.

The specification of $v_x \in \mathcal{P}^a_x$ turns F^a_x into a field $\hat{F}^a_x$ isomorphic to $\mathbb{C}$, since $\frac{v_x}{|v_x|} \cdot \mathbb{C} = F^a_x$. The real axis is $\mathbb{R} \cdot \frac{v_x}{|v_x|}$ and the imaginary one is $\mathbb{R} \cdot \frac{v_x}{|v_x|} \times \frac{a}{|a|}$. We rename these axes by q-axis carried by the unit vector $\bar{q}_x$ and by p-axis carried by the unit vector $\bar{p}_x$, respectively. Clearly, $\bar{p}_x = \bar{q}_x \cdot j^a_x(i)$. Any $h \in F^a_x$ is thus of the form $h = (q, p)$. The Schwartz space of the real axis and its L^2- completion are denoted by $\mathcal{S}(\mathbb{R}, \mathbb{C})$ and $L^2(\mathbb{R}, \mathbb{C})$, respectively. The Schrödinger representation ρ_x of G^a_x acts on each complex-valued $\psi \in \mathcal{S}(\mathbb{R}, \mathbb{C}) \subset L^2(\mathbb{R}, \mathbb{C})$ by

$$(13) \qquad \rho_x(z + h)(\psi)(\tau) := z \cdot e^{p\cdot\tau\cdot i} \cdot e^{-\frac{1}{2}\cdot p\cdot q\cdot i} \cdot \psi(\tau - q) \qquad \forall\, \tau \in \mathbb{R}$$

for all $z + h \in G^a_x$ with $h = (q, p)$ (cf. [11], [13] and [7]). Clearly,

$$-p \cdot q \cdot i = \omega^a_x((0, p), (q, 0)) \cdot i \quad \text{and} \quad z = e^{\vartheta \cdot \frac{a}{|a|}}$$

for some $\vartheta \in \mathbb{R}$. By the Stone-von Neumann theorem ρ_x is irreducible (cf. [13] and [7]). Setting $q = |v_x|$, for any $p \in \mathbb{R}$, equation (13) turns into

$$\rho_x(z + (|v_x|, p))(\psi)\left(\tau + \frac{|v_x|}{2}\right) = z \cdot e^{p\cdot\tau\cdot i} \cdot \psi\left(\tau - \frac{|v_x|}{2}\right) \qquad \forall\, \tau \in \mathbb{R}.$$

Operators of this form generate $\rho_x(G^a_x)$, of course. In case $2\pi\nu$ with the frequency ν (justified by (12)) is different from one, for each $p \in \mathbb{R}$ equation (13) turns into

$$(14) \qquad \rho_\nu\left(e^{t\cdot\frac{a}{|a|}} + (|v_x|, p)\right)(\psi)\left(\tau + \frac{|v_x|}{2}\right) = e^{2\pi\nu\cdot(t-p\cdot\tau)\cdot i} \cdot \psi\left(\tau - \frac{|v_x|}{2}\right)$$

for every $\tau, t \in \mathbb{R}$.

This shows that $2\pi\nu(t - p \cdot s)$ in the exponent of the factor $e^{2\pi\nu(t-p\cdot s)\cdot i}$ for $s = \tau$ is characteristic for the circular polarized wave described in (14) and determines the Schrödinger representation. Thus the geometry on the collection $\mathcal{P}^a|_\beta$ of all internal variables along β is directly transfered to the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$ via the Schrödinger representation. Differently formulated, the Schrödinger representation has a geometric counterpart, namely $\mathcal{P}^a$ together with its geometry, which is, for example, used for holography. The counterpart of i in quantum mechanics is the imaginary unit $\frac{a}{|a|} \in \mathbb{H}$.

On the other hand the $U^a_x(1)$-valued function $\tau \longrightarrow e^{2\pi\nu(t-p\cdot\tau)\cdot\frac{a}{|a|}}$ entirely describes the periodic lift γ, rotating with frequency ν and passing through v_x, as expressed in (13). Thus the circular polarized wave w is characterized by the unitary linear transformation $\rho_\nu(e^{t\cdot\frac{a}{|a|}} + (|v_x|, p))$ on $L^2(\mathbb{R}, \mathbb{C})$. Due to the Stone-von Neumann theorem, the equivalence class of ρ_ν is uniquely determined by ν and vice versa. Therefore, we state:

THEOREM 1. *Let a be a non-vanishing constant. Any periodic lift γ of β on $\mathcal{P}^a|_\beta$ with initial conditions $\gamma(0) = v_x$ and momentum p is uniquely characterized by the unitary linear* *transformation $\rho_x(1 + (|v_x|, p))$ of $L^2(\mathbb{R}, \mathbb{C})$ with $(1 + (|v_x|, p)) \in G_x^a$ and vice versa. Thus $v_x \in \mathcal{P}_x^a$ determines a unitary representation ρ on $L^2(\mathbb{R}, \mathbb{C})$ characterizing the collection $C_{v_x}^a$ of all periodic lifts of β passing through v_x. The unitary linear transformation $\rho_\nu(e^{t\cdot\frac{a}{|a|}} + (|v_x|, p))$ of $L^2(\mathbb{R}, \mathbb{C})$ characterizes the circular polarized wave w on $\mathcal{P}^a|_{im\ \beta}$ with frequency $\nu \neq 0$ generated by γ and vice versa. The frequency determines the equivalence class of ρ_ν.*

As a consequence we have

COROLLARY 1. *The Schrödinger representation ρ_ν of G_x^a describes the transport of any piece of information $(|v_x|, p) \in T_{(v_x,0)}\mathcal{P}^a|_\beta$ along the field line β, with $\mathbb{R} \cdot \frac{a}{|a|}$ as information transmission channel.*

The mechanism by which each geodesic is associated with a Schrödinger representation as expressed in theorem 1 is generalized for the solar field as follows (cf. [12]): Let $O = E \setminus \{0\}$. Given $im\ \beta$ of an integral curve β, we consider the Heisenberg algebra $\mathbb{R} \cdot \frac{a}{|a|} \oplus F^{\frac{a}{|a|}}$ equipped with the symplectic structure determined by $\frac{a}{|a|}$. Now let γ be a geodesic on $\mathcal{P}^a|_{im\ \beta}$ and $\psi \in \mathcal{S}(\mathbb{R}, \mathbb{C})$. Then the Schrödinger representation ρ_{sol} of the solar field on the Heisenberg group G_x^a is given by

$$\rho_{sol}(z, \mathbf{x}(s))(\psi)(\tau)(s) := z \cdot e^{u(s)\cdot\tau\cdot i} \cdot e^{-\frac{1}{2}\cdot u(s)\cdot v(s)\cdot i} \cdot \psi(\tau - v(s))$$

for all s in the domain of γ and any $\tau \in \mathbb{R}$.

8. Periodic lifts of β on $\mathcal{P}^a|_\beta$, the metaplectic group $Mp(F_x^a)$ and quantization

Let ρ_x be given as in (13), meaning that Planck's constant is set to one. For $v_x \in \mathcal{P}_x^a$ and $\dot{\gamma}_{v_x}(0)$ of a periodic lift γ_{v_x} of β,

$$\dot{\gamma}_{v_x}(0) = \dot{\gamma}_{v_x}(0)^{F_x^a} + \dot{\beta}_{v_x}^{hor}(0)$$

is an orthogonal splitting of the velocity of γ_{v_x} at 0. Clearly, the F_x^a-component of $\dot{\gamma}_{v_x}(0)$ is $\dot{\gamma}_{v_x}(0)^{F_x^a} = p \cdot \bar{p}_x$, where p is the momentum. Thus the momenta of periodic lifts of β passing through v_x are in a one-to-one correspondence with elements in $T_{v_x}\mathcal{P}_x^a$.

Therefore, the collection $\bar{C}_x^a$ of all periodic lifts of β on $\mathcal{P}^a|_\beta$ is in a one-to-one correspondence with $T\mathcal{P}_x^a$ (being diffeomorphic to a cylinder) via a map $f : \bar{C}_x^a \longrightarrow T\mathcal{P}_x^a$, say. Let

$$j : T\mathcal{P}_x^a|_\beta \longrightarrow F_x^a$$

be given by $j := T\tilde{j}$ where $\tilde{j} : \mathcal{P}_x^a \longrightarrow \mathcal{P}_x^a$ is the antipodal map. Thus

$$j(w_x, \lambda) = j(w_{-x}, \lambda) = \lambda$$

for every $(w_x, \lambda) \in T_{w_x}\mathcal{P}_x^a$ with $w_x \in \mathcal{P}_x^a$ and $\lambda \in \mathbb{R}$. Clearly, j is two-to-one. Setting $\dot{F}_x^a = F_x^a \setminus \{0\}$, the map

$$j \circ f : \bar{C}_x^a \longrightarrow \dot{F}_x^a$$

is two-to-one, turning $\bar{C}_x^a$ into a two-fold covering of $\dot{F}_x^a$. $j \circ f$ describes the correspondence between periodic lifts in $\bar{C}_x^a$ and their momenta. The symplectic group $Sp(F_x^a)$ acts transitively on F_x^a equipped with ω^a as symplectic structure. Therefore, the metaplectic group $Mp(F_x^a)$, which is the two-fold covering of $Sp(F_x^a)$, acts transitively on $T\mathcal{P}_x^a$.

Thus given $u \in F_x^a$, there is a smooth map

$$\Phi : Sp(F_x^a) \longrightarrow F_x^a$$

given by $\Phi(A) := A(u)$ for all $A \in Sp(F_x^a)$. Since $j \circ f(u_{w_x}) = j \circ f(u_{-w_x})$ for all $u_{w_x} \in T\mathcal{P}^a|_{\beta(0)}$, the map Φ lifts smoothly to

$$\tilde{\Phi} : Mp(F_x^a) \longrightarrow \bar{C}_x^a$$

such that

$$(j \circ f) \circ \tilde{\Phi} = \tilde{\mathrm{pr}} \circ \Phi$$

where $\tilde{\mathrm{pr}} : Mp(F_x^a) \longrightarrow Sp(F_x^a)$ is the covering map. Clearly, the orbit of $Mp(F_x^a)$ on $\bar{C}_x^a$ is all of $\bar{C}_x^a$, and $Mp(F_x^a)$ acts on F_x^a with a one-dimensional stabilizing group (cf. [14]). Now let us sketch the link between this observation and the quantization on $\mathbb{R}$. $Sp(F_x^a)$ operates as an automorphism group on the Heisenberg group G_x^a (leaving the centre fixed) via

$$A(z+h) = z + A(h) \qquad \forall\, z+h \in G_x^a.$$

Any $A \in Sp(F_x^a)$ determines the irreducible unitary representation ρ_A defined by

$$\rho_A(z+h) := \rho_x(z+A(h)) \qquad \forall\,(z+h) \in G_x^a.$$

Due to the Stone-von Neumann theorem it must be equivalent to ρ_x itself, meaning that there is an intertwining unitary operator U_A on $L^2(\mathbb{R}, \mathbb{C})$, determined up to a complex number of absolute value one in $\mathbb{C}_x^a$, such that $\rho_A = U_A \circ \rho \circ U_A^{-1}$ and $U_{A_1} \circ U_{A_2} = coc(A_1, A_2) \cdot U_{A_1 \circ A_2}$ for all $A_1, A_2 \in Sp(F_x^a)$. Here coc is a cocycle with value $coc(A_1, A_2) \in \mathbb{C}\backslash\{0\}$. Thus U is a projective representation of $Sp(F_x^a)$ and hence lifts to a representation W of $M_p(F_x^a)$. Since the Lie algebra of $M_p(F_x^a)$ is isomorphic to the Poisson algebra of homogenous quadratic polynomials, dW provides the quantization procedure of quadratic homogeneous polynomials on $\mathbb{R}$ and moreover describes the transport of information in $\mathcal{P}^a$ along the field line β, as described in [4].

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