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QUADRATIC SPLINE QUASI-INTERPOLANTS ON BOUNDED DOMAINS OF $\mathbb{R}^d$, $d = 1, 2, 3$

Abstract. We study some C^1 quadratic spline quasi-interpolants on bounded domains $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. These operators are of the form $Qf(x) = \sum_{k \in K(\Omega)} \mu_k(f) B_k(x)$, where $K(\Omega)$ is the set of indices of B-splines B_k whose support is included in the domain Ω and $\mu_k(f)$ is a discrete linear functional based on values of f in a neighbourhood of $x_k \in supp(B_k)$. The data points x_j are vertices of a uniform or nonuniform partition of the domain Ω where the function f is to be approximated. Beyond the simplicity of their evaluation, these operators are uniformly bounded independently of the given partition and they provide the best approximation order to smooth functions. We also give some applications to various fields in numerical approximation.

1. Introduction and notations

In this paper, we continue the study of some C^1 quadratic (or d-quadratic) spline *discrete quasi-interpolant* (dQIs) on bounded domains $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ initiated in [36]. These operators are of the form $Qf(x) = \sum_{k \in K(\Omega)} \mu_k(f) B_k(x)$, where $K(\Omega)$ is the set of indices of B-splines B_k whose support is included in the domain Ω and $\mu_k(f)$ is a *discrete linear functional* $\sum_{i \in I(r)} \lambda_k(i) f(x_{i+k})$, with $I(r) = \mathbb{Z}^d \cap [-r, r]^d$ for $r \in \mathbb{N}$ fixed (and small). The data points x_j are vertices of a uniform or nonuniform partition of the domain Ω where the function f is to be approximated. Such operators have been widely studied in recent years (see e.g. [4], [6]-[11],[14], [23], [24], [31], [38], [40]), but in general, except in the univariate or multivariate tensor-product cases, they are defined on the *whole space* $\mathbb{R}^d$: here we restrict our study to *bounded domains* and to C^1 *quadratic spline* dQIs. Their main interest lies in the fact that they provide approximants having the best approximation order and small norms while being easy to compute. They are particularly useful as initial approximants at the first step of a multiresolution analysis. First, we study univariate dQIs on uniform and non-uniform meshes of a bounded interval of the real line (Section 2) or on bounded rectangles of the plane with a uniform or non-uniform criss-cross triangulation (Section 3). We use

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quadratic B-splines whose Bernstein-Bézier (abbr. BB)-coefficients are given in technical reports [37], [38] and which extend previous results given in [12]. In the same way, in section 4, we complete the study of a bivariate blending sum of two univariate dQIs of Section 1 on a rectangular domain. Finally, in Section 5, we do the same for a trivariate blending sum of a univariate dQI (Section 1) and of the bivariate dQI described in Section 2. For blending and tensor product operators, see e.g. [2], [3], [16], [18], [19], [20], [21], [30]. For some of these operators, we improve the estimations of infinite norms which are *bounded independently of the given partition* of the domain. Using the fact that the dQI S is exact on the space $\mathbb{P}_2 \in \mathcal{S}_2$ of quadratic polynomials and a classical result of approximation theory: $\| f - Sf \| \leq (1 + \|S\|) d(f, \mathcal{S}_2)$ (see e.g. [15], chapter 5), we conclude that $f - Sf = O(h^3)$ for f smooth enough, where h is the maximum of diameters of the elements (segments, triangles, rectangles, prisms) of the partition of the domain. But we specify upper bounds for some constants occuring in inequalities giving error estimates for functions and their partial derivatives of total order at most 2. Finally, in Section 6, we present some applications of the preceding dQIs, for example to the computation of multivariate integrals, to the approximate determination of zeros of functions, to spectral-type methods and to the solution of integral equations. They are still in progress and will be published elsewhere.

2. Quadratic spline dQIs on a bounded interval

Let $X = \{x_0, x_1, \ldots, x_n\}$ be a partition of a bounded interval $I = [a, b]$, with $x_0 = a$ and $x_n = b$. For $1 \leq i \leq n$, let $h_i = x_i - x_{i-1}$ be the length of the subinterval $I_i = [x_{i-1}, x_i]$. Let $\mathcal{S}_2(X)$ be the $n + 2$-dimensional space of C^1 quadratic splines on this partition. A basis of this space is formed by quadratic B-splines $\{B_i, 0 \leq i \leq n + 1\}$. Define the set of evaluation points

$$\Theta_n = \{\theta_0 = x_0, \ \ \theta_i = \frac{1}{2}(x_{i-1} + x_i), \quad for \ \ 1 \leq i \leq n, \ \ \theta_{n+1} = x_n\}.$$

The simplest dQI associated with Θ_n is the *Schoenberg-Marsden* operator (see e.g. [25], [36]):

$$S_1 f := \sum_{i=0}^{n+1} f(\theta_i) B_i$$

This operator is exact on $\mathbb{P}_1$. Moreover $S_1 e_2 = e_2 + \sum_{i=1}^{n} \frac{1}{4} h_i^2 B_i$. We have studied in [1] and [36] the unique dQI of type

$$S_2 f = f(x_0) B_0 + \sum_{i=1}^{n} \mu_i(f) B_i + f(x_n) B_{n+1}$$

whose coefficient functionals are of the form

$$\mu_i(f) = a_i f(\theta_{i-1}) + b_i f(\theta_i) + c_i f(\theta_{i+1}), \quad 1 \leq i \leq n$$

and which is exact on the space $\mathbb{P}_2$ of quadratic polynomials. Using the following notations and the convention $h_0 = h_{n+1} = 0$, we finally obtain, for $1 \le i \le n$:

$$\sigma_i = \frac{h_i}{h_{i-1} + h_i}, \quad \sigma_i' = \frac{h_{i-1}}{h_{i-1} + h_i} = 1 - \sigma_i,$$

$$a_i = -\frac{\sigma_i^2 \sigma_{i+1}'}{\sigma_i + \sigma_{i+1}'}, \quad b_i = 1 + \sigma_i \sigma_{i+1}', \quad c_i = -\frac{\sigma_i (\sigma_{i+1}')^2}{\sigma_i + \sigma_{i+1}'}.$$

Defining the fundamental functions of S_2 by

$$\begin{aligned}
\tilde{B}_0 &= B_0 + a_1 B_1, \\
\tilde{B}_i &= c_{i-1} B_{i-1} + b_i B_i + a_{i+1} B_{i+1}, \ 1 \le i \le n, \\
\tilde{B}_{n+1} &= c_n B_n + B_{n+1},
\end{aligned}$$

we can express $S_2 f$ in the following form

$$S_2 f = \sum_{i=0}^{n+1} f(\theta_i) \tilde{B}_i.$$

In [26] (see also [22] and [32], chapter 3), Marsden proved the existence of a unique *Lagrange interpolant* Lf in $\mathcal{S}_2(X)$ satisfying $Lf(\theta_i) = f(\theta_i)$ for $0 \le i \le n+1$. He also proved the following

THEOREM 1. *For f bounded on I and for any partition X of I, the Chebyshev norm of the Lagrange operator L is uniformly bounded by* 2*.*

Now, we will prove a similar result for the dQI S_2 defined above. It is well known that the infinite norm of S_2 is equal to the Chebyshev norm of the Lebesgue function $\Lambda_2 = \sum_{i=0}^{n+1} |\tilde{B}_i|$ of S_2.

THEOREM 2. *For f bounded on I and for any partition X of I, the infinite norm of the dQI S_2 is uniformly bounded by* 2.5*.*

Proof. Each function $|\tilde{B}_i|$ being bounded above by the continuous quadratic spline $\bar{B}_i$ whose BB-coefficients are absolute values of those of $\tilde{B}_i$, we obtain $\Lambda_2 \le \bar{\Lambda}_2 = \sum_{i=0}^{n+1} \bar{B}_i$. So, we have to find an upper bound of $\bar{\Lambda}_2$. First, we need the BB-coefficients of the fundamental functions: they are computed as linear combinations of the BB-coefficients of B-splines. In order to avoid complicated notations, we denote by $[a, b, c]$ the triplet of BB-coefficients of the quadratic polynomial $a(1-u)^2 + 2bu(1-u) + cu^2$ for $u \in [0, 1]$. Any function $g \in \mathcal{S}_2(X)$ can be written in this form on each interval $[x_{i-1}, x_i]$, $1 \le i \le n$, with the change of variable $u = (x - x_{i-1})/h_i$. So, the BB-coefficients of g consist of a list of n triplets. Let us denote by $L(i)$ the list associated with the function $\tilde{B}_i$ (we do not write the triplets of null BB-coefficients). Setting, for $1 \le i \le n-1$:

$$d_i = c_i \sigma_{i+1} + b_{i+1} \sigma_{i+1}', \quad e_i = b_i \sigma_{i+1} + a_{i+1} \sigma_{i+1}',$$

we obtain for the three first functions $\tilde{B}_0$, $\tilde{B}_1$, $\tilde{B}_2$:

$$L(0) = [1, a_1, a_1\sigma_2], [a_1\sigma_2, 0, 0]$$

$$L(1) = [0, b_1, e_1], [e_1, a_2, a_2\sigma_3], [a_2\sigma_3, 0, 0]$$

$$L(2) = [0, c_1, d_1], [d_1, b_2, e_2], [e_2, a_3, a_3\sigma_4], [a_3\sigma_4, 0, 0]$$

For $3 \le i \le n-2$ (general case), we have $supp(\tilde{B}_i) = [x_{i-3}, x_{i+2}]$ and

$$L(i) = [0, 0, c_{i-1}\sigma'_{i-1}], [c_{i-1}\sigma'_{i-1}, c_{i-1}, d_{i-1}], [d_{i-1}, b_i, e_i],$$

$$[e_i, a_{i+1}, a_{i+1}\sigma_{i+2}], [a_{i+1}\sigma_{i+2}, 0, 0]$$

Finally, for the three last functions $\tilde{B}_{n-1}$, $\tilde{B}_n$, $\tilde{B}_{n+1}$, we get:

$$L(n-1) = [0, 0, c_{n-2}\sigma'_{n-2}], [c_{n-2}\sigma'_{n-2}, c_{n-2}, d_{n-2}], [d_{n-2}, b_{n-1}, e_{n-1}], [e_{n-1}, a_n, 0]$$

$$L(n) = [0, 0, c_{n-1}\sigma'_{n-1}], [c_{n-1}\sigma'_{n-1}, c_{n-1}, d_{n-1}], [d_{n-1}, b_n, 0]$$

$$L(n+1) = [0, 0, c_n\sigma'_n], [c_n\sigma'_n, c_n, 1]$$

We see that $d_i \ge 0$ (resp. $e_i \ge 0$), for it is a convex combination of c_i and b_{i+1} (resp. of b_i and a_{i+1}), with $b_i \ge 1$ and $|c_i|$ and $|a_i| \le 1$ for all i. Therefore, the absolute values of the above BB-coefficients (i.e. the BB-coefficients of the $\bar{B}_i's$) are easy to evaluate. Now, it is easy to compute the BB-coefficients of the continuous quadratic spline $\bar{\Lambda}_2 = \sum_{i=0}^{n+1} \bar{B}_i$. On each interval $[x_{i-1}, x_i]$, for $2 \le i \le n-1$, we obtain

$$[\lambda_{i-1}, \mu_i, \lambda_i] = [-a_{i-1}\sigma_i + d_{i-1} + e_{i-1} - c_i\sigma'_i, b_i - a_i - c_i, -a_i\sigma_{i+1} + d_i + e_i - c_{i+1}\sigma'_{i+1}]$$

For the first (resp. the last) interval, we have $\lambda_0 = 1$ (resp. $\lambda_n = 1$) For the central BB-coefficient, we get, since σ_i and σ'_i are in $[0, 1]$ for all indices:

$$\mu_i = b_i - (a_i + c_i) = 2b_i - 1 = 1 + 2\sigma_i\sigma'_{i+1} \le 3$$

For the extreme BB-coefficients, we have, since $a_i + b_i + c_i = 1$:

$$\lambda_i = (1 - 2a_i)\sigma_{i+1} + (1 - 2c_{i+1})\sigma'_{i+1} = 1 + \frac{2(\sigma_i)^2\sigma_{i+1}\sigma'_{i+1}}{\sigma_i + \sigma'_{i+1}} + \frac{2\sigma_{i+1}\sigma'_{i+1}(\sigma'_{i+2})^2}{\sigma_{i+1} + \sigma'_{i+2}}.$$

Let us consider the rational function f defined by $\lambda_i = 1 + f(\sigma_i, \sigma_{i+1}, \sigma_{i+2})$:

$$f(x, y, z) = \frac{2x^2y(1-y)}{1 + x - y} + \frac{2y(1-y)(1-z)^2}{1 + y - z},$$

the three variables x, y, z lying in the unit cube. Its maximum is attained at the vertices $\{(0, 1, 0), (1, 0, 0), (1, 0, 1), (1, 1, 0)\}$ and it is equal to 1. This proves that $\lambda_i \le 2$ for all i. Therefore, in each subinterval (after the canonical change of variable), $\bar{\Lambda}_2$ is bounded above by the parabola:

$$\pi_2(u) = 2(1-u)^2 + 6u(1-u) + 2u^2$$

whose maximum value is $\pi_2(\frac{1}{2}) = \frac{5}{2} = 2.5$.

$\square$

Now, we consider the case of a *uniform partition*, say with integer nodes for simplification (e.g. $I = [0, n]$, $X = \{0, 1, \ldots, n\}$). In that case, we have

$$\sigma_1 = 1, \quad \sigma_1' = 0; \quad \sigma_i = \sigma_i' = \frac{1}{2} \quad for \quad 2 \le i \le n; \quad \sigma_{n+1} = 0, \quad \sigma_{n+1}' = 1,$$

from which we deduce:

$$a_1 = c_n = -\frac{1}{3}, \quad b_1 = b_n = \frac{3}{2}, \quad c_1 = a_n = -\frac{1}{6},$$

and, for $2 \le i \le n-1$:

$$a_i = c_i = -\frac{1}{8}, \quad b_i = \frac{5}{4}.$$

It is easy to see that, in order to compute $\|S_2\|_\infty$, it suffices to evaluate the maximum of the Lebesgue function on the subinterval $J = [0, 4]$. Here are the lists $L(i)$ of the BB-coefficients of the fundamental functions $\{\tilde{B}_i, 0 \le i \le 6\}$ whose supports have at least a common subinterval with J. As in the nonuniform case, we only give the triplets associated with subintervals of $supp(\tilde{B}_i) \cap J$:

$$supp(\tilde{B}_0) \cap J = [0, 2], \quad L(0) = \left[1, -\frac{1}{3}, -\frac{1}{6}\right], \left[-\frac{1}{6}, 0, 0\right]$$

$$supp(\tilde{B}_1) \cap J = [0, 3], \quad L(1) = \left[0, \frac{3}{2}, \frac{11}{16}\right], \left[\frac{11}{16}, -\frac{1}{8}, -\frac{1}{16}\right], \left[-\frac{1}{16}, 0, 0\right],$$

$$supp(\tilde{B}_2) \cap J = [0, 4], \quad L(2) = \left[0, -\frac{1}{6}, \frac{13}{24}\right], \left[\frac{13}{24}, \frac{5}{4}, \frac{9}{16}\right], \left[\frac{9}{16}, -\frac{1}{8}, -\frac{1}{16}\right], \left[-\frac{1}{16}, 0, 0\right],$$

$$supp(\tilde{B}_3) \cap J = [0, 2], \quad L(3) = \left[0, 0, -\frac{1}{16}\right], \left[-\frac{1}{16}, -\frac{1}{8}, \frac{9}{16}\right], \left[\frac{9}{16}, \frac{5}{4}, \frac{9}{16}\right], \left[\frac{9}{16}, -\frac{1}{8}, -\frac{1}{16}\right],$$

$$supp(\tilde{B}_4) \cap J = [1, 4], \quad L(4) = \left[0, 0, -\frac{1}{16}\right], \left[-\frac{1}{16}, -\frac{1}{8}, \frac{9}{16}\right], \left[\frac{9}{16}, \frac{5}{4}, \frac{9}{16}\right],$$

$$supp(\tilde{B}_5) \cap J = [2, 4], \quad L(5) = \left[0, 0, -\frac{1}{16}\right], \left[-\frac{1}{16}, -\frac{1}{8}, \frac{9}{16}\right],$$

$$supp(\tilde{B}_6) \cap J = [3, 4], \quad L(6) = \left[0, 0, -\frac{1}{16}\right],$$

Drawing Λ_2 reveals that the abscissa $\bar{x}$ of its maximum lies in the interval $[0.6, 1]$. In this interval, we obtain successively:

$$\Lambda_2(x) = -\tilde{B}_0(x) + \tilde{B}_1(x) + \tilde{B}_2(x) - \tilde{B}_3(x) = -(1-x)^2 + \frac{10}{3}x(1-x) + \frac{35}{24}x^2$$

whence $\Lambda_2'(x) = \frac{1}{12}(64 - 69x)$ and $\bar{x} = \frac{64}{69}$. This leads to

$$\|S_2\|_\infty = \|\Lambda_2\|_\infty = \Lambda_2(\bar{x}) = \frac{305}{207} \approx 1.4734.$$

□

So, we have proved the following result:

THEOREM 3. *For uniform partitions of the interval I, the infinite norm of S_2 is equal to $\frac{305}{207} \approx 1.4734$.*

REMARK 1. Further results on various types of dQIs will be given in [21].

Now, we will give some *bounds for the error* $f - S_2 f$. Using the fact that the dQI S_2 is exact on the subspace $\mathbb{P}_2 \subset \mathcal{S}_2$ of quadratic polynomials and a classical result of approximation theory (see e.g. [17], chapter 5), we have for all partitions X of I in virtue of Theorem 4:

$$\| f - S_2 f \|_\infty \leq (1 + \|S_2\|_\infty)\, dist(f, \mathcal{S}_2)_\infty \leq 3.5\, dist(f, \mathcal{S}_2)_\infty$$

So, the approximation order is that of the best quadratic spline approximation. For example, from [17], we know that for any continuous function f

$$dist(f, \mathcal{S}_2)_\infty \leq 3\, \omega(f, h)_\infty$$

where $h = max\{h_i, 1 \leq i \leq n\}$, so we obtain

$$\| f - S_2 f \|_\infty \leq 10.5\, \omega(f, h)_\infty$$

But a direct study allows to decrease the constant in the right-hand side.

THEOREM 4. *For a continuous function f, there holds:*

$$\| f - S_2 f \|_\infty \leq 6\, \omega(f, h)_\infty$$

Proof. For any $x \in I$, we have

$$f(x) - S_2 f(x) = \sum_{i=0}^{n+1} [f(x) - f(\theta_i)]\tilde{B}_i(x)$$

Assuming $n \geq 5$ and $x \in I_p = [x_{p-1}, x_p]$, for some $3 \leq p \leq n-2$, this error can be written, since $supp(\tilde{B}_i) = [x_{i-3}, x_{i+2}]$:

$$f(x) - S_2 f(x) = \sum_{i=p-2}^{p+2} [f(x) - f(\theta_i)]\tilde{B}_i(x).$$

As $\theta_i = \frac{1}{2}(x_{i-1} + x_i)$, we have $|x - \theta_i| \le r_i h$, with $r_i = |p - i| + 0.5$. Using a well known property of the modulus of continuity of f, $\omega(f, r_i h) \le (1 + r_i)\omega(f, h)$, we deduce

$$|f(x) - S_2 f(x)| \le \left[\sum_{i=p-2}^{p+2} (1 + r_i)\bar{B}_i(x)\right] \omega(f, h).$$

Without going into details, we use the local BB-coefficients of $\bar{B}_i$, $p - 2 \le i \le p + 2$ in the subinterval $[x_{p-1}, x_p]$, and we can prove that for all partitions of I, we have

$$\sum_{i=p-2}^{p+2} (1.5 + |p - i|)\bar{B}_i(x) \le 6$$

so, we obtain finally a lower constant (but not the best one) in the right-hand side of the previous inequality:

$$\| f - S_2 f \|_\infty \le 6\, \omega(f, h)_\infty$$

□

Now, let us assume that $f \in C^3(I)$, then we have the following

THEOREM 5. *For all function $f \in C^3(I)$ and for all partitions X of I, the following error estimate holds, with $C_0 \le 1$:*

$$\| f - S_2 f \|_\infty \le C_0 h^3 \| f^{(3)} \|_\infty$$

Proof. Given $x \in I_p$ fixed and $t \in [x_{p-3}, x_{p+2}]$, we use the Taylor formula with integral remainder

$$f(t) = f(x) + (t - x) f'(x) + \frac{1}{2}(t - x)^2 f''(x) + \frac{1}{2}\int_x^t (t - s)^2 f^{(3)}(s) ds$$

As $p_1(t) = t - x$ and $p_2(t) = (t - x)^2$ are in $\mathbb{P}_2$, we have $S_2 p_1 = p_1$ and $S_2 p_2 = p_2$, which can be written explicitly as

$$S_2 p_1(t) = t - x = \sum_{i=0}^{n+1} (\theta_i - x)\tilde{B}_i(t), \quad S_2 p_2(t) = (t - x)^2 = \sum_{i=0}^{n+1} (\theta_i - x)^2 \tilde{B}_i(t)$$

and this proves that $S_2 p_1(x) = S_2 p_2(x) = 0$. Therefore it remains:

$$S_2 f(x) - f(x) = \frac{1}{2} \sum_{i=p-2}^{p+2} \left[\int_x^{\theta_i} (\theta_i - s)^2 f^{(3)}(s) ds\right] \tilde{B}_i(x)$$

As $|\int_x^{\theta_i}(\theta_i - s)^2 ds| \le \frac{1}{3}|x-\theta_i|^3$, we get the following upper bound:

$$\begin{aligned} |S_2 f(x) - f(x)| &\le \frac{1}{6}\| f^{(3)}\|_\infty \sum_{i=p-2}^{p+2} |x-\theta_i|^3 \bar{B}_i(x) \\ &\le \frac{h^3}{6}\| f^{(3)}\|_\infty \sum_{i=p-2}^{p+2} (|p-i| + \frac{1}{2})^3 \bar{B}_i(x) \end{aligned}$$

As in the proof of theorem above, and without going into details, one can prove that the last sum in the r.h.s. is uniformly bounded by 6 for any partition of I. So, we obtain finally:

$$|S_2 f(x) - f(x)| \le h^3 \| f^{(3)}\|_\infty$$

□

By using the same techniques, the results of theorem 5 can be improved when X is a *uniform partition* of I:

THEOREM 6. *(i) For $f \in C(I)$, there holds:*

$$|S_2 f(x) - f(x)| \le 2.75\, \omega(f, \frac{h}{2})_\infty$$

(ii) for $f \in C^3(I)$ and for all $x \in I$ there holds:

$$|S_2 f(x) - f(x)| \le \frac{h^3}{3}\| f^{(3)}\|_\infty$$

$$|(S_2 f)'(x) - f'(x)| \le 1.2\, h^2 \| f^{(3)}\|_\infty$$

and locally, in each subinterval of I:

$$|(S_2 f)''(x) - f''(x)| \le 2.4\, h \| f^{(3)}\|_\infty$$

3. Quadratic spline dQIs on a bounded rectangle

In this section, we study some C^1 quadratic spline dQIs on a nonuniform criss-cross triangulation of a rectangular domain. More specifically, let $\Omega = [a_1, b_1] \times [a_2, b_2]$ be a rectangle decomposed into mn subrectangles by the two partitions

$$X_m = \{x_i, \ \ 0 \le i \le m\}, \ \ Y_n = \{y_j, \ \ 0 \le j \le n\}$$

respectively of the segments $I = [a_1, b_1] = [x_0, x_m]$ and $J = [a_2, b_2] = [y_0, y_n]$. For $1 \le i \le m$ and $1 \le j \le n$, we set $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$, $I_i = [x_{i-1}, x_i]$, $J_j = [y_{j-1}, y_j]$, $s_i = \frac{1}{2}(x_{i-1} + x_i)$ and $t_j = \frac{1}{2}(y_{j-1} + y_j)$. Moreover

$s_0 = x_0, s_{m+1} = x_m, t_0 = y_0, t_{n+1} = y_n$. In this section and the next one, we use the following notations:

$$\sigma_i = \frac{h_i}{h_{i-1}+h_i}, \quad \sigma_i' = \frac{h_{i-1}}{h_{i-1}+h_i} = 1-\sigma_i,$$

$$\tau_j = \frac{k_j}{k_{j-1}+k_j}, \quad \tau_j' = \frac{k_{j-1}}{k_{j-1}+k_j} = 1-\tau_j,$$

for $1 \le i \le m$ and $1 \le j \le n$, with the convention $h_0 = h_{m+1} = k_0 = k_{n+1} = 0$.

$$a_i = -\frac{\sigma_i^2\sigma_{i+1}'}{\sigma_i+\sigma_{i+1}'}, \quad b_i = 1+\sigma_i\sigma_{i+1}', \quad c_i = -\frac{\sigma_i(\sigma_{i+1}')^2}{\sigma_i+\sigma_{i+1}'},$$

$$\bar{a}_j = \frac{\tau_j^2\tau_{j+1}'}{\tau_j+\tau_{j+1}'}, \quad \bar{b}_j = 1+\tau_j\tau_{j+1}', \quad \bar{c}_j = -\frac{\tau_j(\tau_{j+1}')^2}{\tau_j+\tau_{j+1}'}.$$

for $0 \le i \le m+1$ and $0 \le j \le n+1$. Let $\mathcal{K}_{mn} = \{(i,j) : 0 \le i \le m+1, \quad 0 \le j \le n+1\}$, then the data sites are the mn intersection points of diagonals in subrectangles $\Omega_{ij} = I_i \times J_j$, the $2(m+n)$ midpoints of the subintervals on the four edges, and the four vertices of Ω, i.e. the $(m+2)(n+2)$ points of the following set

$$\mathcal{D}_{mn} := \{M_{ij} = (s_i, t_j), \quad (i,j) \in \mathcal{K}_{mn}\}.$$

As in Section 2, the simplest dQI is the bivariate Schoenberg-Marsden operator:

$$S_1 f = \sum_{(i,j)\in\mathcal{K}_{mn}} f(M_{ij}) B_{ij}$$

where

$$\mathcal{B}_{mn} := \{B_{ij}, 0 \le i \le m+1, \quad 0 \le j \le n+1\}$$

is the collection of $(m+2)(n+2)$ B-splines (or generalized box-splines) generating the space $\mathcal{S}_2(\mathcal{T}_{mn})$ of all C^1 piecewise quadratic functions on the criss-cross triangulation $\mathcal{T}_{mn}$ associated with the partition $X_m \times Y_n$ of the domain Ω (see e.g. [14], [13]). There are mn *inner B-splines* associated with the set of indices

$$\hat{\mathcal{K}}_{mn} = \{(i,j), 1 \le i \le m, 1 \le j \le n\}$$

whose restrictions to the boundary Γ of Ω are equal to zero. To the latter, we add $2m+2n+4$ *boundary B-splines* whose restrictions to Γ are univariate quadratic B-splines. Their set of indices is

$$\tilde{\mathcal{K}}_{mn} := \{(i,0),(i,n+1), 0 \le i \le m+1; \ (0,j),(m+1,j), \quad 0 \le j \le n+1\}$$

The BB-coefficients of inner B-splines whose indices are in $\{(i,j), 2 \le i \le m-1, 2 \le j \le n-1\}$ are given in [32]. The other ones can be found in the technical reports [37] (uniform partition) and [38](non-uniform partitions). The B-splines are positive and form a partition of unity (blending system). The boundary B-splines are *linearly independent* as the univariate ones. But the inner B-splines are *linearly dependent*, the dependence relationship being:

$$\sum_{(i,j)\in\hat{\mathcal{K}}_{mn}} (-1)^{i+j} h_i k_j B_{ij} = 0$$

It is well known that S_1 is exact on bilinear polynomials, i.e.

$$S_1 e_{rs} = e_{rs} \quad for \quad 0 \le r, s \le 1$$

In [36], we obtained the following dQI, which is exact on $\mathbb{P}_2$:

$$S_2 f = \sum_{(i,j)\in\mathcal{K}_{mn}} \mu_{ij}(f) B_{ij}$$

where the coefficient functionals are given by

$$\begin{aligned} \mu_{ij}(f) &= (b_i + \bar{b}_j - 1) f(M_{ij}) + a_i f(M_{i-1,j}) + c_i f(M_{i+1,j}) \\ &+ \bar{a}_j f(M_{i,j-1}) + \bar{c}_j f(M_{i,j+1}). \end{aligned}$$

As in Section 2, we introduce the fundamental functions:

$$\tilde{B}_{ij} = (b_i + \bar{b}_j - 1) B_{ij} + a_{i+1} B_{i+1,j} + c_{i-1} B_{i-1,j} + \bar{a}_{j+1} B_{i,j+1} + \bar{c}_{j-1} B_{i,j-1}.$$

We also proved the following theorems, by bounding above the Lebesgue function of S_2:

$$\Lambda_2 = \sum_{(i,j)\in\mathcal{K}_{mn}} |\tilde{B}_{ij}|$$

THEOREM 7. *The infinite norm of S_2 is uniformly bounded independently of the partition $\mathcal{T}_{mn}$ of the domain:*

$$\|S_2\|_\infty \le 5$$

THEOREM 8. *For uniform partitions, we have the following bound:*

$$\|S_2\|_\infty \le 2.4$$

These bounds are probably not optimal and can still be slightly reduced.

4. A biquadratic blending sum of univariate dQIs

In this section, we study a biquadratic dQI on a rectangular domain $\Omega = [a_1, b_1] \times [a_2, b_2]$ which is a blending sum of bivariate extensions of quadratic spline dQIs of Section 2. We use the same notations as in Section 2 for the domain Ω, the partitions of $I = [a_1, b_1]$, $J = [a_2, b_2]$ and data sites. The partition considered on Ω is the tensor product of partitions of I and J. We use the two sets of univariate B-splines

$$\{B_i(x), 0 \le i \le m+1\}, \quad \{B_j(y), 0 \le j \le n+1\}$$

and the two sets of univariate fundamental functions introduced in Section 2:

$$\{\tilde{B}_i(x), 0 \le i \le m+1\}, \quad \{\tilde{B}_j(y), 0 \le j \le n+1\}$$

The associated *extended bivariate* dQIs are respectively (see e.g. [14] for bivariate extensions of univariate operators)

$$P_1 f(x,y) := \sum_{i=0}^{m+1} f(s_i, y) B_i(x), \quad P_2 f(x,y) := \sum_{i=0}^{m+1} f(s_i, y) \tilde{B}_i(x)$$

$$Q_1 f(x,y) := \sum_{j=0}^{n+1} f(x, t_j) B_j(y), \quad Q_2 f(x,y) := \sum_{j=0}^{n+1} f(x, t_j) \tilde{B}_j(y)$$

The bivariate dQI considered in this section is now defined as the blending sum

$$R := P_1 Q_2 + P_2 Q_1 - P_1 Q_1$$

and it can be written in the following form

$$Rf(x,y) = \sum_{(i,j)\in\mathcal{K}_{mn}} f(M_{ij}) \bar{B}_{ij}(x,y)$$

where the biquadratic fundamental functions are defined by

$$B^{\flat}_{ij}(x,y) := B_i(x)\tilde{B}_j(y) + \tilde{B}_i(x)B_j(y) - B_i(x)B_j(y)$$

In terms of tensor-product B-splines $B_{ij}(x,y) = B_i(x)B_j(y)$, we have:

$$Rf(x,y) = \sum_{(i,j)\in\mathcal{K}_{mn}} \mu_{ij}(f) B_{ij}(x,y),$$

where the coefficient functionals are given by

$$\begin{aligned} \mu_{ij}(f) \quad := \quad & a_i f(M_{i-1,j}) + c_i f(M_{i+1,j}) + \bar{a}_j f(M_{i,j-1}) \\ + \quad & \bar{c}_j f(M_{i,j+1}) + (b_i + \bar{b}_j - 1) f(M_{ij}) \end{aligned}$$

We have proved in [36] the following

THEOREM 9. *The operator R is exact on the 8-dimensional subspace* $(\mathbb{P}_{12}[x,y]) \oplus (\mathbb{P}_{21}[x,y])$ *of biquadratic polynomials. Moreover, its infinite norm is bounded above independently of the nonuniform partition $X_m \otimes Y_n$ of the domain Ω*

$$\|R\|_\infty \le 5$$

5. A trivariate blending sum of univariate and bivariate quadratic dQIs

In this section, we study a trivariate dQI on a parallelepiped $\Omega = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ which is a blending sum of trivariate extensions of univariate and bivariate dQIs seen in Sections 2 and 3. We consider the three partitions

$$X_m := \{x_i, \quad 0 \le i \le m\}, \quad Y_n = \{y_j, \quad 0 \le j \le n\}, \quad Z_p := \{z_k, \quad 0 \le k \le p\}$$

respectively of the segments $I = [a_1, b_1] = [x_0, x_m]$, $J = [a_2, b_2] = [y_0, y_n]$ and $K = [a_3, b_3] = [z_0, z_p]$. For the projection $\Omega' = [a_1, b_1] \times [a_2, b_2]$ of Ω on the $xy - plane$, the notations are those of Section 3. For the projection $\Omega'' = [a_3, b_3]$ of Ω on the $z - axis$, we use the following notations, for $1 \le k \le p$:

$$l_k = z_k - z_{k-1}, \quad K_k = [z_{k-1}, z_k], \quad u_k = \frac{1}{2}(z_{k-1} + z_k),$$

with $u_0 = z_0$ and $u_{p+1} = z_p$. For mesh ratios of subintervals, we set respectively

$$\omega_k = \frac{l_k}{l_{k-1} + l_k}, \quad \omega'_k = \frac{l_{k-1}}{l_{k-1} + l_k} = 1 - \omega_k$$

for $1 \le k \le p$, with $l_0 = l_{p+1} = 0$ (all these ratios lie between 0 and 1), and

$$\hat{a}_k = -\frac{\omega_k^2 \omega'_{k+1}}{\omega_k + \omega'_{k+1}}, \quad \hat{b}_k = 1 + \omega_k \omega'_{k+1}, \quad \hat{c}_k = -\frac{\omega_k (\omega'_{k+1})^2}{\omega_k + \omega'_{k+1}.}$$

Let $\mathcal{K} = \mathcal{K}_{mnp} = \{(i, j, k), \quad 0 \le i \le m+1, \ 0 \le j \le n+1, \ 0 \le k \le p+1\}$, then the set of data sites is

$$\mathcal{D} = \mathcal{D}_{mnp} = \{N_{ijk} = (x_i, y_j, z_k), (i, j, k) \in \mathcal{K}_{mnp}\},$$

The partition of Ω considered here is the tensor product of partitions on Ω' and Ω'', i.e. a partition into *vertical prisms with triangular horizontal sections.* Setting $\mathcal{K}'_{mn} = \{(i, j), 0 \le i \le m+1, \ 0 \le j \le n+1\}$, we consider the bivariate B-splines and fundamental splines on $\Omega' = [a_1, b_1] \times [a_2, b_2]$ defined in Section 3 above:

$$\{B_{ij}(x, y), \quad (i, j) \in \mathcal{K}'_{mn}\}, and \quad \{\tilde{B}_{ij}(x, y), \quad (i, j) \in \mathcal{K}'_{mn}\}$$

and the univariate B-splines and fundamental splines on $[a_3, b_3]$ defined in Section 2:

$$\{B_k(z), \quad 0 \le k \le p+1\} \ \ and \ \ \{\tilde{B}_k(z), \quad 0 \le k \le p+1\}.$$

The extended trivariate dQIs that we need for the construction are the following

$$P_1 f(x, y, z) := \sum_{(i,j) \in \mathcal{K}'_{mn}} f(s_i, t_j, z) B_{ij}(x, y),$$

$$P_2 f(x, y, z) := \sum_{(i,j) \in \mathcal{K}'_{mn}} f(s_i, t_j, z) \tilde{B}_{ij}(x, y),$$

$$Q_1 f(x,y,z) := \sum_{k=0}^{p+1} f(x,y,u_k) B_k(z), \quad Q_2 f(x,y,z) := \sum_{k=0}^{p+1} f(x,y,u_k) \tilde{B}_k(z).$$

For the sake of clarity, we give the expressions of P_2 and Q_2 in terms of B-splines:

$$P_2 f(x,y,z) = \sum_{(i,j)\in\mathcal{K}'_{mn}} \mu_{ij}(f) B_{ij}(x,y)$$

$$\mu_{ij}(f) = a_i f(s_{i-1}, t_j, z) + c_i f(s_{i+1}, t_j, z)) + \bar{a}_j f(s_i, t_{j-1}, z) + \bar{c}_j f(s_i, t_{j+1}, z)$$
$$+(b_i + \bar{b}_j - 1) f(s_i, t_j, z)$$

$$Q_2 f(x,y,z) := \sum_{k=0}^{p+1} \{\hat{a}_k f(x,y,u_{k-1}) + \hat{b}_k f(x,y,u_k) + \hat{c}_k f(x,y,u_{k+1})\} B_k(z)$$

We now define the trivariate blending sum

$$R = P_1 Q_2 + P_2 Q_1 - P_1 Q_1$$

Setting

$$B^\flat_{ijk}(x,y,z) = B_{ij}(x,y)\tilde{B}_k(z) + \tilde{B}_{ij}(x,y) B_k(z) - B_{ij}(x,y) B_k(z)$$

we obtain

$$Rf = \sum_{(i,j,k)\in\mathcal{K}_{mnp}} f(N_{ijk}) B^\flat_{ijk}$$

In terms of tensor product B-splines $B_{ijk} = B_{ij} B_k$, one has

$$Rf = \sum_{(i,j,k)\in\mathcal{K}_{mnp}} \nu_{ijk}(f) B_{ijk}$$

where $\nu_{ijk}(f)$ is based on the 7 neighbours of N_{ijk} in $\mathbb{R}^3$:

$$\begin{aligned}\nu_{ijk}(f) \quad = \quad & \hat{a}_k f(N_{i,j,k-1}) + \hat{c}_k f(N_{i,j,k+1}) + a_i f(N_{i-1,j,k}) + c_i f(N_{i+1,j,k}) \\ & + \bar{a}_j f(N_{i,j-1,k}) + \bar{c}_j f(N_{i,j+1,k}) + (b_i + \bar{b}_j + \hat{c}_k - 1) f(N_{ijk}).\end{aligned}$$

In [36], we proved the following

THEOREM 10. *The operator R is exact on the 15-dimensional subspace $(\mathbb{P}_1[x,y]\otimes\mathbb{P}_2[z]) \oplus (\mathbb{P}_2[x,y]\otimes\mathbb{P}_1[z])$ of the 18-dimensional space $\mathbb{P}_2[x,y]\otimes\mathbb{P}_2[z]$. Moreover, its infinite norm is bounded above independently of the nonuniform partition of the domain Ω*

$$\|R\|_\infty \le 8.$$

6. Some applications

We present some applications of the preceding sections. For sake of simplicity, we give results for uniform partitions only. Let Q be any of the previous dQIs.
1) *Approximate integration.* Approximating $\int_\Omega f$ by $\int_\Omega Qf$ gives rise to several interesting quadrature formulas (abbr. QF) in $\mathbb{R}^d$, mainly for $d = 2, 3$. For $d = 1$ and for a uniform partition of I with meshlength h, we obtain the QF:

$$QF_n(f) = \int_a^b S_2 f = h(\frac{1}{9} f_0 + \frac{7}{8} f_1 + \frac{73}{72} f_2 + \sum_{i=3}^{n-2} f_i + \frac{73}{72} f_{n-1} + \frac{7}{8} f_n + \frac{1}{9} f_{n+1}),$$

where $f_i = f(\theta_i)$ for $0 \le i \le n+1$. This formula is exact for $\mathbb{P}_3$, like composite Simpson's formula, i.e. $QF_n(f) = \int_a^b f$ for all $f \in \mathbb{P}_3$. Therefore $\int_a^b f - QF_n(f) = O(h^4)$ for functions $f \in C^4(I)$. Numerical experiments show that it is better than Simpson's formula based on $n+1$ points (n even). Moreover, the errors associated with the two QFs have often opposite signs, thus giving upper and lower values of the exact integral.
2) *Approximate differentiation: pseudo-spectral methods.* One can approximate the first (partial) derivatives of f by those of Qf at the data sites. We thus obtain differentiation matrices which can be also used for second derivatives and for pseudo-spectral methods. Let us give an example for $d = 1$ and for a uniform partition of meshlength h of the interval I. Denoting $g = S_2 f$, then we get:

$$\begin{aligned}
g'(\theta_0) &= \frac{1}{h}\left(-\frac{8}{3} f_0 + 3 f_1 - \frac{1}{3} f_2\right),\\
g'(\theta_1) &= \frac{1}{h}\left(-\frac{7}{6} f_0 + \frac{11}{16} f_1 + \frac{13}{24} f_2 - \frac{1}{16} f_3\right),\\
g'(\theta_2) &= \frac{1}{h}\left(\frac{1}{6} f_0 - \frac{3}{4} f_1 + \frac{1}{48} f_2 + \frac{5}{8} f - 3 - \frac{1}{16} f_4\right),\\
g'(\theta_{n-1}) &= \frac{1}{h}\left(\frac{1}{16} f_{n-3} - \frac{5}{8} f_{n-2} - \frac{1}{48} f_{n-1} + \frac{3}{4} f_n - \frac{1}{6} f_{n+1}\right),\\
g'(\theta_n) &= \frac{1}{h}\left(\frac{1}{16} f_{n-2} - \frac{13}{24} f_{n-1} - \frac{11}{16} f_n + \frac{7}{6} f_{n+1}\right),\\
g'(\theta_{n+1}) &= \frac{1}{h}\left(\frac{1}{3} f_{n-1} - 3 f_n + \frac{8}{3} f_{n+1}\right),
\end{aligned}$$

and for $3 \le i \le n-2$:

$$g'(\theta_i) = \frac{1}{h}\left(\frac{1}{16} f_{i-2} - \frac{5}{8} f_{i-1} + \frac{5}{8} f_{i+1} - \frac{1}{16} f_{i+2}\right).$$

3) *Approximation of zeros of polynomials.* We have tested the approximation of the *Legendre polynomial* $f(x) = P_8(x)$ and of its zeros in the interval $I = [-1, 1]$ by the dQI $S_2 f$ of Section 1 based on Chebyshev points with $n = 32$. We obtain

$\| f - S_2 f \|_\infty \approx 0.0034$. There is practically no difference between the approximation $S_2 f$ and the Marsden interpolant of f which needs the solution of a linear system of $n+2$ equations. We obtain also quite good approximations of the eight roots of f in the interval. For bivariate or trivariate functions, the advantage of using dQIs over interpolants is still bigger since one avoids the solution of large linear systems. Moreover, at least in the bivariate case, one can use the nice properties of piecewise quadratic surfaces (see e.g. the results given by M.J.D. Powell in [29]).
4) *Integral equations* The dQIs can be used for various types of approximation of the solution of Fredholm type integral equation with a regular or a weakly singular kernel. This work is still in progress (see e.g. [15]).

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