Hyperbolic Equations in the Twentieth Century

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Abstract

The subject began with Huygens's theory of wave fronts as envelopes of smoother waves, and subsequent work by Euler, d'Alembert and Riemann. Singularities at the wave fronts were not understood before Hadamard's theory of "partie finie" at the beginning of this century. Contributions by Herglotz and Petrovsky and the theory of distributions created in the forties by Laurent Schwartz greatly illuminated the study of singularities of solutions of hyperbolic PDE's. Solutions of Cauchy's problem given by Hadamard, Schauder, Petrovsky, and the author are discussed. More recently, microlocal analysis, initiated by M. Sato and L. Hörmander led to important advances in understanding the propagation of singularities. Functional analysis together with distributions and microlocal analysis are expected to be useful well into the next century.

Résumé

Le sujet débute avec la théorie de Huygens qui considère les fronts d'onde comme des enveloppes d'ondes plus régulières, et se poursuit par les travaux de Euler, d'Alembert et Riemann. Les singularités des fronts d'onde n'ont pas été comprises avant la théorie de la « partie finie » de Hadamard au début de ce siècle. Les contributions de Herglotz, Petrovsky et dans les années quarante, la théorie des distributions de Laurent Schwartz ont éclairé l'étude des singularités des solutions des EDP hyperboliques. On passe en revue les solutions au problème de Cauchy

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données par Hadamard, Schauder, Petrovsky et l'auteur. Plus récemment, l'analyse microlocale de M. Sato et L. Hörmander a permis de grandes avancées dans la compréhension de la propagation des singularités. L'analyse fonctionnelle, les distributions et l'analyse microlocale seront certainement des outils importants du prochain siècle.

1. Introduction

The first example of a hyperbolic equation was the wave equation

$$u_{tt} - \Delta u = 0.$$

In one space variable n, the solutions describe free movements with velocity 1 in a perfectly elastic medium. A nonlinear version appears in one dimensional hydrodynamics. Riemann's 1860 treatment was later completed by the Rankine-Hugoniot jump conditions and conditions of entropy. Further examples of hyperbolic equations and systems appeared in the theory of electricity and magnetism and elasticity.

Originally, the adjective hyperbolic marked the connection between the wave equation and a hyperbolic conoid. When applied to general partial differential operators or systems the term now indicates that one of the variables is time $t = t(x)$ and that the solutions of the system describe wave propagation with finite velocity in all directions. More precisely, the solution u of Cauchy's problem with no source function and with data given for $t =$ const. should have the property that the value of u at a point depends continuously on the values of the data and their derivatives in a compact set. For an operator $P(D)$ with constant coefficients this means that there is a fundamental solution $E(x)$, i.e. a distribution such that $P(D)E(x) = \delta(x)$, whose support is contained in a proper, closed cone.

In the first half of the twentieth century, local existence by classical analysis of solutions to Cauchy's problem for hyperbolic equations with smooth data was the main problem. Soon after, functional analysis and distributions came into play and the introduction around 1970 of pseudodifferential operators and microlocal analysis of distributions was followed by a period of important results on the propagation of singularities, both free and under reflection in a boundary. Later this study was extended to nonlinear equations. Another question, latent during the period, is the problem of global existence of solutions for nonlinear equations close to linear ones. It took a new turn with the study of blow-up times by Fritz John.

Only a sample of the main results can be mentioned here. In particular, I refrain from the various hyperbolic aspects of hydrodynamics and the theory of scattering in spectral analysis.

The development of the theory of hyperbolic equations from 1900 cannot be understood without a review of some of the main results from the time before 1900. It is done here briefly under the heading of Prehistory.[1]

2. Prehistory

With three space variables the wave equation describes free propagation of light in physical space with velocity 1. For this equation, Poisson proved what in modern terms amounts to the fact that the wave operator $\Box = \partial_t^2 - \Delta$ has a fundamental solution

$$E(t,x) = \frac{1}{2\pi} H(t)\delta(t^2 - |x|^2)$$

with support on the forward lightcone $t = |x|$. It was then only too easy to believe this to be a general phenomenon, for instance that the equations for the propagation of light in media with double refraction follow the same rule known under the name of Huygens principle:[2] *all* light from a point-source is concentrated to the surface given by the rules of geometric optics. Both G. Lamé and Sonya Kovalevski made this mistake till the use of Fourier analysis proved that the existence of diffuse light outside such surfaces is the rule and the contrary an exception (for a historical review, see [Gårding 1989]).

A fundamental solution of the wave operator for two space variables was found by Volterra and, at the turn of the century, Tedone tried the general case, but could only construct what amounts to sufficiently repeated integrals with respect to time of purported fundamental solutions. Behind these difficulties is the fact that, in contrast to the properties of Laplace's operator, the fundamental solutions of the wave operator are distributions with singularities outside the pole which get worse as the number n of space variables increases. Before the theory of distributions, this was a formidable difficulty.

3. "Partie finie"

The obstacle which stopped Tedone, was surmounted by Hadamard in his theory of *partie finie*, found before 1920 and exposed in [Hadamard 1932].

[1] The remarks and notes of Hadamard's book 1932give a fuller account.

[2] Huygens's minor premise according to Hadamard [1932].

His operator is the wave operator with smooth, variable coefficients and has the form

$$L(x, \partial_x) = \sum a_{jk}(x)\partial_j\partial_k + \text{lower terms} \tag{3.1}$$

where the metric form $\sum a_{jk}\xi_j\xi_k$ has Lorentz signature $+, -...-$. A direction for which the inverse metric form is positive, zero or negative is said to be time-like, light-like and space-like respectively. Surfaces with time-like and space-like normals are said to be space-like and time-like respectively. The light rays are the geodesics of length zero. A time function $t(x)$ with $t'(x)$ time-like is given.

The light rays with a positive time direction issued from a point y constitute the forward light cone C_y with its vertex at y. Inside this light cone, the fundamental solution with its pole at y has the same form as in the elliptic case

$$f(x, y)d(x, y)^{2-n} \tag{3.2}$$

where f is a smooth function and d is the geodesic distance between x and y. The difficulty is that $d(x, y) = 0$ when $x \in C_y$. The *partie finie* can be said to be a renormalization procedure which extends this formula for n odd to a distribution which is also a fundamental solution. For n even, Hadamard uses what is called the method of descent. In the work by M. Riesz [1949] the exponent $2 - n$ of (3.2) is replaced by $\alpha - n$ where α is a complex paramater. At the same time f is made to depend on α and a denominator $\Gamma(\alpha/2)\Gamma((\alpha + 2 - n)/2)$ is introduced. The stage is then set for an analytical continuation with respect to α. In this way and for selfadjoint operators L, Riesz constructs kernels of the complex powers of L.

In his case, Hadamard could give a complete local solution of Cauchy's problem with data on a space-like surface, but the corresponding mixed problem with reflection in a time-like surface presented insurmountable difficulties.

4. Friedrichs-Lewy energy density, existence proofs by Schauder and Petrovsky

The discovery of Friedrichs and Lewy [1928] that $\partial_1 u \Box u$ with u real is the divergence of a tensor with a positive energy density on space-like surfaces produced both uniqueness results and *a priori* energy estimates, decisive for the later development.

A great step forward was taken by Schauder [1935, 1936a,b] who proved local existence of solutions of Cauchy's problem and also the mixed problem for quasilinear wave operators. The method is to use approximations starting from the case of analytic coefficients and analytic data. The success of these papers depends on stable energy estimates derived from the energy tensor and the use of the fact that square integrable functions with square integrable derivatives up to order n form a ring under multiplication.[3]

Only a year after Schauder, Petrovsky [1937] extended his results for Cauchy's problem to strongly hyperbolic systems, in the simplest case

$$u_t + \sum_1^n A_k(t,x)u_k + Bu = v, \quad u_k = \partial u/\partial x_k, \tag{4.1}$$

and the corresponding quasilinear versions. Here the coefficients are square matrices of order m and the strong hyperbolicity with respect to the time variable t means that all m velocities c given by

$$\det(cI + \sum \xi_k A_k(t,x)) = 0 \tag{4.2}$$

are real and separate for all real $\xi \neq 0$. The method is that of Schauder starting from the analytic case, but Petrovsky had to find his own energy estimate. For this he used the Fourier transform, but the essential point is to be found in thirty rather impenetrable pages. Note that if the system (4.1) is symmetric, *i.e.*, the matrices A_k are Hermitian symmetric, then (4.2) holds except that the velocities need not be separate. Moreover,

$$\partial_t|u(t,x)|^2 + \sum \partial_k(A_k u(t,x), u(t,x)) = O(|u(t,x)|^2 + |u(t,x)||v(t,x)|)$$

under suitable conditions on the coefficients. Hence the proper energy density on $t = \text{const}$ is here simply $|u(t,x)|^2 dx$.

Petrovsky's paper was followed by a study [Petrovsky 1938] of conditions for the continuity of Cauchy's problem for operators whose coefficients depend only on time.

5. Fundamental solutions, Herglotz and Petrovsky

Herglotz [1926-28] and Petrovsky [1945] used the Fourier transform to construct fundamental solutions $E(P,t,x)$ for constant coefficient homogeneous differential operators $P = P(\partial_t, \partial_x)$ of degree m which are strongly hyperbolic with respect to t. Every such fundamental solution E is analytic outside a

[3]Soon after, Sobolev proved that one gets a ring also when n is replaced by $(n+1)/2$ when n is odd and by $(n+2)/2$ when n is even.

wave front surface $W(P)$, which is the real dual of the real surface $P(\tau, \xi) = 0$, and vanishes for $t < 0$ and outside the outer sheet of $W(P)$.[4] Petrovsky also found explicit formulas for derivatives of order $> m - n$ of a fundamental solution in terms of Abelian integrals, integrated over cycles $c(x)$ of real dimension $n - 3$ in the complex projective intersection I of $P(\xi) = 0$ and $(x, \xi) = 0$. The cycles depend on the parity[5] of n and the component T of $C(P)\backslash W$ where x is situated. When $\alpha(x)$ is homologous to zero in I, the region T is a lacuna, *i.e.*, the fundamental solution is a polynomial of degree $m - n$ in T and hence vanishes when $m < n$. The point of the paper is that the vanishing of the cycle is necessary when the lacuna is stable under small deformations of the operator.[6]

The intriguing paper [1937] by Petrovsky became the starting point for the development after 1950 of a general theory of hyperbolic differential operators by Leray and others and the paper [Petrovsky 1945] was generalized and clarified by Atiyah, Bott and Gårding [1970, 1973].

A decisive factor in the further development was the full use of the distributions of Laurent Schwartz and later by pseudodifferential operators and microlocal analysis.

6. Hyperbolicity for constant coefficients

Inspired by Petrovsky [1938], Gårding [1950] gave an intrinsic definition of the hyperbolicity of differential operator $P(D)$ with constant coefficients and principal part P_m as follows. The operator is said to be *hyperbolic* with respect to a hyperplane $(x, N) = 0$ or to be in a class $\mathrm{hyp}(N)$ if

(6.1) all smooth solutions u of $Pu = v$ tend to zero locally uniformly in the halfspace $(x, N) > 0$ when all their derivatives tend to zero locally uniformly in the hyperplane $(x, N) = 0$ and all derivatives of v tend to zero locally uniformly when $(x, N) \geq 0$.

It is implicit in this definition that the value of a solution u of $Pu = 0$ at a point only depends on the values of u and its derivatives in a compact subset of the initial plane.

Applying this condition to exponential solutions $e^{i(x,\zeta)}$ with $P(\zeta) = 0$ and suitable ζ, an equivalent algebraic condition was found, namely that $P_m(N) \neq 0$ and that $P(\xi + tN) \neq 0$ for all real ξ when $\operatorname{Im} t$ is large enough

[4]The real dual is generated by $\operatorname{grad} P(\xi)$ when $P(\xi) = 0$ and has m sheets. Its intersection with $t \geq 0$ has $[\frac{m+1}{2}]$ sheets.

[5]When n is even, $\alpha(x)$ is just the real intersection.

[6]In his work, Petrovsky analysed the homology in middle dimension of a general algebraic hypersurface.

negative.[7] It follows easily that P_m belongs to the class $\mathrm{Hyp}(N)$ of homogenous elements in $\mathrm{hyp}(N)$, that $P_m(\xi)/P_m(N)$ is real for real argument and that the real, homogeneous hypersurface $P_m = 0$ consists of of m sheets meeting the lines $\xi = tN + \mathrm{const}$ in m points. When these points are always separate unless all zero, *i.e.*, the real surface $P_m(\xi) = 0$ is non-singular outside the origin, P is said to be strongly (strictly) hyperbolic. In this case, $P_m + R$ belongs to $\mathrm{hyp}(N)$ for any polynomial R of degree $< m$. In the general case, $P_m + R$ is hyperbolic if and only if $R(\xi + iN)/P_m(\xi + iN)$ is bounded for all real ξ [Svensson 1969].

The hyperbolicity cone $\Gamma(N)$, defined as the connected component of $P_m(\xi) \neq 0$ that contains N, is open and convex and has the property that $P \in \mathrm{hyp}(\eta)$ for all $\eta \in \Gamma$.

Every $P \in \mathrm{hyp}(N)$ has a fundamental solution, the distribution

$$E(P, N, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{e^{i(x,\xi+i\eta)}}{P(\xi + i\eta)} d\xi \qquad (6.2)$$
$$\eta \in -cN - \Gamma,\ c > 0,\ \text{suff. large.}$$

The Fourier-Laplace integral on the right does not depend on the choice of η. As a function of x it is supported in a propagation cone $C(P, N)$, dual to Γ and consisting of all x such that $(x, \Gamma) \geq 0$. This cone is proper, closed and convex and has only the origin in common with all hyperplanes $(x, \eta) = 0$, $\eta \in \Gamma$. The existence of such a fundamental solution is equivalent to the condition (6.1). Note that a square matrix $M(D)$ of partial differential operators whose determinant $P(D)$ belongs to $\mathrm{hyp}(N)$ is itself hyperbolic. In fact, there is a matrix $M'(D)$ such that $M(D)M'(D) = P(D)I$ with I a unit matrix and then $M(D)$ has a fundamental solution $M'(D)E(P, N, x)$ with support in the propagation cone of P.[8]

7. The theory of lacunas

Leray's Princeton lectures [1953] and the paper by Atiyah *et al.* [1970] were both written in an effort to understand [Petrovsky 1945]. The second one extends his results to arbitrary $P \in \mathrm{Hyp}(N)$ which are complete, *i. e.*, not expressible in fewer than n variables. For this, it is important to consider also the local hyperbolicity cones $\Gamma(P_\xi, N) \supset \Gamma(P, N)$ where $P_\xi(\eta) \in \mathrm{Hyp}(N)$ is

[7] It is not difficult to see that $\mathrm{hyp}(-N) = \mathrm{hyp}(N)$.

[8] If the class C^∞ in (3.1) is replaced by a smaller Gevrey class, the class $\mathrm{Hyp}(N)$ is the same, but the class $\mathrm{hyp}(N)$ may permit more lower terms. Actually there is quite a number of papers dealing with hyperbolicity in Gevrey classes, but they will be disregarded here.

the first non-vanishing homogeneous term in the Taylor expansion of $P_m(\xi + \eta)$. Note that $P_\xi(N)$ is all of $\mathbb{R}^n$ when $P_m(\xi) \neq 0$ and a half-space when $P_m(\xi) = 0, \mathrm{grad} P_m(\xi) \neq 0$. The wave front surface $W(P, N)$ is now defined to be the union of the local propagation cones $C(P_\xi, N)$, dual to the local hyperbolicity cones $\Gamma(P_\xi, N)$. Modulo constant factors, the resulting formulas for derivatives ∂_x^ν of E of order $|\nu|$ are

$$\partial_x^\nu E(P, N, x) \sim \int_{\alpha^*} (x, \xi)^q \xi^\nu P(\xi)^{-1} \omega(\xi)$$

when $q = m - n - |\nu| \geq 0$ and

$$\partial_x^\nu E(P, N, x) \sim \int_{t_x \partial\alpha^*} (x, \xi)^q \xi^\nu P(\xi)^{-1} \omega(\xi) \tag{7.1}$$

when $q < 0$. Here

$$\omega = \sum (-1)^{j-1} d\xi_1 ... d\hat{\xi}_j ... d\xi_n$$

so that the integrands are rational $(n-1)$-forms of homogeneity zero on $Z = \mathbb{C}^n$ and hence also closed forms of maximal degree on the $n-1$-dimensional projective space Z^*. They are holomorphic in $Z^* - P_*$ and $Z^* - P^* \cap X^*$ respectively where P^*, X^* are the complex, projective surfaces $P(\zeta) = 0$ and $X : (x, \zeta) = 0$ respectively. The forms are integrated over certain homology classes α^* and $t_x \partial\alpha^*$. Their description is based on the existence of a continuous map $\xi \to \xi - iv(\xi)$ where

$$v(\xi) \in \Gamma(P_\xi, N) \cap \mathrm{Re} X, \quad \forall \xi \neq 0.$$

The class $\alpha^* \in H_{n-1}(Z^* - P^*, X^*)$ is twice the projective image of this map oriented by $(x, \xi)\omega(\xi) > 0$. The class $\partial\alpha^* \in H_{n-2}(X^* - X^* \cap P^*)$ is an absolute class and $t_x \partial\alpha^*$ denotes a tube around it.[9]

Connected components c of $C(P, N) - W(P, N)$ where the fundamental solution $E(P, N, x)$ is a polynomial, necessarily homogeneous of degree $m - n$, are called Petrovsky lacunas. The formula (7.1) shows c is a Petrovsky lacuna if the Petrovsky condition $\partial\alpha^* = 0$ holds for some $x \in c$. The main point of Atiyah *et al.* [1973] was to prove the converse of this statement by proving the completeness of the rational cohomology used.[10]

[9] When possible, residues in the last integral down into $X^* \cap P^*$ give integrals over the original Petrovsky cycles.

[10] It has been shown that $W(P, N)$ may be bigger than the singular support of $E(P, N, x)$ in $C(P, N)$ when P is not strongly hyperbolic, but the answer is no for at most double characteristics [Hörmander 1977].

8. Cauchy's problem for strongly hyperbolic operators with variable coefficients

In his lectures, Leray [1953] solved Cauchy's problem for smooth scalar differential operators and systems which are strongly hyperbolic in the sense that the corresponding characteristic polynomials are strongly hyperbolic with respect to some direction. A surface is said to be space-like when the operator is hyperbolic with respect to its normals.

Assuming uniform hyperbolicity of $P(x, D) = D_1^m + ...$ with respect to x_1 in some band $a \leq x_1 \leq b$, Leray devised a suitable global energy form for constant coefficients which he extended to variable coefficients by Gårding's inequality [1953]. This permitted him to construct solutions of Cauchy's problem with initial data on planes $x_1 =$ const. by approximations from the analytic case. Leray's paper also marks the first appearance of distributions in the theory of hyperbolic equations, to be used ever after.

In Gårding [1956, 1958], the energy tensor of Friedrichs and Lewy was extended to scalar, strongly hyperbolic operators with variable coefficients in the following way, opened up by Leray [1953].

When $|\beta| = m - 1, |\alpha| = m$, the product $\partial^\alpha u(x)\partial^\beta u(x)$ with real u is a divergence $\sum \partial_k C_k(u, u)$ where every C_k is a quadratic form in the derivatives of u of order $m - 1$. It follows that if $P(x, D)$ and $Q(x, D)$ are differential operators of degrees m and $m - 1$, then

$$\text{Im} Q(x, D)u\overline{P(x, D)u} = \sum \partial_k C_k(x, u, u) + C_0(x, u, u) \tag{8.1}$$

where all C_k are hermitian forms in the derivatives of u of order at most $m-1$, C_0 containing only derivatives of order $\leq m - 1$.

When $P_m(x, D) = D_1^m +$ lower terms has constant coefficients and is strongly hyperbolic with respect to x_1, and $Q(x, D) = \partial P_m(x, D)/\partial D_1$, a Fourier transform in the variables $x' = (x_2, ..., x_n)$ shows that

$$\int C_1(u, u)dx' \geq c \int \sum_{|\alpha|=m-1} |D^\alpha u(x)|^2 dx', \quad c > 0, \tag{8.2}$$

when the right side converges. If $P(x, D)$ of order m is uniformly strongly hyperbolic in a band $B : 0 \leq x_1 \leq a$ with time function x_1, if the coefficients are bounded and if the highest coefficients satisfy a uniform Lipschitz condition, this formula with an additional term of lower order extends to $P(x, D)$

[Gårding 1953]. The result is an inequality for $t > 0$,[11]

$$\| D^{m-1}u(t,.) \| \leq C \int_0^t \|Pu(x_1,.)\| \, dx_1 + Ce^{ct} \| D^{m-1}u(0,.) \| \tag{8.3}$$

for some $C, c > 0$ where

$$\| D^k u(t,.) \|^2 = \int \sum_{|\alpha|\leq k} |D^\alpha u(t,x')|^2 dx'. \tag{8.4}$$

The inequality (8.3) also has a local version for lens-shaped subsets of B bounded from below by space-like surfaces. It follows in particular that solutions of $P(x,D)u = 0$ which vanish at order $m-1$ on a space-like surface, vanish identically.

When the left side of (8.3) is finite, the vector $T^k u = u(t,.), ..., D_t^k u(t,.)$ belongs to a certain Hilbert space H^k. Let $C(H^k), L^1(H^k), L^\infty(H^k)$ denote functions of t such that, as a function of t, $T^k u(t,.)$ is continuous, integrable and essentially bounded respectively with values in H^k.

Associated to (8.3) is the following Cauchy's problem

$$Pu = v \quad \text{when} \quad 0 < t < a, \quad T^{m-1}u(0,.) = T^{m-1}w(0,.). \tag{8.5}$$

Here $w \in C(H^{m-1})$ and $v \in L^1(H^0)$. This problem has a unique solution in $C(H^{m-1})$. The proof by Gårding [1956, 1958] improved on earlier ones by using only functional analysis and the inequality (8.3).

The existence of a solution can also be expressed as an inequality

$$\| u \|_{\infty,0} \leq c \sup_v \frac{|(u,Pv)|}{\| v \|_{\infty,m-1}}, \quad c > 0. \tag{8.6}$$

Here all functions are defined on a band $0 \leq t \leq a$, $u \in L^\infty(H^0)$ with the corresponding norm and v, equipped with the norm of $L^\infty(H^{m-1})$, runs through the space C_0 of all smooth compactly supported functions vanishing close to $t = 0$. The inequality says in particular that PC_0 is dense in $L^1(H^0)$. The analogous inequality

$$\| u \|_{\infty,0} \leq c \sup_v \frac{|(u,Av)|}{\| v \|_{\infty,0}}, \quad c > 0,$$

[11] It is proved in [Ivrii and Petkov 1974] that this inequality implies that $P(x,D)$ is strongly hyperbolic when its coefficients are sufficiently differentiable.

where $A = D_1 + A_2D_2 + ... + A_nD_n + C$ is strongly hyperbolic as in (4.2), is a consequence of its scalar counterpart (8.6). In fact, the left side is not increased if we replace v by Bv where

$$B = D_1 + B_2D_2... + B_nD_n$$

has the property that $B(x,\xi)A'(x,\xi) = I\,\det A'(x,\xi)$ where I is the $m \times m$ unit matrix and A' is the principal part of A. By hypothesis, $\det A(x,\xi)$ is uniformly strongly hyperbolic and hence $A(x,D)B(x,D) \equiv I\,\det A(x,D)$ modulo bounded terms of order $< m$. Since $\| Bv \|_{\infty,0} \leq \| D^{m-1}v \|_{\infty,0}$, the result follows.

Under smoothness assumptions about the coefficients, the inequality (8.3) was extended by Hörmander [1963] to the case when the norm square (8.4) is replaced by

$$\| D^{k,s}u(t,.) \|^2 = \int \sum_{|\alpha|\leq k} |D^\alpha(1+|D'|)^s u(t,x')|^2 dx' \tag{8.7}$$

where s is any real number and the right side is defined by the Fourier transform in the varaible x'. In this way, also functions with distributional values in the x' direction are taken into account. This inequality permitted Hörmander [1963] to solve the corresponding Cauchy's problem very simply by a duality argument. In particular, when the coefficients of P are smooth enough, the operator P has a fundamental solution $E(x,y)$: $P(x,D)E(x,y) = \delta(x-y)$ which vanishes when $x_1 < y_1$.

Cauchy's problem on a manifold. The inequality (8.3) for lens-shaped regions proves the basic uniqueness theorem for strongly hyperbolic operators P on a manifold: if $Pu = 0$ in some neighborhood of x_0 and the Cauchy data of u vanish on some smooth space-like surface $S : s(x) = s(x_0)$, then $u = 0$ close to x_0.

To deal with more global situations it is convenient to require the existence of smooth, real time functions $t(x)$ such that $P(x,\zeta) \in \mathrm{hyp}(\mathrm{grad}\, t(x))$ for all x.[12] The condition $\mathrm{grad}\, s(x) \in \pm\Gamma(P_m(x,.), t_x)$ with a fixed sign for smooth, real $s(x)$ defines two opposite classes $T_\pm$ of time functions. A region where some time function is in T_+ is positive or negative is called a future and a past respectively and a surface where some time function is constant is said to be space-like. The manifold X is said to be complete relative to P if every compact set is contained in an intersection of a past and a future with

[12] By assuming the existence of time functions, Christodoulou and Klainerman [1993] were able to prove global existence for Einstein's equations with small data.

compact closure.[13] The intersection of all futures (pasts) containing a point x then defines two propagation conoids $C_{\pm}(x)$ issuing from x. Leray [1953] found suitable Sobolev spaces for the construction of inverses $P_{\pm}^{-1}$ of a strongly hyperbolic operator P on a manifold, complete with respect to P such that $P_{\pm}^{-1}u$ vanishes outside the union of the corresponding propagation conoids issuing from $\operatorname{supp} u$, supposed to be compact.

Nonlinear equations, hyperbolic conservation laws. The control of lower order derivatives in Cauchy's problem for linear, strongly hyperbolic equations, makes it possible to use successive approximations to prove local existence for Cauchy's problem and quasilinear, and even nonlinear, strongly hyperbolic equations. The proofs are almost as simple as in the second order case, but involve a judicious use of Sobolev inequalities. The initial work by Petrovsky [1937] and Leray [1953] was carried further by Dionne [1962].

Global existence is a problem beset with difficulties. Discontinuites may appear and solutions may cease to exist. This is clear from the much studied case of nonlinear hyperbolic conservation laws in two variables t, x

$$u_t + f(u)_x = 0, \quad u, f(u) \in \mathbb{R}^n,$$

where f is smooth and nonlinear and the matrix $\partial f(u)/\partial u$ has real, separate eigenvalues. Burger's equation for $n = 1$, $u_t + uu_x = 0$ is a model case exhibiting collisions and rarefaction waves depending on initial data for $t = 0$. The use of weak solutions [Lax 1957b] motivates jump conditions, the classical Rankine-Hugoniot jump conditions, and existence proofs have to use various entropy conditions. The case of arbitrary n has a refined existence proof for initial data of small bounded variation [Glimm 1965] with a recent amelioration by Young [1993]. When the initial total variation is not small and $n > 2$ blow-up may occur (see Young [1995]). A short text cannot do justice to the complicated nature and history of hyperbolic conservation laws. There is ample material in [Smoller 1983].

9. Mixed boundary problems

Let $P(D)$ be a differential operator, hyperbolic with respect to the first variable x_1, and consider boundary problems for P in a quarter space $x_1 \geq 0, x_2 \geq 0$ with a source function, Cauchy data C on $x_1 = 0$ and some other linear data F on a non-characteristic plane $x_2 = 0$. If the problem is correctly posed, the reduced problem with vanishing source and non-vanishing

[13] The full Cauchy's problem with data on a space-like surface requires this condition.

Cauchy data should also be correctly posed. Hence the data F in the reduced problem ought to propagate away in the positive x_1 and x_2 directions [Agmon 1962, Hersh 1963]. In particular, if $n = 2$ and

$$P(D) = \prod_1^m (D_1 + a_k D_2), \quad \forall a_k \neq 0,$$

these solutions should be a linear combination of functions of $x_2 - a_k x_1$ with $a_k < 0$. For $n = 2$, this principle determines the form of mixed problems for hyperbolic operators in regions limited by polygons (see [Campbell and Robinson 1955] and [Thomée 1957]).

In the general case, the principle says that the reduced mixed boundary problem should not have exponential solutions $e^{i(x,\xi)}$ with $P(\xi) = 0$ which are exponentially large for $x_1 > 0, x_2 > 0$ when the solution is bounded when $x_1 = 0, x_2 \geq 0$ and $x_1 \geq 0, x_2 = 0$. This means that $\operatorname{Im} \xi_1 > 0, \operatorname{Im} \xi_2 > 0$. This criterion is workable since it follows from the hyperbolicity that the polynomial

$$\xi_2 \to P(\xi), \quad \operatorname{Im}\xi_1 > 0, \quad \xi_3, ... \text{real},$$

has no real zeros and hence a fixed number m_+ of zeros with $\operatorname{Im} \xi_2 > 0$. The remaining, forbidden ones have $\operatorname{Im} \xi_2 < 0$. It is therefore reasonable that F can only have m_+ independent data. Appropriate polynomial boundary conditions on $x_2 = 0$ have the form

$$Q_1(D)u = g_1, ..., Q_{m_+}(D)u = g_{m_+}$$

where $Q_1, ..., Q_{m_+}$ should be linearly independent modulo the product of the permitted factors[14] of the polynomial $\xi_2 \to P(\xi)$. There is a corresponding determinant, the Lopatinski determinant, which should be hyperbolic in a certain sense with respect to the first variable. As shown by Reiko Sakamoto [1974] and exposed in [Hörmander 1983b, pp. 162-179], these conditions are both necessary and sufficient for the mixed problem for strongly hyperbolic operators to be correctly posed in the C^∞ sense. The waves from the Cauchy data at the boundary $x_2 = 0$ are reflected in a way consistent with the boundary condition.

In a wellknown paper by H.-O. Kreiss [1970], the problem above was put for first order operators, strongly hyperbolic with respect to the first variable,

$$D_1 + A_2 D_2 + ... + A_n D_n,$$

whose coefficients are $m \times m$ matrices. The matrix A_2 is supposed to be diagonal with m_+ positive and $m - m_+$ negative eigenvalues which gives m_+

[14] with zeros such that $\operatorname{Im} \xi_2 > 0$ when $\operatorname{Im}\xi_1 > 0$.

linear boundary operators. It is shown that a strengthening of the condition above to no solutions with $\mathrm{Im}\xi_1 \leq 0$ gives L^2 bounds of the solution in terms of similar bounds for the data.

10. Hyperbolicity for variable coefficients

It is proved in [Ivrii and Petkov 1974] that an inequality (8.3) implies that $P(x, D)$ is strongly hyperbolic when its coefficients are sufficiently differentiable. The same paper also offers necessary conditions for the hyperbolicity for operators with variable coefficients as defined by an obvious localization of (6.1) to a neighbourhood N of a point x_0 and its intersection I with a plane $(x - x_0, \theta) = 0$. It is required that u tends to zero close to x_0 when all the derivatives tend to zero locally uniformly in I and Pu tends to zero in the same way in N. The verification of this property involves existence and uniqueness of a suitable Cauchy's problem.

By the construction of suitable asymptotic solutions it is shown that $P(x_0, D)$ must be hyperbolic with respect to θ. The proofs have been simplified by Hörmander [1985a, pp. 400-403]. Earlier proofs by the same method due to Lax [1957a] for analytic coefficients and Mizohata [1961-62] for first order systems supposed that θ is not characteristic.

In the Cauchy's problem for the operator $D_1^2 - x_1^2 D_2^2 + bD_2$, studied by Oleinik [1970], the regularity of the solution requires more and more regularity of the Cauchy data the smaller b is. This is the motivation in [Ivrii and Petkov 1974] to define regular hyperbolicity (effective hyperbolicity according to Hörmander [1977]) as hyperbolicity under addition of arbitrary lower order terms in the operator. The authors then prove the following interesting result. For an operator $P(x, D)$ to be effectively hyperbolic in an open set it is necessary that the fundamental matrix (Hamiltonian map)

$$\begin{pmatrix} p_{\xi x} & p_{\xi\xi} \\ -p_{xx} & -p_{x\xi} \end{pmatrix}, \quad p = P_m(x, \xi), \tag{10.1}$$

skewsymmetric in symplectic structure given by $dx \wedge d\xi$, has a pair of nonvanishing real eigenvalues at every point where $dp = 0$ but $d^2p \neq 0$. When this condition is not satisfied, there are conditions on the lower terms, exhibited in [Hörmander 1977]. Finally, it is proved that the condition that

$$dP_m(x, \xi) \neq 0$$

is both necessary and sufficient for hyperbolicity with a fixed relation between the regularity of the data and that of the solution independent of lower order terms. This condition implies strong hyperbolicity in an open set and at most double zeros of $P_m(x,\xi)$ on a bounding space-like surface. Tricomi's operator $D_1^2 - x_1(D_2^2 + ... + D_n^2)$ in the region $x_1 \geq 0$ is here a classical example (see [Hörmander 1985b, section 23.4]).

In contrast to this situation, the sufficiency of effective hyperbolicity for hyperbolicity is a delicate problem. A positive answer is known only for equations of order two [Iwasaki 1984, Nishitani 1984a,b]) and under a certain restriction in the general case Ivrii [1978], removed by Melrose [1983]. The fact that the condition (10.1) is invariant under canonical maps is used by all these authors to get suitable normal forms of the operators which then must involve pseudodifferential operators. The canonical maps are realized by Fourier integral operators, a tool created by Hörmander [1971] (see below).

Outside of effective hyperbolicity, there are microlocal conditions at multiple characteristics which make the Cauchy's problem correctly set in the sense given above (see [Kajitani and Wakabayashi 1994] and the literature quoted there).

Systems. Necessary conditions for hyperbolicity with respect to the time variable x_1 for first order hyperbolic operators

$$L(x,D) + B(x), \quad L(x,D) = ED_1 + L_2(x)D_2 + ... + L_n(x)D_n.$$

is a much studied subject. The coefficients are smooth $m \times m$ matrices and E is the unit matrix. It is of course necessary that the determinant $h(x,\xi) = detL(x,\xi)$ be hyperbolic at every x, but this is not enough. A zero of order r of $h(x,\xi)$ must give a zero of order $r-2$ of the cofactor matrix $M(x,\xi) = (m_{ij}(x,\xi))$ and if L is effectively hyperbolic in the sense above, then every $h(x,D) + m_{ij}(x,D)$ must be hyperbolic with respect to x_1 [Nishitani 1993].

11. Fundamental solutions by asymptotic series

It was clear from the formulas of Herglotz and Petrovsky that the singularities of the fundamental solutions of homogeneous, strongly hyperbolic operators $P(D) \in \mathrm{Hyp}(N)$ of degree m lie on the wave front surface, consisting of $[(m+1)/2]$ sheets issued from the origin and contained in the dual to the characteristic surface $P(\xi) = 0$.[15] But the abstract existence proofs for variable coefficients did not give this kind of information, nor is it expected

[15] The dual of $P(\xi) = 0$ is generated by $x = \mathrm{grad}P(\xi)$ when $P(\xi) = 0$. It has m sheets and is invariant under reflection in the origin. The wave front surface is the restriction to $(x,N) \geq 0$ and has the number of sheets stated.

unless the coefficients are smooth. But for the case of infinitely differentiable coefficients, there are very precise results.

The construction of fundamental solutions of strongly hyperbolic operators by means of oscillating integrals [Lax 1957a, Ludwig 1960] gave the first answer.[16] The oscillatory integrals used have the following general form introduced by Hörmander [1971],

$$u(x) = \int a(x,\theta)e^{i\lambda(x,\theta)}d\theta. \tag{11.1}$$

The amplitude $a(x,\theta)$ is a smooth function with x in some open subset of $\mathbb{R}^n$ and $\theta \in \mathbb{R}^N$. It is assumed that $\partial_\theta^\alpha a(x,\theta) = O(|\theta|^{m-|\alpha|})$ for large θ, locally uniformly in x. The phase function $\lambda(x,\theta)$ is supposed to be smooth and real and homogeneous of degree 1 in θ. The assumption that $d\lambda \neq 0$ makes u a distribution which is a smooth function of x unless $\lambda_\theta(x,\theta) = 0$ for some θ. In practice, the amplitude $a(x,\theta)$ is often polyhomogeneous, *i.e.*, an asymptotic sum of terms of decreasing integral homogeneity in θ for large values of this variable.

When $P(x,D)$ of order m is strongly hyperbolic with respect to x_1, its principal symbol $p(x,\xi)$ is a product of m factors $p_k(x,\xi)$ of homogeneity 1 in ξ. The phase functions used in Lax's paper are solutions $\lambda_k(x,\theta)$ of the equations $p_k(x, \mathrm{grad}\lambda_k) = 0$ such that $\lambda_k = \mu = \sum x_k\theta_k$, $k > 1$ when $x_1 = 0$. These functions exist only for small x_1, but permit an extension of an oscillating integral $w_k(x_2, ..., x_n)$ with a polyhomogeneous amplitude and phase function μ (and hence singular only at the origin) to an oscillating integral $W_k(x)$ with polyhomogeneous amplitude and phase function λ_k such that $P(x,D)w_k$ is arbitrarily smooth. By a suitable choice of $w_1, ..., w_m$, the difference between a fundamental solution $E(x)$ supported in $x_1 \geq 0$ and the sum $W_1 + ... + W_m$ can be made arbitrarily smooth. It follows that $E(x)$ is regular at x except when the θ-gradient of some $\lambda_k(x,\theta)$ vanishes, in particular only at the origin when $x_1 = 0$. Since $d\lambda_k$ invariant under the characteristics $dx/dt = p_\xi(x,\lambda_x)$ of the equation $p(x,\lambda_x) = 0$, the fundamental solution is singular only at the locus of these curves issued from the origin.

For larger times, the locus of characteristics may develop singularities, the caustics of geometrical optics may occur. Oscillating integrals which represent the fundamental solution beyond the caustics were constructed in Ludwig's paper.

[16] Both authors treat hyperbolic systems.

12. Microlocal analysis, wave front sets, pseudodifferential operators

All the results above are clarified by microlocal analysis which deals with localization in space and frequency of distributions and operators. A beginning was made by Maslov [1964]. There is also a microlocal analysis for hyperfunctions initiated by Sato [1969] and later developed by his students and others. However, we shall stick to distributions, following Hörmander [1971].[17]

The setting of microlocal analysis is the cotangent bundle $T^*(X)$ of a differentiable manifold X with local coordinates x, ξ and invariant differential form $\omega = (dx, \xi)$. Let u be a distribution on $\mathbb{R}^n$ and let $f \in C_0^\infty$. Simple arguments show that the growth at infinity of the Fourier transform $v(\xi)$ of fu gets smaller in all directions when f is replaced by a product fg and $g \in C_0^\infty$. Hence there is for instance a natural localization $H^s(x, \xi)$ of the classical space H^s at a point $x, \xi (\xi \neq 0)$ invariant under multiplication by smooth functions and consisting of distributions u such that $(1 + |\xi|)^s v(\xi)$ belongs to L^2 in some conical neighborhood of x, ξ for some $f \in C_0^\infty$ whose support contains x. Another interesting object is the wave front set $\mathrm{WF}(u)$ of a distribution u, equal to the complement of all x, ξ such that $v(\xi)$ has fast decrease in some conical neighborhood of x, ξ for some f as above. The wave front set is a closed, conical subset of the cotangent bundle $T^*(X)$. The projection of $\mathrm{WF}(u)$ on X is the singular support $S(u)$ of u. All these notions extend to distributions on a manifold.

An important example of wave front set is the following. The wave front set of the oscillatory integral (11.1) is contained in the set of pairs x, ξ such that $\lambda_\theta(x, \theta) = 0$. When the phase function is regular, *i.e.*, the differentials $d\lambda_\theta$ are linearly independent, this equation defines a conical Lagrangian manifold, a submanifold of $T^*(\mathbb{R}^n)$ of maximal dimension were the differential form (ξ, dx) vanishes. One important result of Hörmander [1971] is that two oscillatory integrals with regular phase functions with the same Lagrangian produce the same distributions modulo smooth functions, at least when the conical support of the amplitudes are small.

When the phase function λ of (11.1) has the form $\lambda(x, y, \theta), x \in \mathbb{R}^n, y \in \mathbb{R}^m$, the integral $I(x, y)$ represents the kernel of what is called a Fourier integral operator [Hörmander 1971]. Generally speaking, the corresponding operator will map distributions u to distributions v such that

$$\mathrm{WF}(v) \subset C(\mathrm{WF}(u))$$

[17] Only the simplest version of microlocal analysis can be given here. For full exposition, see Hörmander's monumental four volumes [Hörmander 1983a,b, 1985a,b].

where $C = (x, \xi, y, -\eta)$ is a canonical relation such that (x, ξ, y, η) belongs to the Lagrangian defined by I. This fact makes Fourier integral operators a powerful tool in microlocal analysis which permits a change of variables in the cotangent bundle which mixes its two ingredients.

When the phase function above has the form $(x - y, \theta)$ where the three dimensions are the same, C reduces to the identity and the corresponding operators are pseudodifferential operators in the form developed by Kohn and Nirenberg [1965]. They were originally given as singular integrals by Calderón and Zygmund [1957].

A pseudodifferential operator has the form

$$P(x, D)u(x) = (2\pi)^{-n} \int P(x, \xi)\hat{u}(\xi)d\xi, \quad \hat{u}(\xi) = \int e^{-i(x,\xi)}u(x)dx.$$

Here the left side is a definition, $u \in C_0^\infty$ and the symbol $P(x, \xi)$ of P is a smooth amplitude with properties as in (11.1), for instance polyhomogeneous. When $P(x, \xi)$ is a polynomial in the second variable, $P(x, D)$ is a differential operator. The first non-zero term in the expansion of P is the principal symbol $p(x, \xi)$ of P. Pseudodifferential operators act on Schwartz's space S and, by duality also on S'. In each case they form an algebra, the map from P to its principal symbol being a homomorphism. The calculus of pseudodifferential operators extends to distributions on a manifold X. Its symbols are then defined on the cotangent bundle $T^*(X)$ with local coordinates (x, ξ).

One has $\mathrm{WF}(Pu) \subset \mathrm{WF}(u)$ with equality when $P(x, D)$ is elliptic, *i.e.*, when $\mathrm{Char}P = \emptyset$, and then also $\mathrm{WF}(u) = \emptyset$ when $Pu \in C^\infty$. A proper reduction of singularity may occur at the characteristic set $\mathrm{Char}(P)$ where $p(x, \xi) = 0$ and $\xi \neq 0$. One of the uses of pseudodifferential operators is the factorization of the principal parts of hyperbolic operators into a product of pseudodifferential operators of degree 1.

Pseudodifferential operators give a short, equivalent definition of $\mathrm{WF}(u)$ for a distribution on a manifold X, namely

$$\bigcap_{Pu \in C^\infty} \mathrm{Char}P.$$

This is also the original definition in Hörmander [1970].

13. Propagation of singularities in boundary problems

A pseudodifferential operator $P(x, D)$ is said to be of real principal type when its principal symbol $p(x, \xi)$ is real and $\partial_\xi p(x, \xi) \neq 0$ in $\mathrm{Char}P$. The operator P has the characteristic equation $p(x, \varphi_x) = 0$ which in turn has the characteristic curves

$$x_t = p_\xi(x, \xi),\ \xi_t = -p_x(x, \xi), \quad p(x, \xi) = 0, \tag{13.1}$$

called (null) bicharacteristics of P. By geometrical optics theory they leave both Char$P$ and the restriction to Char$P$ of the differential form ω invariant.

A basic general result proved by Hörmander [1970] gives to the wave front sets of solutions of $Pu = 0$ a geometrical optics structure when P is a pseudodifferential operator of principal type. It says that $\mathrm{WF}(u)\backslash\mathrm{WF}(Pu)$ is invariant under the bicharacteristic flow (13.1) so that, in other words,

(13.2) $\mathrm{WF}(u)\backslash\mathrm{WF}(Pu)$ is a union of bicharacteristics.

To prove this result it suffices to show that a bicharacteristic interval I outside $\mathrm{WF}(Pu)$ is outside $\mathrm{WF}(u)$ when its endpoints are. When P has order 1 and its symbol vanishes outside a neighborhood of I, the proof is not difficult and the general situation can be reduced to this case. In another version ([Duistermaat and Hörmander 1972], [Hörmander 1985b, p. 57]) the condition that $\partial_\xi p(x, \xi) \neq 0$ on Char p is eliminated, there is a radical reduction to the case $P = D_1$.

If P is a differential operator which is strongly hyperbolic with respect to some θ, it follows from the general results above that the wave front set W outside y of the fundamental solution $E(P, x, y, \theta)$ with pole at y and support in the halfspace $(x - y, \theta) \geq 0$ consists of all bicharacteristics issued from y and directed into this space. The fiber of the wave front set over y is $\mathbb{R}^n\backslash 0$. In fact this is the fiber over y of $\delta(x-y)$ and $PE(x, y, \theta) = \delta(x-y)$. Caustics appear when the projection of W on x-space is not invertible.

In the Cauchy's problem for a hyperbolic operator in a half-space, the source and the data on the boundary may be distributions and the question of the singularities of the solution arises. The gross answer is that its wave front set outside that of the source is generated by null bicharacteristics issuing into the halfspace from the wave fronts of the source and the data. The precise answer involves a calculus of pseudodifferential operators on a manifold with boundary introduced by Melrose [1981] and exposed by Hörmander [1985a, pp. 112-141].

The question of singularities of the solution of a mixed problem involve reflections at a time-like boundary. The propagation of singularities in this case involves some serious microlocal analysis and is the subject of papers by Chazarain [1973], Melrose [1975], Taylor [1976], Andersson and Melrose [1977], Eskin [1977], Melrose and Sjöstrand [1978, 1982], Ivrii [1980] and others.

To take an example, let u be the solution of a second order equation $Pu = f$ in the interior of a manifold X with boundary ∂X where $u = u_0$ and consider the wave front set W of u outside the union of the wave front sets of f and u_0. The simplest case is reflection of a bicharacteristic by the law of geometrical optics. In addition, the boundary may contain a glancing set where the incoming bicharacteristic is tangent to the boundary. The bicharacteristic may then still be diffracted off the boundary, but there may also be gliding rays which are limits of rays which are reflected many times. In all cases, these bicharacteristics are part of the wave front set. A somewhat final result [Melrose and Sjöstrand 1978, 1982] says roughly that a bicharacteristic in the wave front set outside that of the source can always be continued except at points over the boundary where the Hamilton field is radial.

Propagation of polarization. The notion of characteristic set $\mathrm{Char}(P)$ of a scalar differential operator extends to a matrix operator $P(x, D)$ of type $M \times N$ with principal symbol $p(x, \xi)$. It is defined as the set of triples $(x, \xi, w \in C^N)$ such that $p(x, \xi)w = 0, \xi \neq 0$.

The polarization set $W_{\mathrm{pol}}(u)$ of a distribution $u(x)$ with k components is then defined as the intersection of all $\mathrm{Char}(P)$ with P of type $1 \times k$, for which $Pu \in C^\infty$. Polarization of electromagnetic waves fits this definition. The projection of the polarization set is the wave front set $\mathrm{WF}(u)$ defined as the union of the wave front sets of the components of u.

In simple cases, for instance for strongly hyperbolic systems, polarization propagates along certain Hamilton orbits which are unique liftings of bicharacteristics. The propagation of polarization, not restricted to hyperbolic equations, has been studied in a series of papers by N. Dencker [1982, 1995].

14. Propagation of singularities for nonlinear equations

If $u \in H^s(x, \xi) \cap H^t(x, \eta), s > n/2$, the properties of the Fourier transform of u^2 show that it may happen that $u^2 \in H^{s+t-n/2}(x, \zeta)$ when ζ is a convex linear combination of ξ, η. It is therefore natural that new and weaker singularities appear in solutions of equations when nonlinearities are introduced. These new singularities will then propagate along bicharacteristics which in turn may meet to give still weaker singularities and so on. According to the number of steps, this process will be called selfspreading of first order, second order, etc.

Selfspreading is made explicit in the paper by Rauch and Reed [1982]. It deals with solutions $u = (u_1, ..., u_m)$ of strongly hyperbolic first order semilinear systems in two variables t, x with right hand sides which are smooth functions of t, x, u. The initial value $u(0, x)$ is supposed to be of class H^s in an interval I and smooth outside. In the linear case, the singularities lie on $2m$ forward characteristics from the endpoints of I which form a net with crossings. In the semilinear case, new forward characteristics occur from the lowest crossing points and so on. The end result is an explicit rule giving the regularity of u in regions bounded by bicharacteristics. Roughly speaking, the regularity increases with the distance to the origin. For more than two variables this process of selfspreading of singularities may result in uniformly distributed singularity. Beals [1983] constructed solutions $u(t, .) \in H^s$, with $0 < t < 1, s > (n+1)/2$, of the wave equation in $1 + n > 2$ variables with a suitable nonlinear term $f(x)u^3$ and initial data in H^s, H^{s-1}, singular only at the origin. The singular support of one such solution was shown to contain the part of the forward light cone where $t \leq 1$ and the solution is regular there at least of the order $3s - n + 1 + 0$.

The method of paradifferential calculus by Bony [1981] (see also the review article [Bony 1989]) has given some very general results about the propagation of singularities of nonlinear strongly hyperbolic equations. The calculus is based on smooth functions $\varphi(\xi)$ supported in some annulus $A_k : 1/k \leq |\xi| \leq k$ with $k > 1$ such that the dyadic sum $\sum_0^\infty \varphi(2^{-j}\xi)$ equals $1 - \psi(\xi)$ where ψ is smooth and supported in $|\xi| < 1$. The action of paramultiplication T_u on a distribution v is defined by the formula

$$\sum \theta(2^{-j}D)(u\varphi(2^{-j}D)v)$$

with $\varphi(\xi)$ as above and $\theta(x)$ smooth and equal to 1 in A_k with support in a slightly larger annulus. The crucial property of paramultiplication is that if $u \in H^s, v \in H^t$, $s + t > 0$ then $uv = T_u v + T_v u + R(u, v)$ where R maps $H^s \times H^t$ continuously to $H^{s+t-n/2}$ and similar properties for substitution.

Bony proved that a nonlinear differential operator of order m, $F(u) = F(x, u, Du, ..., D^m u)$, has a paralinearization L given by

$$Lu = \sum T_{\partial F/\partial^\alpha u} \partial^\alpha u$$

such that Lu belongs to $H_{\text{loc}}^{2s-2m-n/2}$ when $u \in H^s$, $F(u) = 0$ and $s > m+n/2$. The operator L and the ordinary linearization $L_F = \sum (\partial F/\partial^\alpha u)\partial^\alpha$ have the same principal parts.

The preceding result can be applied to the situation when L_F is strongly hyperbolic with some time variable t and $F(u) = 0$ in some region, $u \in H^s$. Outside Char$L$ the solution is locally in $H^{2s-m-n/2}$ and regularity one step or more lower is propagated along bicharacteristics. There are also analogous results about the reflection and diffraction of bicharacteristics (see [Bony 1989]).

When the equation $F(u) = 0$ is semilinear and the interaction between singularities are taken into account, more precise results have been obtained by Chemin [1988] using a refined paradifferential calculus. Briefly, his results say that if $u \in H^s_{\mathrm{loc}}$ and s exceeds some s_0 depending on the equation, then singularity of the order at most $3s+O(1)$ is propagated by a a certain modified second order selfspreading. Chemin also shows that this result is close to best possible.

More precise propagation of singularities, closer to the linear case, can be obtained with special initial data describing simple waves. These are the conormal distributions. The singular support of such a distribution is a smooth hypersurface $S : s(x) = 0$ and the regularity of u does not decrease under the action of smooth vector fields tangent to S. Example: $u = f(s(x))$ where f is a homogeneous distribution on the line. For wave equations with nonlinear lower terms, initial data of this form are propagated close to the linear case and the first order selfspreading suffices for a precise description. Problems of this kind, the propagation caused by intersecting hypersurfaces, by a hypersurface developing a swallowtail, by reflexion of a simple wave in a wall, etc. are the subject of many papers of progressing complexity which are still being published by, among others, Melrose and collaborators (see e.g. [Bony 1989, Lebeau 1989, Melrose and Ritter 1985, 1987, Melrose and Barreto 1994, Barreto 1995]). These and many similar papers bear out Bony's remark that nonlinear singularities require more microlocal analysis, not less, than linear ones.

15. Blow-up and global existence for wave equations

The subject of global solutions of semilinear wave equations got new life in the eighties. The impetus came from Fritz John's papers about life-time and blow-up of semilinear wave equations with small initial data [John *Papers*, part. IV]. The best studied equations have the form

$$u_{tt} - \sum a_{jk}(u')\partial_j\partial_k u - f(u') = 0, \qquad u' = \mathrm{grad} u, \quad \partial_j = \partial/\partial x_j, \tag{15.1}$$

or

$$\Box u = g(u, u', u''), \quad \Box = \partial_t^2 - \Delta \tag{15.2}$$

in $1+n$ variables t, x with compactly supported initial data $u(0, x) = \varepsilon u_0(x)$, $u_t(0, x) = \varepsilon u_1(x)$ where $\varepsilon > 0$ is small. It is also assumed that the coefficients are smooth and that the equations deviate little from $\Box u = 0$ so that $a_{jk}(u') - \delta_{jk}$ vanishes of order zero and $f(u, u', u'')$ of order 1 for vanishing arguments. The lifetime T of the solution is the maximal time below which the solution is reasonably smooth. The work done with these equations is ample confirmation of Schauder's remark that the solution of nonlinear equations means getting optimal bounds on solutions of linear equations. In particular, the improvement below of the blow-up time with increasing n depends on the increasing dispersion of initial data for the linear equation.

John worked with both equations above but mostly with the case $n = 3$ where $\Box$ has a fundamental solution ≥ 0. One of his many results [John 1979] says that $T \sim \varepsilon^{-2}$ for the equation $\Delta u = u^2$. Improving on [John 1976], John and Klainerman [1984] proved for equations (15.2) that $T > e^{c/\varepsilon}$ when $n = 3$. For $n > 4$ this was improved by Klainerman [1985a] to existence for all sufficiently small $\varepsilon > 0$ when g does not depend on u. The case $n = 3$ requires an extra condition on the main term, the null condition, found by Klainerman [1986] and Christodoulou [1986]. For $n = 4$ and $g = g(u', u'')$, Hörmander proved [1991] that $T_\varepsilon \geq e^{c/\varepsilon}$. The same estimate with ε^2 was obtained later by Li Ta-Tsien and Zhou Yi [1995]. Corresponding results for nonlinear pertubations of the Klein-Gordon equation $u_{tt} - \Delta u + u = F(u, u')$, where the linear equation has a better energy density, are less delicate [Klainerman 1985b].

Recent interest has been focussed on the details of the blow-up. Caffarelli and Friedman [1986] found a space-like smooth blow-up surface for the wave equation with right side $F(u) \sim Au^p, A > 0, p > 1$. Lindblad [1990a] proved that the rescaled solution $U_\varepsilon(t, x) = \varepsilon^{-4} u(t/\varepsilon^2, x/\varepsilon^2)$ of (15.2) in 1+3 variables and $f = u^2$ has a distribution limit v which solves (15.2) with the right side $v^2 + \mu$ in some interval $0 < t < T$. Here μ is a measure carried by the forward lightcone. More precise life times T for two space variables are given in [Alinhac 1994, 1995] where it also conjectured — under certain regularity assumptions— that the quotient $1/(T - t)$ describes the growth of the L^2 norm of grad$u$ at the point t close to T. Alinhac suggests that singularities may appear as folds after a suitable change of variables and proposes better approximations of the quasilinear equation than just the linear part. Such methods were also used in [Hörmander 1989].

16. Concluding remarks

The development of the theory of hyperbolic partial differential equations in the twentieth century is a continuing effort to master the singularities of solutions of such equations. In this process new analysis was used and as old problems were solved, new ones have appeared.

The difficulty that the fundamental solution of a second order hyperbolic operator has singularities of high order outside the pole was circumvented by Hadamard in his use of the *partie finie*. The full use of the Fourier transform has permitted the construction of fundamental solutions of homogeneous higher order hyperbolic partial differential operators with constant coefficients, in the beginning with an incomplete treatment of the singularities. The algebraic definition of hyperbolicity has been motivated intrinsically by the requirements of finite propagation velocity and continuity. New energy densities have made possible existence proofs for Cauchy's problem and mixed problems for linear and nonlinear hyperbolic differential equations using a passage to the limit from the analytic case. Afterwards, the theory of distributions gave a better understanding of the nature of singularities and functional analysis has given simple existence proofs for Cauchy's problem and mixed problems both for smooth and not smooth data. For nonlinear equations, the control of lower order derivatives make local existence proofs possible. As shown by the theory of hyperbolic conservation laws, global existence and uniqueness are much more difficult problems. For quasilinear equations, the problem of the lifetime of solutions with small initial data has recently received much attention.

Microlocal analysis is a new tool for the study of the propagation of singularities of solutions of hyperbolic partial differential equations. For linear equations and a variety of boundary problems, this study has given almost definitive results, at least for smooth coefficients. Recent efforts are directed towards the analysis of singularities of solutions of nonlinear hyperbolic equations. Here the nonlinearity itself generates singularities which have been successfully treated for equations close to linear ones.

Bibliography

Agmon (S.)

[1962] Problèmes mixtes pour les équations hyperboliques d'ordre supérieur. In *Coll. Int CNRS*, vol. 117, Paris , 1962, pp. 13–18.

Alinhac (S.)

[1994] Temps de vie et comportement explosif des solutions d'équations d'ondes quasilinéaires en dimension deux II, *Duke Math. J.*, 73 (1994), pp. 543–560.

[1995] Temps de vie et comportement explosif des solutions d'équations d'ondes en dimension deux I, *Ann. Sci. Ecole Norm. Sup.*, (IV) 28 (1995), pp. 225–250.

ANDERSSON (K.) and MELROSE (R.)

[1977] The propagation of singularities along gliding rays, *Invent. Math.*, 41 (1977), pp. 197–232.

ATIYAH (M.), BOTT (R.) and GÅRDING (L.)

[1970] Lacunas for hyperbolic differential equations with constant coefficients I, *Acta Math.*, 124 (1970), pp. 109–189.

[1973] Lacunas for hyperbolic differential equations with constant coefficients II, *Acta Math.*, 131 (1973), pp. 145–206.

BARRETO (A.)

[1995] Evolution of semilinear waves with swallowtail singularities. Preprint.

BEALS (M.)

[1983] Self-spreading and strength of singularities, *Ann. of Math.*, 118 (1983), pp. 187–214.

BONY (J.M.)

[1981] Calcul symbolique et propagation des singularités pour les équations partielles non linéaires, *Ann. Sci. Ec. Norm. Sup.*, (IV) 14 (1981), pp. 209–246.

[1989] *Analyse microlocale des équations aux dérivées partielles non linéaires*, vol. 1495 of *Lecture Notes in Mathematics*, Springer, 1989.

CAFFARELLI (L.) and FRIEDMAN (A.)

[1986] The blow-up boundary for nonlinear wave equations, *Trans. Amer. Math. Soc.*, 297 (1986), pp. 223–241.

CALDERÓN (A.) and ZYGMUND (A.)

[1957] Singular integrals and differential equations, *Amer. J. Math.*, 79 (1957), pp. 901–921.

CAMPBELL (L.) and ROBINSON (A.)

[1955] Mixed problems for hyperbolic partial diffferential equations, *Proc. London Math. Soc.*, (III) 5 (1955), pp. 129–147.

CHAZARAIN (J.)

[1973] Construction de la paramétrix du problème mixte hyperbolique pour l'équation des ondes, *C. R. Acad. Sci. Paris*, 276 (1973), pp. 1213–1215.

CHEMIN (J.Y.)

[1988] Interaction contrôlée dans les e.d.p. non linéaires strictement hyperboliques, *Bull. Soc. Math. France*, 116 (1988), pp. 346–383.

Christodoulou (D.)

[1986] Global solutions of nonlinear hyperbolic equations for small initial data, *Comm. Pure Appl. Math.*, 39 (1986), pp. 276–282.

Christodoulou (D.) and Klainerman (S.)

[1993] *The globasl stability of Minskowski space.* Princeton Univ. Press, 1993.

Dencker (N.)

[1982] The propagation of polarization for systems of real principal type, *J. Funct. Anal.*, 46 (1982), pp. 351–372.

[1995] The propagation of polarization for systems of transversal type, *Ark. Mat.*, 33 (1995), pp. 249–279.

Dionne (P.A.)

[1962] Sur les problèmes de Cauchy hyperboliques bien posés, *J. Anal. Math.*, 10 (1962), pp. 1–90.

Duistermaat (J.) and Hörmander (L.)

[1972] Fourier integral operators II, *Acta Math.*, 128 (1972), pp. 183–269.

Eskin (G.)

[1977] Parametrix and propagation of singularities for the interior mixed hyperbolic problem, *J. Anal. Math.*, 40 (1977), pp. 17–62.

Friedrichs (K.) and Lewy (H.)

[1928] Über die Eindeutigkeit und das Abhängigkeitsgebiet der Lösungen beim Anfangswertproblem linearer hyperbolischer Differentialgleichungen, *Math. Ann.*, 98 (1928), pp. 192–204.

Gårding (L.)

[1950] Linear hyperbolic partial differential operators with constant coefficients, *Acta Math.*, 85 (1950), pp. 1–62.

[1953] Dirichlet's problem for linear elliptic partial differential operators, *Math. Scand.*, 1 (1953), pp. 55–72.

[1956] Solution directe du problème de Cauchy pour les équations hyperboliques, *Coll. Int. CNRS Nancy, 1956*, pp. 71–89.

[1958] Cauchy's problem for hyperbolic equations, corr. ed., *Univ. of Chicago*, 1958.

[1989] History of the mathematics of double refraction, *Arch. Hist. Exact Sci.*, 40 (1989), pp. 355–385.

Glimm (J.)

[1965] Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.*, 18 (1965), pp. 697–732.

HADAMARD (J.)

[1932] *Le problème de Cauchy et les équations hyperboliques linéaires*, Paris: Hermann, 1932.

HERGLOTZ (G.)

[1926-28] Über die Integration linearer partieller Differentialgleichungen mit konstanten Koeffizienten I-III, *Ber. der Sächsischen Akad. der Wiss.*, 78 (1926), pp. 93–126 and 287–318; *Ibid.*, 80 (1928), pp. 69–114.

HERSH (R.)

[1963] Mixed problems in several variables, *J. Math. Mech.*, 12 (1963), pp. 317–334.

[1964] Boundary conditions for equations of evolution, *Arch. Rational Mech. Anal.*, 16 (1964), pp. 242–264.

HÖRMANDER (L.)

[1963] *Linear Partial Differential Operators*, vol. 116 of Grundlehren, Berlin-Heidelberg: Springer, 1963.

[1965] Pseudodifferential operators, *Comm. Pure Appl. Math.*, 18 (1965), pp. 89–152.

[1970] Linear differential operators, in *Actes Congr. Int. Math., Nice 1970*, vol. 1, pp. 121–133.

[1971] Fourier integral operators I, *Acta Math.*, 127 (1971), pp. 79–183.

[1977] The Cauchy problem for differential equations with double characteristics, *J. Anal. Math.*, 32 (1977), pp. 118–196.

[1983a] *The Analysis of Linear Partial Differential Operators I.* Springer, 1983.

[1983b] *The Analysis of Linear Partial Differential Operators II.* Springer, 1983.

[1985a] *The Analysis of Linear Partial Differential Operators III.* Springer, 1984.

[1985b] *The Analysis of Linear Partial Differential Operators IV*. Springer, 1985.

[1988] Non-linear hyperbolic differential equations, Lectures 1986-87, Dept. Math. Lund Univ., 1988.

[1989] The fully non-linear Cauchy problem with small data, *Bol. Soc. Brasil Mat.*, 20 (1989), pp. 1–27.

[1991] The fully non-linear Cauchy problem with small data II, in *Microlocal Analysis and Nonlinear Waves* (edited by M. Beals, R. Melrose and J. Rauch), vol. 30 of IMA Volumes in Mathematics and its Applications, Berlin: Springer-Verlag, pp. 51–81.

[1993] Hyperbolic systems with double characteristics, *Comm. Pure Appl. Math.*, 46 (1993), pp. 261–301.

IAGOLNITZER (D.)

[1975] Microlocal essential support of distributions and decomposition theorems - An introduction, in *hyperfunctions and theoretical physics*, vol. 449 of Lecture Notes in Mathematics, Springer, 1975, pp. 121–132.

IVRII (V.)

[1978] Sufficient conditions for regular and completely regular hyperbolicity, *Trans. Moscow Math. Soc.*, 1 (1978), pp. 1–65.

[1980] Wave fronts for solutions of boundary problems for a class of symmetric hyperbolic systems, *Siberian Math. J.*, 21 (1980), pp. 527–534.

IVRII (V.) and PETKOV (M.)

[1974] Necessary conditions for the correctness of the Cauchy problem, *Uspekhi Mat. Nauk*, 29-5 (1974), pp. 3–70.

IWASAKI (N.)

[1984] The Cauchy problem for effectively hyperbolic equations (A standard type), *Publ. RIMS*, (A) 20 (1984), pp. 543–584.

JOHN (F.)

[*Papers*] *Collected papers*, vol. 1. Boston-Basel: Birkhäuser, 1985.

[1976] Delayed singularity formation for solutions of nonlinear partial differential equations in higher dimensions, *Comm. Pure Appl. Math.*, 29 (1976), pp. 649–682; *Papers* I, pp. 498–531.

[1979] Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.*, 28 (1979), pp. 235–268; *Papers* I, pp. 532–565.

[1983] Lower bounds for the life span of solutions of nonlinear wave equations in three dimensions, *Comm. Pure Appl. Math.*, 36 (1983), pp. 1–35; *Papers* I, pp. 589–623.

JOHN (F.) and KLAINERMAN (S.)

[1984] Almost global existence to nonlinear wave equations in three space dimensions, *Comm. Pure Appl. Math.*, 37 (1984), pp. 443–455.

KAJITANI (K.) and WAKABAYASHI (S.)

[1994] Microlocal à priori estimates and the Cauchy problem II, *Jap. J. Math.*, 20 (1994), pp. 1–71.

KLAINERMAN (S.)

[1985a] Uniform decay estimates and the Lorentz invariance of the classical wave equation, *Comm. Pure Appl. Math.*, 38 (1985), pp. 321–332.

[1985b] Existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions, *Comm. Pure Appl. Math.*, 38 (1985), pp. 631–641.

[1986] The null condition and global existence to nonlinear wave equations, in *Nonlinear systems of partial differential equations in applied mathematics*, vol. 23-1 of Lectures in Applied Mathematics, AMS, 1986, pp. 293–326.

KOHN (J.J.) and NIRENBERG (L.)

[1965] On the algebra of pseudodifferential operators, *Comm. Pure Appl. Math.*, 18 (1965), pp. 269–305.

KREISS (H.O.)

[1970] Initial boundary value problems for hyperbolic systems, *Comm. Pure Appl. Math.*, 23 (1970), pp. 277–298.

LAX (P.D.)

[1954] Weak solutions of nonlinear hyperbolic equations and their numerical computation, *Comm. Pure Appl. Math.*, 7 (1954), pp. 159–193.

[1957a] Asymptotic solutions of initial value problems, *Duke Math. J.*, 24 (1957), pp. 627–246.

[1957b] Hyperbolic conservation laws II, *Comm. Pure Appl. Math.*, 10 (1957), pp. 537–566.

LEBEAU (G.)

[1989] Equations des ondes semilinéaires II, Controle des singularités et caustiques non linéaires, *Invent. Math.*, 95 (1989), pp. 277–323.

LERAY (J.)

[1933] Études de diverses équations intégrales non-linéaires et de quelques problèmes que pose l'Hydrodynamique, *J. Math. Pures Appl.*, (IX) 12 (1933), pp. 1–82.

[1953] *Hyperbolic Differential Equations*, Princeton: Inst. Adv. Study, 1953.

LI TA-TSIEN and ZHOU YI

[1995] A note on the life-span of classical solutions to nonlinear wave equations in four space dimensions, *Indiana Univ. Math. J.*, 44 (1995), pp. 1207–1248.

LINDBLAD (H.)

[1990a] Blow-up for solutions of $\Box u = |u|^p$ with small initial data, *Comm. Partial Differential Equations*, 15 (1990), pp. 757–821.

[1990b] On the lifespan of solutions of nonlinear wave equations with small initial data, *Comm. Pure Appl. Math.*, 43 (1990), pp. 445–471.

LUDWIG (D.)

[1960] Exact and asymptotic solutions of Cauchy's problem, *Comm. Pure Appl. Math.*, 13 (1960), pp. 473–508.

Maslov (V.P.)

[1964] *Theory of perturbations and asymptotic methods.* Moscow, 1964. (in Russian).

Melrose (R.B.)

[1975] Microlocal parametrix for diffractive boundary problems, *Duke Math. J.*, 42 (1975), pp. 605–635.

[1981] Transformation of boundary problems, *Acta Math.*, 147 (1981), pp. 149–236.

[1983] The Cauchy problem for effectively hyperbolic operators, *Hokkaido Math. J.*, 12 (1983), pp. 371–391.

Melrose (R.B.) and Barreto (A.)

[1994] Non-linear interaction of a cusp and a plane. Preprint.

Melrose (R.B.) and Ritter (N.)

[1985] Interactions of progressive waves for semilinear wave equations I, *Ann. of Math.*, 121 (1985), pp. 187–213.

[1987] Interactions of progressive waves for semilinear wave equations II, *Ark. Mat.*, 25 (1987), pp. 91–114.

Melrose (R.B.) and Sjöstrand (J.)

[1978] Singularities of boundary value problems I, *Comm. Pure Appl. Math.*, 31 (1978), pp. 593–617.

[1982] Singularities of boundary value problems II, *Comm. Pure Appl. Math.*, 35 (1982), pp. 129–168.

Mizohata (S.)

[1961-62] Some remarks on the Cauchy problem, *J. Math. Kyoto Univ.*, 1 (1961-62), pp. 109–127.

Nishitani (T.)

[1984a] A necessary and sufficient condition for the hyperbolicity of second order equations with two independent variables, *J. Math. Kyoto Univ.*, 24 (1984), pp. 91–104.

[1984b] Local energy integrals for effectively hyperbolic operators I, II, *J. Math. Kyoto Univ.*, 24 (1984), pp. 623–658 and 659–698.

[1993] Necessary conditions for strong hyperbolicity of first order systems, *Journal d'Anal. Math.*, 61 (1993), pp. 181–228.

Oleinik (O.A.)

[1970] On the Cauchy problem for weakly hyperbolic equations, *Comm. Pure Appl. Math.*, 23 (1970), pp. 569–586.

PETROVSKY (I.G.)

[1937] Über das Cauchysche Problem für Systeme von partiellen Differentialgleichungen, *Mat. Sb.*, 2-44 (1937), pp. 815–870.

[1938] Über das Cauchysce Problem für Systeme linearer Differentialgleichungen und nichtanalytische Funktionen, *Bull. Moscow Univ. Series math. and mech.*, 1-7 (1938), pp. 1–79.

[1945] On the diffusion of waves and lacunas for hyperbolic equations, *Mat. Sb.*, 17-59 (1945), pp. 289–370.

RAUCH (J.) and REED (M.)

[1982] Nonlinear microlocal analysis of semilinear hyperbolic systems in one space dimension, *Duke Math. J.*, 49 (1982), pp. 397–475.

RIEMANN (B.)

[1860] Über die Fortpflanzug ebener Luftwellen von endlicher Schwingungsweite, *Abh. Kgl. Ges. Wiss. Göttingen*, 8 (1860); *Gesammelte Mathematische Werke*, Springer/Teubner, 1990, pp. 157–175.

RIESZ (M.)

[1949] L'intégrale de Riemann-Liouville et le problème de Cauchy, *Acta Math.*, 81 (1949), pp. 1–223.

SAKAMOTO (R.)

[1974] E-wellposedness for hyperbolic mixed problems with constant coefficients, *J. Math. Kyoto Univ.*, 14 (1974), pp. 93–118.

SATO (M.)

[1969] Hyperfunctions and partial differential operators. In *Proc. Int. Conf. on Functional Analysis and Related Topics, Tokyo 1969*, pp. 91–94.

[1970] Regularity of hyperfunction solutions of partial differential equations. In *Actes Congr. Int. Math., Nice 1970*, vol. 2, pp. 785–794.

SCHAUDER (J.)

[1935] Das Anfangwertproblem einer hyperbolischen Differentialgleichung zweiter Ordnung in beliebiger Anzahl von unabhängigen Veränderlichen, *Fund. Math.*, 24 (1935), pp. 213–246.

[1936a] Quasilineare Differentialgleichungen zweiter Ordnung vom hyperbolischen Typus. Gemischte Randwertaufgabe, *Studia Math.*, 6 (1936), pp. 162–189. (with Krzyzanski).

[1936b] Gemischte Randwertaufgaben bei partiellen Differentialgleichungen vom hyperbolischen Typus, *Studia Math.*, 6 (1936), pp. 190–198.

Smoller (J.A.)

[1983] *Shock waves and reaction-diffusion equations*, vol. 258 of Gundlehren, Springer, 1983.

Svensson (L.)

[1969] Necessary and sufficient conditions for hyperbolicity of polynomials with a hyperbolic principal part, *Ark. Mat.*, 8 (1969), pp. 145–162.

Taylor (M.)

[1976] Grazing rays and reflection of singularities of solution to wave equations, *Comm. Pure Appl. Math.*, 29 (1976), pp. 1–38.

Thomée (V.)

[1957] Estimates of the Friedrichs-Lewy type for mixed problems, *Math. Scand.*, 5 (1957), pp. 93–113.

Young (R.)

[1993] Sup-norm stability of Glimm's scheme, *Comm. Pure Appl. Math.*, 46 (1993), pp. 903–948.

[1995] Exact solutions to degenerate conservation laws. Preprint.