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SOME LOCAL AND NON-LOCAL VARIATIONAL PROBLEMS IN RIEMANNIAN GEOMETRY

by

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Abstract. — In this article we will give a brief summary of some recent work on two variational problems in Riemannian geometry. Although both involve the study of elementary symmetric functions of the eigenvalues of the Ricci tensor, as far as technique and motivation are concerned the problems are actually quite different.

***Résumé* (Problèmes variationnels locaux et non-locaux en géométrie riemannienne)**
Dans cet article nous donnons un aperçu d'un travail récent sur deux problèmes variationnels en géométrie riemannienne. Bien que les deux problèmes soient basés sur l'étude des fonctions symétriques élémentaires des valeurs propres du tenseur de Ricci, les techniques et les motivations sont en réalité différentes.

For since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear. –Leonhard Euler

1. Quadratic Riemannian functionals

The first problem we will discuss represents joint work of the author with Jeff Viaclovsky ([**GV00**]). To describe it, let us begin with some general notions.

Let M be a smooth manifold, $\mathcal{M}$ the space of smooth Riemannian metrics on M, and $\mathcal{G}$ the diffeomorphism group of M. A functional $F : \mathcal{M} \to \mathbb{R}$ is called *Riemannian* if F is invariant under the action of $\mathcal{G}$; i.e., if $F(\phi^* g) = F(g)$ for each $\phi \in \mathcal{G}$ and $g \in \mathcal{M}$. If we endow $\mathcal{M}$ with a natural L^2–Sobolev norm, then we may speak of *differentiable* Riemannian functionals. Letting $S_2(M)$ denote the bundle of symmetric two–tensors, we then say that $F : \mathcal{M} \to \mathbb{R}$ has a *gradient at* $g \in \mathcal{M}$ if $\dfrac{d}{dt} F[g+th]|_{t=0} = \int g(h, \nabla F)\, d\,\mathrm{vol}_g$ for some $\nabla F \in \Gamma(S_2(M))$ and all $h \in \Gamma(S_2(M))$.

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An important example of a Riemannian functional is of course the total scalar curvature

$$S[g] = \int R_g \, d\,\mathrm{vol}_g \tag{1.1}$$

where R_g denotes the scalar curvature of g. For Riemannian geometers, the importance of (1.1) lies in the fact that when M is compact, critical points of $S|_{\mathcal{M}_1}$, where

$$\mathcal{M}_1 = \{g \in \mathcal{M} | \ \mathrm{Vol}(g) = 1\},$$

are Einstein (see [**Bes87**]). In the Lorentzian setting, Hilbert showed that the equations of general relativity can be realized in a similar manner ([**Hil72**]).

Our interest here is in functionals that are obtained by integrating a polynomial which is quadratic in the curvature. By Weyl's invariant theory ([**Wey39**]), a basis for these functionals is

$$\begin{gathered} \mathcal{R}[g] = \int |\,\mathrm{Riem}_g\,|^2 \, d\,\mathrm{vol}_g, \quad \rho[g] = \int |\,\mathrm{Ric}_g\,|^2 \, d\,\mathrm{vol}_g, \\ \tau[g] = \int R_g^2 \, d\,\mathrm{vol}_g, \end{gathered} \tag{1.2}$$

where Riem_g and Ric_g denote respectively the Riemann curvature tensor and Ricci curvature tensor of g. Such functionals arise in certain field theories in physics; in particular $\mathcal{R}$ can be viewed as a Riemannian analogue of Yang–Mills (see [**Bac21**], [**Bou96**], [**Bes87**]).

From the variational point of view, the functionals in (1.2) have the apparent advantage of being bounded below, and thus more amenable to the direct method. However, the associated Euler equations are quite complicated (see [**And97**], [**Bes87**], [**Lam98**]). Indeed, in [**Lam98**] a critical point of $\mathcal{R}$ is constructed on S^3 which does not have constant sectional curvature. Thus, even if successful, it is not clear that such an approach would yield Einstein metrics (under certain geometric and topological constraints there are some exceptions; see [**Gur98**]).

Before we give an exact description of the functional we will be interested in, for the purpose of motivation it may be helpful to first recall a basic fact about the decomposition of the curvature tensor (see [**Bes87**]). Let $\odot$ denote the Kulkarni-Nomizu product, and define the tensor $C_g = \mathrm{Ric} - \frac{R}{2(n-1)} g$. Then the full curvature tensor of g can be decomposed as

$$\mathrm{Riem} = W + \frac{1}{(n-2)} C \odot g,$$

where W denotes the Weyl curvature tensor of g. In three dimensions, we have $C_g = \mathrm{Ric} - \frac{R}{4} g$, and the Weyl tensor vanishes. Thus, the full curvature tensor is actually determined by C_g.

Now if $\sigma_k : \mathbb{R}^3 \to \mathbb{R}$ denote the elementary symmetric functions, then the scalar curvature can be expressed as $R = 4\sigma_1(C)$. It follows that the natural quadratic counterpart to (1.1) is the functional:

$$\mathcal{F}_2[g] = \int_M \sigma_2 \left(C_g \right) \, d \operatorname{vol}_g . \tag{1.3}$$

A simple calculation gives

$$\mathcal{F}_2[g] = \int_M \left(-\frac{1}{2} | \operatorname{Ric} |^2 + \frac{3}{16} R^2 \right) \, d \operatorname{vol}_g . \tag{1.4}$$

$\mathcal{F}_2$ is therefore quadratic in the curvature of g, and is a non–convex linear combination of the functionals in (1.2).

There are interesting parallels between the functionals S and $\mathcal{F}_2$. Like the total scalar curvature, $\mathcal{F}_2$ is neither bounded above nor below on $\mathcal{M}_1$. Further, one can consider a constrained version of $\mathcal{F}_2$ by restricting to a fixed conformal class; see [**Viaa**], [**Viab**]. In these works, the Euler equation for $\mathcal{F}_2|_{[g]_1}$, where

$$[g]_1 = \{ \tilde{g} = e^{2w} g, w \in C^{\infty}(M, \mathbf{R}) | \operatorname{Vol}(\tilde{g}) = 1 \},$$

is shown to be $\sigma_2(\operatorname{Ric} - \frac{1}{4} Rg) \equiv \lambda = \text{constant}$. Remarkably, this scalar equation encodes information about the sectional curvatures of g, provided $\lambda > 0$:

Proposition 1.1. — *Let M be three-dimensional. If $\sigma_2(\operatorname{Ric} - \frac{1}{4} Rg)_x > 0$ then the sectional curvatures of g at x are either all positive or all negative. In particular, critical points of $\mathcal{F}_2|_{[g]_1}$ with $\mathcal{F}_2[g] > 0$ have either strictly positive or strictly negative sectional curvature.*

Moreover, we have the following new characterization of (compact) Einstein three-manifolds:

***Theorem 1.1* ([GV00])**. — *Let M be compact and three–dimensional. Then a metric g with $\mathcal{F}_2[g] \geq 0$ is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ if and only if g has constant sectional curvature.*

Remark

1. The condition $\mathcal{F}_2[g] \geq 0$ in Theorem 1.1 is necessary: if $E = \operatorname{Ric} - \frac{1}{3} Rg$ denotes the trace–free Ricci tensor, then

$$\sigma_2(C) = \sigma_2 \left(\operatorname{Ric} - \frac{1}{4} Rg \right) = -\frac{1}{2} |E|^2 + \frac{1}{48} R^2. \tag{1.5}$$

   Thus, if g has constant curvature, $\sigma_2 = \frac{1}{48} R^2 \geq 0$.
2. The condition $\mathcal{F}_2[g] > 0$ may be thought of as an *ellipticity* assumption. To our knowledge, this is the first example of a quadratic Riemannian functional in three dimensions whose elliptic critical points are necessarily of constant curvature.
3. The case $\mathcal{F}_2[g] = 0$ is the case of degenerate ellipticity, and the proof in this case is much more delicate, as the curvature may change sign.

4. When $\mathcal{F}_2[g] < 0$, we have left the region of ellipticity, and we do not expect a simple classification of these critical points. Indeed, the construction of [**Lam98**] provides an example of a critical metric on S^3 with $\mathcal{F}_2[g] < 0$.

In [**GV00**] we also considered a constrained version of the problem: $\mathcal{F}_2|_{\Xi}$, where

$$\Xi = \left\{ g \in \mathcal{M}_1 \middle| \; \sigma_2(C_g) = \int_M \sigma_2(C_g)\, d\,\mathrm{vol}_g > 0, \text{ and } R_g < 0 \right\}.$$

In analogy with the work of Koiso for the scalar curvature (see [**Koi79**]), one can show that Ξ is in fact a submanifold of $\mathcal{M}_1$. Restricting to Ξ introduces a Lagrange multiplier term into the Euler equation, and like the corresponding problem for the scalar curvature we can show that this term vanishes:

Theorem 1.2 **([GV00])**. — *Let M be compact and three–dimensional. If g is a critical point of $\mathcal{F}_2|_{\Xi}$, then g is hyperbolic.*

The proof of Theorem 1.1 naturally divides into two cases: first, assuming the critical metric g has $\sigma_2(C) > 0$, then the more difficult case of $\sigma_2(C) = 0$. The former case further divides into two parts, according to whether the scalar curvature is strictly positive or strictly negative.

The Euler equation for $\mathcal{F}_2$ is quite complicated; see [**GV00**] for a detailed account of the first variation. The precise formula is:

$$\begin{aligned}(\nabla\mathcal{F}_2)_{ij} = \frac{1}{2}\Delta E_{ij} + \frac{1}{24}\Delta R g_{ij} - \frac{1}{8}\nabla_i\nabla_j R \\ - 2E_{im}E_{mj} - \frac{5}{24}RE_{ij} + \frac{1}{36}R^2 g_{ij} - \frac{3}{2}\sigma_2(C)g_{ij}.\end{aligned} \tag{1.6}$$

For the proof of the case when $\sigma_2(C) > 0$ and $R > 0$ it will be helpful to introduce the tensor $T = -\operatorname{Ric} + \frac{1}{2}Rg$. The significance of T is the following: suppose Π is a non-degenerate tangent plane in T_pM for some $p \in M$. If $u \in T_pM$ is a unit normal to Π, then the sectional curvature of Π is $T(u,u)$. In particular, if $\sigma_2(C) > 0$ and $R > 0$ then by Proposition 1.1 the tensor T is positive definite. In fact, the same argument shows that when $R > 0$ but $\sigma_2(C) \geq 0$, then T is positive semi-definite.

Now suppose that g is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$. Taking the inner product with E on both sides of (1.6) we get

$$\begin{aligned}\frac{1}{4}T^{ij}\nabla_i\nabla_j R = \Delta\sigma_2(C) + |\nabla E|^2 - \frac{1}{24}|\nabla R|^2 \\ + 4\operatorname{tr} E^3 + \frac{5}{12}R|E|^2 + 2g(\nabla\mathcal{F}_2, E),\end{aligned} \tag{1.7}$$

where $\operatorname{tr} E^3 = E_i^j E_j^k E_k^i$. Since g is critical, $\nabla\mathcal{F}_2 = 0$ and $\Delta\sigma_2(C) = 0$, so

$$\frac{1}{4}T^{ij}\nabla_i\nabla_j R = |\nabla E|^2 - \frac{1}{24}|\nabla R|^2 + 4\operatorname{tr} E^3 + \frac{5}{12}R|E|^2. \tag{1.8}$$

To show that $E = 0$ when $\sigma_2(C) > 0$ and $R > 0$ we use the maximum principle, which requires the following lemma:

Lemma 1.1. — *Suppose g is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ and $\sigma_2(C) \geq 0$. Let $U \subset M$ be an open set on which $R > 0$. Then in U,*

$$\frac{1}{4} T^{ij} \nabla_i \nabla_j R \geq \frac{1}{\sqrt{6}} |E|^3. \tag{1.9}$$

Proof. — Since $\sigma_2(C)$ is a non-negative constant, it is easy to see that

$$|\nabla E|^2 \geq \frac{1}{24} |\nabla R|^2. \tag{1.10}$$

If we substitute this into (1.8) we obtain

$$\frac{1}{4} T^{ij} \nabla_i \nabla_j R \geq 4 \operatorname{tr} E^3 + \frac{5}{12} R |E|^2. \tag{1.11}$$

Using the sharp inequality

$$\operatorname{tr} E^3 \geq -\frac{1}{\sqrt{6}} |E|^3, \tag{1.12}$$

we conclude

$$\frac{1}{4} T^{ij} \nabla_i \nabla_j R \geq -\frac{4}{\sqrt{6}} |E|^3 + \frac{5}{12} R |E|^2. \tag{1.13}$$

Since $\sigma_2(C) \geq 0$, we have $R \geq 2\sqrt{6}|E|$, thus

$$\frac{1}{4} T^{ij} \nabla_i \nabla_j R \geq -\frac{4}{\sqrt{6}} |E|^3 + \frac{5}{12} 2\sqrt{6} |E|^3 = \frac{1}{\sqrt{6}} |E|^3. \tag{1.14}$$

□

Now if $\sigma_2(C) > 0$ and $R > 0$ on M, then we can apply Lemma 1.1 on $U = M$. Since $T > 0$, we conclude by the maximum principle that $E \equiv 0$ on M.

The case where $\sigma_2(C) > 0$ and $R < 0$ requires a different idea. The argument in ([**GV00**]) is very much inspired by the work of Koiso ([**Koi78**]) and Bourguignon ([**Bou81**]). Here we will offer a different (but equivalent) argument which seems more natural, in part because it sheds some light on the rather roccoco expression for the gradient in (1.6).

Note that the tensor C, being a section of $S_2(M)$, can alternatively be viewed as a one-form with values in the cotangent bundle T^*M. We will write this as $C \in \Omega^1(T^*M)$. Now consider the complex

$$\Omega^0(T^*M) \to \Omega^1(T^*M) \to \Omega^2(T^*M) \to \cdots \tag{1.15}$$

The Riemannian connection $\nabla : \Omega^0(T^*M) \to \Omega^1(T^*M)$, and induces the exterior derivative $d^\nabla : \Omega^1(T^*M) \to \Omega^2(T^*M)$. We also have the adjoint maps $\delta^\nabla : \Omega^2(T^*M) \to \Omega^1(T^*M)$ and $\nabla^* : \Omega^1(T^*M) \to \Omega^0(T^*M)$. Note that ∇^* is just the usual divergence operator on symmetric two-tensors. Moreover, a manifold is locally conformally flat if and only if the tensor C satisfies $d^\nabla C \equiv 0$.

Using these operators, we can now give an alternative description of the Euler equation for $\mathcal{F}_2$:

Lemma 1.2. — *The gradient of $\mathcal{F}_2$ is given by*

$$\nabla \mathcal{F}_2 = -\frac{1}{4}\delta^{\nabla} d^{\nabla} C + \Lambda_C, \tag{1.16}$$

where Λ_C is a tensor which is given in local coordinates by

$$(\Lambda_C)_{ij} = -\frac{1}{2}C_{ik}C_{jk} + \frac{1}{4}|C|^2 g_{ij} + \frac{1}{8}RC_{ij} - \frac{1}{64}R^2 g_{ij}. \tag{1.17}$$

Aside from aesthetic considerations, the advantage of writing the gradient of $\mathcal{F}_2$ in this form is two-fold: first, it shows that the highest order terms in (1.6) are collectively a Hodge-laplacian defined on the appropriate bundle. The second advantage can been seen from the following lemma:

Lemma 1.3. — *If the scalar curvature $R \leq 0$, then there is a positive constant a such that the following inequality holds:*

$$g(\Lambda_C, C) \leq -a|E|\sigma_2(C). \tag{1.18}$$

Moreover, if equality holds in (1.18), then the trace-free Ricci tensor E of g must have eigenvalues $\lambda, \lambda, -2\lambda$ for some λ.

Combining (1.16) and (1.18), we can now complete the proof of the case when $\sigma_2(C) > 0$ and $R < 0$. Taking the inner product of both sides of (1.16) with C and integrating over M, then appealing to (1.18), we conclude

$$0 \leq \int_M \left(-\frac{1}{4}|d^{\nabla} C|^2 - a|E|\sigma_2(C) \right) d\,\mathrm{vol}\,. \tag{1.19}$$

Therefore, $E \equiv 0$.

When $\sigma_2(C) \equiv 0$, then the above argument does not quite work. The first (and most serious) obstacle is that we have no *a priori* knowledge of the sign of R. However, appealing to Lemma 1.1, we can actually argue that $R \leq 0$ on M. For, suppose $U = \{p \in M : R(p) > 0\}$ is non-empty. Let $p \in U$ be a point at which R attains its maximum. At p the Hessian of R is negative semi-definite, while T is positive semi-definite. It follows from (1.9) that $|E| = 0$ at p. Since $\sigma_2(C) = -\frac{1}{2}|E|^2 + \frac{1}{48}R^2 = 0$, we conclude that $R(p) = 0$.

This fact allows us to appeal to (1.18), and repeat the argument for the case when $R < 0$. Since $\sigma_2(C) \equiv 0$, though, we can't conclude that $|E| \equiv 0$. But we *can* conclude two things: first, (1.19) tells us that $d^{\nabla} C = 0$; i.e., (M, g) is locally conformally flat. Second, equality must hold in (1.18) at each point of M. These observations give us the following characterization of critical metrics with $\sigma_2(C) \equiv 0$:

Theorem 1.3. — *Let M be compact and three-dimensional. Then g is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ with $\mathcal{F}_2[g] = 0$ if and only if $R \leq 0$, g is locally conformally flat, and the eigenvalues of the tensor C_g are $\{0, 0, R/4\}$.*

Using (1.3), we can provide a local classification of such critical metrics:

Theorem 1.4. — *Let M be compact and three-dimensional. If g is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ with $\mathcal{F}_2[g] = 0$ then for $p \in M$, either*
(i) the sectional curvature vanishes at p, or
(ii) there exists a local coordinate system $\{x, y, t\}$ around p mapping a neighborhood of p to a cube in $\mathbf{R}^3$ in which the metric g takes the form

$$g = dx^2 + dy^2 + f(x, y, t)^2 dt^2, \tag{1.20}$$

where

$$f(x, y, t) = a(t)(x^2 + y^2) - 2b(t)x - 2c(t)y + d(t).$$

with $a(t), b(t), c(t), d(t)$ some functions of t.

We are now in a position to at least sketch the proof of the final case of the main theorem. The actual arguments are fairly delicate, so we can only provide an overview. But hopefully some flavor of the complexities will be preserved in this necessarily brief recounting.

We begin by assuming that our critical metric is not flat. It then follows that the scalar curvature attains a global negative minimum at some point $p \in M$. Near p, we can refine the local expression (1.20) somewhat; in particular, f takes the form

$$f(x, y, t) = (x - b(t))^2 + (y - c(t))^2 + d(t), \tag{1.21}$$

with $d(t) > 0$, and $b(0) = c(0) = 0$. We now argue that the coordinates $\{x, y, t\}$ can be extended *locally* in t, but *globally* in $\{x, y\}$. The point is the following: at each point of M where the scalar curvature is negative, the tensor C has precisely two eigenvalues, 0 (multiplicity 2) and $R/4$ (multiplicity 1). Thus, the tangent space splits into two subspaces; this gives rise to two distributions which we call V_0 and $V_{R/4}$. Note that these distributions are well-defined away from the zero set of R. Now, since C is a Codazzi tensor, both V_0 and $V_{R/4}$ are integrable. In particular, the integral manifolds of V_0 induce a foliation by flat, totally geodesic leaves. Using the exponential map, we show that the $\{x, y\}$ coordinates are globally defined on a leaf through a minimum point of R.

Indeed, near such a minimum point, the leaves of the foliation can be parametrized in a natural way. This allows us to define a diffeomorphism between Euclidean three-space and an open union of leaves. Using the $\{x, y, t\}$ coordinates to compute the pull-back of the volume form under this diffeomorphism, we see that the induced volume is infinite. Since M is compact, it must be flat.

Remarks

1. The local formula for the metric in (1.4) can be used to give many examples of complete, non-compact manifolds which are critical for $\mathcal{F}_2$ in the sense that they are stationary over all compactly supported variations.

2. The constrained problem of restricting to the space of metrics with $\sigma_2(C) \equiv$ *const.* > 0 and $R < 0$ requires some technical background. For this reason we will not provide the details here. But the essential point, as we observed above, is that the Lagrange multiplier term vanishes, reducing the analysis to one of the cases above.
3. It is natural to ask under what conditions a three-manifold M admits a metric with $\sigma_2(C) \geq 0$. First, note that if M admits a metric g with $\sigma_2(C) \equiv 0$ but no metric with $\sigma_2(C) > 0$, then g is necessarily critical for $\mathcal{F}_2$. It follows that g is flat. If M admits a metric g with $\sigma_2(C) > 0$, then g has either strictly positive or strictly negative sectional curvature. In the positive case, in view of Hamilton's work ([**Ham82**]), M must also admit a metric of constant positive sectional curvature. So it remains to characterize three-manifolds which admit a metric with $\sigma_2(C) > 0$ and $R < 0$. This is potentially a very important question: it has been conjectured for some time that a negatively curved three-manifold admits a hyperbolic metric. The following conjecture therefore seems reasonable:

   Conjecture. — *A compact three-manifold M admits a metric g with $\sigma_2(C_g) > 0$ if and only if M admits a metric with constant sectional curvature.*

2. Four-manifolds with positively pinched Ricci curvature

In four dimensions, the tensor C appearing in the decomposition of the curvature tensor described above is given by $C = \mathrm{Ric} - \frac{1}{6}Rg$. Moreover, the integral

$$\int_M \sigma_2(C)\, d\,\mathrm{vol} \tag{2.1}$$

is conformally invariant. This just follows from the Chern-Gauss-Bonnet formula for four-manifolds:

$$8\pi^2\chi(M) = \int_M \left(|W|^2 + \sigma_2(C)\right)\, d\,\mathrm{vol}\,. \tag{2.2}$$

While the positivity of $\sigma_2(C)$ no longer imposes a sign condition on the sectional curvature (as it did in three dimensions), it does impose a sign on the *Ricci* curvature:

Lemma 2.1. — *Suppose $\sigma_2(C) > 0$. Then either $R > 0$ or $R < 0$. Moreover,* $\mathrm{Ric} > 0$ *(resp.* $\mathrm{Ric} < 0$*) and $S = -\mathrm{Ric} + \frac{1}{2}Rg > 0$ (resp. $S < 0$) assuming $R > 0$ (resp. $R < 0$).*

Note that the positivity of the tensor S implies that each eigenvalue of Ric is positive, but less than the sum of the other three.

In joint work with S.Y.A. Chang and P. Yang ([**CGY00**]), we have proved an existence result for metrics with $\sigma(C) > 0$, only assuming a sign on the integral in (2.1) and the positivity of the scalar curvature:

Theorem 2.1. — *Assume that (M, g_0) is a compact four-manifold of positive scalar curvature with*

$$\int_M \sigma_2(C_{g_0})\, d\,\mathrm{vol}_{g_0} > 0. \tag{2.3}$$

Then there is a conformal metric $g = e^{2w} g_0$ with $\sigma_2(C_g) > 0$.

In particular, this result gives a method for constructing a large class of conformal four-manifolds with positively pinched Ricci curvature.

The proof of Theorem 2.1 is quite involved, but some essential features are worth noting here. The basic idea is to introduce a regularized problem, then show that one can take the appropriate limit to construct a metric with $\sigma_2 > 0$.

More precisely, we actually consider two additional variational problems, one local and one non-local. The first is the (local) functional

$$g \longmapsto \int R_g^2\, d\,\mathrm{vol}_g\,. \tag{2.4}$$

If we restrict this functional to a fixed conformal class of metrics, then the gradient is given by $-6\Delta R$.

The second functional is non-local, and arises in spectral theory. If we let $L = -6\Delta + R$ denote the conformal Laplacian, then one can introduce a regularized notion of the determinant of L (see [**BO91**]). Fixing a metric g_0, for any conformal metric g we consider the functional

$$g \longmapsto \log \frac{\det(L_g)}{\det(L_{g_0})}. \tag{2.5}$$

The gradient for this functional is $|W|^2 + \frac{1}{3}\Delta R - 2\sigma_2(C)$, where W is the Weyl curvature tensor (see [**CY95**]). Therefore, by taking an appropriate linear combination of these two actions, one arrives at our regularized equation:

$$\sigma_2(C) = \delta \Delta R + c_0 |W|^2 \tag{2.6}$$

where c_0 is a positive constant.

The idea, if not the details, should now be clear: first, we need to show that (2.6) admits a solution for any $\delta > 0$. Second, we need to study what happens as $\delta \to 0$.

For the existence part, we rely on the work of Chang and Yang ([**CY95**]), which gives sufficient conditions for the existence of extremals for (2.5). This corresponds to establishing the existence of solutions to (2.6) with $\delta \gg 0$. For small values of $\delta > 0$ we use the continuity method, and this requires us to first understand the linearized problem.

The linearized operator is fourth order, and the principal symbol depends on δ. Thus, as $\delta \to 0$, invertibility becomes a delicate issue. Moreover, when δ is small, the lower order terms in the regularized equation become important. If we take $\delta = 0$ in (2.6) then the resulting equation is of Monge-Ampere type; thus, our estimates for the regularized equation are very much inspired by the methods developed for real

Monge-Ampere equations. However, our analysis uses integral, and not pointwise, estimates: (2.6) is fourth order, so the maximum principle is not available. In any case, we are eventually able to obtain strong enough Sobolev estimates to allow us to take $\delta \to 0$.

It will be interesting to see whether the methods developed in ([**CGY00**]) an be applied to other geometric variational problems.

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