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THE ELLIPTIC REPRESENTATION OF THE SIXTH PAINLEVÉ EQUATION

by

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Abstract. — We find a class of solutions of the sixth Painlevé equation corresponding to *almost all* the monodromy data of the associated linear system; actually, all data but one point in the space of data. We describe the critical behavior close to the critical points by means of the elliptic representation, and we find the relation among the parameters at the different critical points (connection problem).

Résumé **(Représentation elliptique de l'équation de Painlevé VI).** — Nous exhibons une classe de solutions de l'équation de Painlevé VI prenant en compte *presque toutes* les données de monodromie du système linéaire associé ; en fait, toutes les données sauf un point de l'espace des données de monodromie.

Nous décrivons le comportement critique au voisinage de chaque point critique au moyen de la représentation elliptique. Nous explicitons les relations liant les paramètres aux différents points critiques (problème de connexion).

1. Introduction

In this paper, I review some results [**6, 7**] on the elliptic representation of the general Painlevé 6 equation (PVI in the following). I would like to explain the motivations which brought me to study the elliptic representation, and the problems which such an approach has solved.

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The sixth Painlevé equation is

$$\text{(PVI)}\quad \frac{d^2y}{dx^2}=\frac{1}{2}\left[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right]\frac{dy}{dx}$$
$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right].$$

The generic solution has essential singularities and/or branch points in $0,1,\infty$. These points will be called *critical.* The other singularities, which depend on the initial conditions, are poles. The behavior of a solution close to a critical point is called *critical behavior.* A solution of PVI can be analytically continued to a meromorphic function on the universal covering of $\mathbf{P}^1\backslash\{0,1,\infty\}$. For generic values of the integration constants and of the parameters α,β,γ,δ, it can not be expressed via elementary or classical transcendental functions. For this reason, it is called a *Painlevé transcendent.*

The first analytical problem with Painlevé equations is to determine the critical behavior of the transcendents at the critical points. Such a behavior must depend on two parameters (integration constants). The second problem, called *connection problem*, is to find the relation between the couples of parameters at different critical points.

2. Previous Results

The work of Jimbo [**9**] is the fundamental paper on the subject. For generic values of α, β, γ δ, PVI admits a 2-parameter class of solutions, with the following critical behavior: .

$$y(x)=a^{(0)}x^{1-\sigma^{(0)}}(1+O(|x|^{\epsilon})),\qquad x\to 0, \tag{1}$$

$$y(x)=1-a^{(1)}(1-x)^{1-\sigma^{(1)}}(1+O(|1-x|^{\epsilon})),\qquad x\to 1, \tag{2}$$

$$y(x)=a^{(\infty)}x^{\sigma^{(\infty)}}(1+O(|x|^{-\epsilon})),\qquad x\to\infty, \tag{3}$$

where ϵ is a small positive number, $a^{(i)}$ and $\sigma^{(i)}$ are complex numbers such that $a^{(i)}\neq 0$ and

$$0\leq\Re\sigma^{(i)}<1. \tag{4}$$

We remark that x converges to the critical points *inside a sector* with vertex on the corresponding critical point. The connection problem is to finding the relation among the three pairs $(\sigma^{(i)},a^{(i)})$, $i=0,1,\infty$. In [**9**] the problem is solved by the isomonodromy deformations theory. Actually, PVI is the isomonodromy deformation equation of a Fuchsian system of differential equations [**12, 10, 11**]

$$\frac{dY}{dz}=A(z;x)Y,\qquad A(z;x):=\left[\frac{A_0(x)}{z}+\frac{A_x(x)}{z-x}+\frac{A_1(x)}{z-1}\right].$$

The 2×2 matrices $A_i(x)$ ($i=0,x,1$ are labels) depend on x in such a way that the monodromy of a fundamental solution $Y(z,x)$ does not change for small deformations of x. They also depend on the parameters $\alpha,\beta,\gamma,\delta$ of PVI. Here, we use the same notations of the paper [**9**]: namely, $A_0(x)+A_1(x)+A_x(x)=-\frac{1}{2}\mathrm{diag}(\theta_\infty,-\theta_\infty)$; the eigenvalues of $A_i(x)$ are $\pm\frac{1}{2}\theta_i$, $i=0,1,x$, and

$$\alpha=\frac{1}{2}(\theta_\infty-1)^2,\quad -\beta=\frac{1}{2}\theta_0^2,\quad \gamma=\frac{1}{2}\theta_1^2,\quad \left(\frac{1}{2}-\delta\right)=\frac{1}{2}\theta_x^2. \tag{5}$$

The equations of monodromy-preserving deformation (Schlesinger equations), can be written in Hamiltonian form [**15**] and reduce to PVI, being the transcendent $y(x)$ solution of $A(y(x);x)_{1,2}=0$.

Let M_0, M_1, M_x be the monodromy matrices at $z=0,1,x$, for a given basis in the fundamental group of $\mathbf{P}^1\backslash\{0,1,x,\infty\}$.There is a one to one correspondence[(1)] between a given choice of *monodromy data* θ_0, θ_x, θ_1, θ_∞, $\mathrm{tr}(M_0M_x)$, $\mathrm{tr}(M_0M_1)$, $\mathrm{tr}(M_1M_x)$ and a transcendent $y(x)$ (see [**9, 2, 6**]) . Namely:

$$y(x)=y\big(x;\,\theta_0,\theta_x,\theta_1,\theta_\infty,\mathrm{tr}(M_0M_x),\mathrm{tr}(M_0M_1),\mathrm{tr}(M_1M_x)\big). \tag{6}$$

We remark that $\theta_0,\theta_x,\theta_1,\theta_\infty$ specify the equation. Only two of $\mathrm{tr}(M_0M_x)$, $\mathrm{tr}(M_0M_1)$, $\mathrm{tr}(M_1M_x)$ are independent, because, for a given choice of the basis of loops in $\mathbf{P}^1\backslash\{0,1,x,\infty\}$, we have $M_\infty=M_1M_xM_0$. This implies

$$\begin{aligned}\cos(\pi\theta_0)\mathrm{tr}(M_1M_x)+\cos(\pi\theta_1)\mathrm{tr}(M_0M_x)+\cos(\pi\theta_x)\mathrm{tr}(M_1M_0)\qquad&\\ =2\cos(\pi\theta_\infty)+4\cos(\pi\theta_1)\cos(\pi\theta_0)\cos(\pi\theta_x).&\end{aligned}$$

A transcendent in the class (1) (2) (3) above, coincides with a transcendent (6), for:

$$\begin{aligned}2\cos(\pi\sigma^{(0)})&=\mathrm{tr}(M_0M_x),\\ 2\cos(\pi\sigma^{(1)})&=\mathrm{tr}(M_1M_x),\\ 2\cos(\pi\sigma^{(\infty)})&=\mathrm{tr}(M_0M_1)\end{aligned} \tag{7}$$

and

$$a^{(i)}=a^{(i)}\big(\sigma^{(i)};\theta_0,\theta_x,\theta_1,\theta_\infty,\mathrm{tr}(M_0M_x),\mathrm{tr}(M_0M_1),\mathrm{tr}(M_1M_x)\big),\qquad i=0,1,\infty. \tag{8}$$

Formula (8) for $a^{(0)}$, can be derived from (1.8), (1.10) and (2.15) of [**9**][(2)]. It can be derived also from (A.6), (A.28), (A.29) of [**7**] (note that in [**7**] I miss-printed (A.30),

(1) If $\theta_0,\ \theta_x,\ \theta_1,\ \theta_\infty\ \not\in\mathbf{Z}$.

(2) The connection problem is solved in [**9**] for generic values of α, β, γ, δ . More precisely, by *generic case* we mean:

$$\theta_0,\ \theta_x,\ \theta_1,\ \theta_\infty\ \not\in\mathbf{Z};\qquad \frac{\pm\sigma^{(i)}\pm\theta_1\pm\theta_\infty}{2},\ \frac{\pm\sigma^{(i)}\pm\theta_0\pm\theta_x}{2}\ \not\in\mathbf{Z}. \tag{9}$$

The signs $\pm$ vary independently. This is a technical condition which can be abandoned. For example, the non-generic case $\beta=\gamma=1-2\delta=0$ and α any complex number was analyzed in [**2**], for its relevant applications to Frobenius manifolds. Its elliptic representation is discussed in [**6**].

which can be anyway corrected using (A.28), (A.29). Also in formula (1.8) of [**9**] there is a miss-print, I think: the last sign is $\pm$ and not $\mp$.).

(10)
$$
\begin{aligned}
a^{(0)} = \frac{1}{4}\,&\frac{[(\theta_x+\sigma^{(0)})^2-\theta_0^2][\theta_\infty+\theta_1+\sigma^{(0)}]}{{\sigma^{(0)}}^2[\theta_\infty+\theta_1-\sigma^{(0)}]}\\
&\times\frac{\Gamma(1+\sigma^{(0)})^2\Gamma\left(\frac{1}{2}(\theta_0+\theta_x-\sigma^{(0)})+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0-\sigma^{(0)})+1\right)}{\Gamma(1-\sigma^{(0)})^2\Gamma\left(\frac{1}{2}(\theta_0+\theta_x+\sigma^{(0)})+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0+\sigma^{(0)})+1\right)}\\
&\qquad\times\frac{\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1-\sigma^{(0)})+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty-\sigma^{(0)})+1\right)}{\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1+\sigma^{(0)})+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty+\sigma^{(0)})+1\right)}\times\frac{V}{U}\\
U :=& \left[\frac{i}{2}\sin(\pi\sigma^{(0)})\mathrm{tr}(M_1M_x)-\cos(\pi\theta_x)\cos(\pi\theta_\infty)-\cos(\pi\theta_0)\cos(\pi\theta_1)\right]e^{i\pi\sigma^{(0)}}\\
&\qquad+\frac{i}{2}\sin(\pi\sigma^{(0)})\mathrm{tr}(M_0M_1)+\cos(\pi\theta_x)\cos(\pi\theta_1)+\cos(\pi\theta_\infty)\cos(\pi\theta_0)\\
V :=& 4\sin\frac{\pi}{2}(\theta_0+\theta_x-\sigma^{(0)})\sin\frac{\pi}{2}(\theta_0-\theta_x+\sigma^{(0)})\\
&\qquad\times\sin\frac{\pi}{2}(\theta_\infty+\theta_1-\sigma^{(0)}))\sin\frac{\pi}{2}(\theta_\infty-\theta_1+\sigma^{(0)}).
\end{aligned}
$$

The formulas of $a^{(1)}$, $a^{(\infty)}$, are given in Remark 2 below. The monodromy data are restricted by the following condition, equivalent to (4):

$$\mathrm{tr}(M_iM_j)\notin(-\infty,-2],\qquad j=0,1,x. \tag{11}$$

I take the occasion to say that in [**7**] the condition (1.30) is wrong, the right one being (11).

Remark 1. — PVI depends holomorphically on θ_0, θ_1, θ_x, θ_∞; and so does $y(x)$. On the other hand, the matrices of the Fuchsian system have a pole in $\theta_\infty=0$. This is a non-generic case, which must be treated separately. The non-generic cases have been studied, for the equation with $\theta_0=\theta_x=\theta_1=0$ and arbitrary θ_∞. The reader is referred to [**14, 2, 6**]. Also in these cases, $y(x)$ is shown to depend holomorphically on θ_∞[3].

We also remark that formula (10) is to be modified when $\sigma^{(0)}=0$. We refer to [**9**].

(3)From the technical point of view, one has to solve a Riemann-Hilbert problem, to construct the fuchsian system associated to PVI from the given set of monodromy data. If θ_∞ is not integer, the monodromy at infinity is similar to the matrix $\mathrm{diag}(e^{-i\pi\theta_\infty},e^{i\pi\theta_\infty})$. But if the condition $\theta_\infty\in\mathbf{Z}$ is broken, the monodromy contains non diagonal terms. The solution of the problem is possible case by case, and it is reduced to a connection problem for hyper-geometric equations with logarithmic solutions and non-generic monodromy.

Remark 2. — To describe the symmetries of PVI, it may be convenient to choose

$$\alpha = \frac{1}{2}\theta_\infty^2. \tag{12}$$

PVI is invariant for the change of variables $y(x) = 1-\tilde{y}(t)$, $x = 1-t$ and simultaneous permutation of θ_0, θ_1. This means that $y(x)$ solves PVI if and only if $\tilde{y}(t)$ solves PVI with permuted parameters and independent variable t. Similarly, PVI is invariant for $y(x) = 1/\tilde{y}(t)$, $x = 1/t$ and simultaneous permutation of θ_∞, θ_0. It is invariant for $y(x) = (\tilde{y}(t)-t)/(1-t)$, $x = t/(t-1)$ and simultaneous permutation of θ_0, θ_x. By composing the third, first and again third symmetries, we get $y(x) = \tilde{y}(t)/t$, $t = 1/x$ with the permutation of θ_1, θ_x. Therefore, the critical points $0, 1, \infty$ are equivalent. This means that it is enough to know (8) for $a^{(0)}$, to write the analogous for $a^{(1)}$ and $a^{(\infty)}$. Explicitly, to compute $a^{(1)}$ one has to do the following substitution in the formula of $a^{(0)}$:

$$\sigma \mapsto \sigma^{(1)}$$

$$\theta_0 \mapsto \theta_1, \quad \theta_1 \mapsto \theta_0 \tag{13}$$

$$\mathrm{tr}(M_0M_x) \mapsto \mathrm{tr}(M_1M_x), \quad \mathrm{tr}(M_1M_x) \mapsto \mathrm{tr}(M_0M_x), \tag{14}$$

$$\begin{aligned}\mathrm{tr}(M_0M_1) \mapsto 4\left[\cos(\pi\theta_0)\cos(\pi\theta_1) + \cos(\pi\theta_\infty)\cos(\pi\theta_x)\right] + \\ - \left(\mathrm{tr}(M_0M_1) + \mathrm{tr}(M_0M_x)\mathrm{tr}(M_1M_x)\right)\end{aligned} \tag{15}$$

to compute $a^{(\infty)}$ one has to do the following substitution in the formula of $a^{(0)}$:

$$\sigma \mapsto \sigma^{(\infty)}$$

$$\theta_x \mapsto \theta_1, \quad \theta_1 \mapsto \theta_x \tag{16}$$

$$\mathrm{tr}(M_0M_x) \mapsto \mathrm{tr}(M_0M_1), \tag{17}$$

$$\begin{aligned}\mathrm{tr}(M_0M_1) \mapsto 4\big[\cos(\pi\theta_x)\cos(\pi\theta_0) + \cos(\pi\theta_\infty)\cos(\pi\theta_1)\big] + \\ - \left(\mathrm{tr}(M_0M_x) + \mathrm{tr}(M_1M_x)\mathrm{tr}(M_0M_1)\right).\end{aligned} \tag{18}$$

In the above formula we used the definition (5) for θ_∞.

3. Two Questions

Problem 1. — Let PVI be given; namely, let θ_0, θ_1, θ_x, θ_∞ be given. We would like to study *all* the solutions of the given PVI. As a consequence of the one-to-one correspondence (6) between monodromy data and transcendents, we need to compute the critical behavior and solve the connection problem for all values $\mathrm{tr}(M_iM_j)$, $j = 0, 1, x$[4].

This problem was for me the first motivation to study the elliptic representation.

[4] In exceptional cases (θ_0, θ_x, θ_1, $\theta_\infty \in \mathbf{Z}$) the one-to-one correspondence is broken. They can be treated separately. See for example [**14**].

The problem is then to study the critical behavior and the connection problem if the quantities $\mathrm{tr}(M_iM_j)$ break the condition (11). Not only the desire to get the most general results justifies such a study. We need such results in the theory of Frobenius manifolds. It is actually possible to construct a 3-dimensional Frobenius structure starting from Painlevé transcendents with any α, and $\beta=\gamma=1-2\delta=0$ ([**1, 5**]). There are important examples of Frobenius manifolds which are associated to Painlevé transcendents with $\mathrm{tr}(M_iM_j)<-2$, like the quantum cohomology of the 2 dimensional complex projective space (the quantities $\mathrm{tr}(M_iM_j)$ are computed in terms of binomial coefficients [**4, 1**]).

If we break (11), we face the problem to understand what happens to the behaviors (1) (2) (3) when $\Re\sigma=1$. What can we expect? Naively speaking, *if we could extend* the results above to, say, $\Re\sigma^{(0)}=1$, then the leading terms $a^{(0)}x^{1-\sigma^{(0)}}$, $x\to 0$, would become oscillatory. Moreover, if $\sigma^{(0)}=1$, the leading term is constant: we might expect that the transcendent decays very slowly as $x\to 0$.

In general, we should expect critical behaviors which may be completely different from (1) (2) (3). For example, in [**14**] the case $\mathrm{tr}(M_0M_1)=\mathrm{tr}(M_0M_x)=\mathrm{tr}(M_xM_1)=-2$ (namely, $\sigma^{(i)}=1$) is worked out, for values of $\alpha=2m^2$, $m\in\mathbf{Z}$, $m\neq 0$, and $\beta=\gamma=0$, $\delta=1/2$. In this case, for any given m, there exists a 1-parameter family of *classical* solutions, which have critical behaviors:

$$y(x)=\begin{cases} -\ln(x)^{-2}(1+O(\ln(x)^{-1})), & x\to 0\\[2ex] 1+\ln(1-x)^{-2}(1+O(\ln(1-x)^{-1})), & x\to 1\\[2ex] -x\ln(1/x)^{-2}(1+O(\ln(1/x)^{-1}), & x\to\infty\end{cases}$$

This is actually the behavior of a branch, specified by $|\arg(x)|<\pi$, $|\arg(1-x)|<\pi$. The variable x approaches a critical point within a sector. This behavior is completely different from (1) (2) (3). These solutions were called *Chazy solutions* in [**14**], because they can be computed as functions of solutions of the Chazy equation. We observe that, in this case, the one-to-one correspondence between monodromy data and transcendents is lost.

Problem 2. — The equations (7) are invariant for the transformation

$$\sigma^{(i)}\mapsto\pm\sigma^{(i)}+2N,\qquad N\in\mathbf{Z}\tag{19}$$

Therefore, it is a natural question to ask if a given transcendent (6) may have a variety of critical behaviors, with exponents $\pm\sigma^{(i)}+2N$.

This was the second motivation for the analytic study of the elliptic representation.

This can not be done naively. The proofs of (1) (2) (3) and of the connection formulas in [**9**] do not work if we break the hypothesis $0\leq\Re\sigma^{(i)}<1$. Moreover, we have a contradiction: for example, let us choose a transcendent such that the vanishing behavior (1) at $x = 0$ is true for $0 \le \Re\sigma^{(0)} < 1$. Then, we would have a divergent behavior when we change, for example, $\sigma^{(0)} \mapsto \sigma^{(0)} + 2$. But we can not have divergent and vanishing behavior at the same time!

We recall that (1) (2) (3) are critical behaviors of a branch of a transcendent $y(x)$. In other words, x approaches a critical point inside a sector. If we regard x as a point of the universal covering of $\mathbf{P}^1\backslash\{0, 1, \infty\}$, then x can approach $0, 1, \infty$ along any path; for example, along a spiral. The critical behaviors may depend on the path along which x approaches the critical point. So, we may expect no contradiction if there are different exponents $\pm\sigma^{(i)} + 2N$, depending on the paths. We'll show that this is the case.

4. Another Previous Result

Before introducing the elliptic representation, we explain a result by S.Shimomura [**18, 8**]. This is a result of local analysis, namely, it does not touch the connection problem. It explains what happens on the universal covering.

Let $\tilde{C}_0$ be the universal covering of $\mathbf{C}\backslash\{0\}$. S. Shimomura proved the following statement for PVI with any value of the parameters $\alpha, \beta, \gamma, \delta$.

For any complex number k and for any $\sigma \notin (-\infty, 0] \cup [1, +\infty)$ there is a sufficiently small r such that the Painlevé VI equation for given $\alpha, \beta, \gamma, \delta$ has a holomorphic solution in the domain

$$\mathbf{D}_s(r; \sigma, k) = \{x \in \tilde{C}_0 \mid |x| < r,\ |e^{-k} x^{1-\sigma}| < r,\ |e^k x^\sigma| < r\}$$

with the following representation:

$$y(x; \sigma, k) = \frac{1}{\cosh^2(\frac{\sigma-1}{2}\ln x + \frac{k}{2} + \frac{v(x)}{2})},$$

where

$$v(x) = \sum_{n\ge1} a_n(\sigma) x^n + \sum_{n\ge0,\ m\ge1} b_{nm}(\sigma) x^n (e^{-k} x^{1-\sigma})^m + \\ + \sum_{n\ge0,\ m\ge1} c_{nm}(\sigma) x^n (e^k x^\sigma)^m,$$

$a_n(\sigma), b_{nm}(\sigma), c_{nm}(\sigma)$ are rational functions of σ and the series defining $v(x)$ is convergent (and holomorphic) in $\mathbf{D}_s(r; \sigma, k)$. Moreover, there exists a constant $M = M(\sigma)$ such that

$$|v(x)| \le M(\sigma)\ \left(|x| + |e^{-k} x^{1-\sigma}| + |e^k x^\sigma|\right). \tag{20}$$

The domain $\mathbf{D}(r; \sigma, k)$ is specified by the conditions:

$$|x| < r, \quad \Re\sigma \ln|x| + [\Re k - \ln r] < \Im\sigma \arg(x) < (\Re\sigma - 1)\ln|x| + [\Re k + \ln r]. \tag{21}$$

This is an open domain in the plane $(\ln|x|, \arg(x))$.

Shimomura's representation gives the critical behavior when $x \to 0$ along a path, starting from a point x_0 belonging to the domain. If $\Im\sigma = 0$ any path to 0 is allowed (the domain is simply $|x| < r$). Otherwise, we consider a family of paths, depending on a parameter Σ:

$$|x| \le |x_0| < r, \quad \arg x = \arg x_0 + \frac{\Re\sigma - \Sigma}{\Im\sigma} \ln \frac{|x|}{|x_0|}, \quad 0 \le \Sigma \le 1. \tag{22}$$

They are contained in the domain. If $\Im\sigma = 0$, the behavior (1) is obtained. Suppose then that $\Im\sigma \neq 0$.

a) $0 \le \Sigma < 1$. We observe that $|x^{1-\sigma}e^{-k}| \to 0$ as $x \to 0$ along (22). Then,

$$\begin{aligned} y(x;\sigma,k) &= \frac{1}{\cosh^2(\frac{\sigma-1}{2}\ln x + \frac{k}{2} + \frac{v(x)}{2})} \\ &= \frac{4}{x^{\sigma-1}e^k e^{v(x)} + x^{1-\sigma}e^{-k}e^{-v(x)} + 2} = \frac{4e^{-k}e^{-v(x)}x^{1-\sigma}}{(1+e^{-k}e^{-v(x)}x^{1-\sigma})^2} \\ &= 4e^{-k}e^{-v(x)}x^{1-\sigma}\ \left(1 + e^{-v(x)}O(|e^{-k}x^{1-\sigma}|)\right). \end{aligned}$$

Two sub-cases:

a.1) $\Sigma \neq 0$.
Then, $|x^\sigma e^k| \to 0$ and $v(x) \to 0$ (see (20)) and thus,

$$y(x;\sigma,k) = 4e^{-k}x^{1-\sigma}\left(1 + O(|x| + |e^k x^\sigma| + |e^{-k}x^{1-\sigma}|)\right).$$

This is again (1).

a.2) $\Sigma = 0$.
Then, $|x^\sigma e^k| \to$ constant$< r$; so, $|v(x)|$ does not vanish and thus,

$$y(x) = a(x)x^{1-\sigma}\ \left(1 + O(|e^{-k}x^{1-\sigma}|)\right), \qquad a(x) = 4e^{-k}e^{-v(x)}.$$

Note that $a(x)$ may be oscillatory.

b) $\Sigma = 1$. Now, $|x^{1-\sigma}e^{-k}| \to$ (constant$\neq 0$) $< r$. Therefore, $y(x)$ does not vanish as $x \to 0$. We keep the representation

$$y(x;\sigma,k) = \frac{1}{\cosh^2(\frac{\sigma-1}{2}\ln x + \frac{k}{2} + \frac{v(x)}{2})} \equiv \frac{1}{\sin^2(i\frac{\sigma-1}{2}\ln x + i\frac{k}{2} + i\frac{v(x)}{2} - \frac{\pi}{2})}.$$

$v(x)$ does not vanish and $y(x)$ is oscillating as $x \to 0$, with no limit. Figure 1 synthesizes points **a.1)**, **a.2)**, **b)**.

As an application, we consider the case $\Re\sigma = 1$, namely $\sigma = 1 - i\nu$, $\nu \in \mathbf{R}\backslash\{\mathbf{0}\}$. Then, the path corresponding to $\Sigma = 1$ is a *radial* path in the x-plane and

$$y(x;1-i\nu,k) = \frac{1 + O(x)}{\sin^2\left(\frac{\nu}{2}\ln(x) + \frac{ik}{2} - \frac{\pi}{2} + \frac{i}{2}\sum_{m\ge1} b_{0m}(\sigma)(e^{-k}x^{1-\sigma})^m\right)}.$$

The result is local. It can be repeated at $x = 0, 1, \infty$, with integration constants $\sigma^{(i)}$ and k_i, $i = 0, 1, \infty$. In [**6**], we proved that Shimomura's is a representation of a

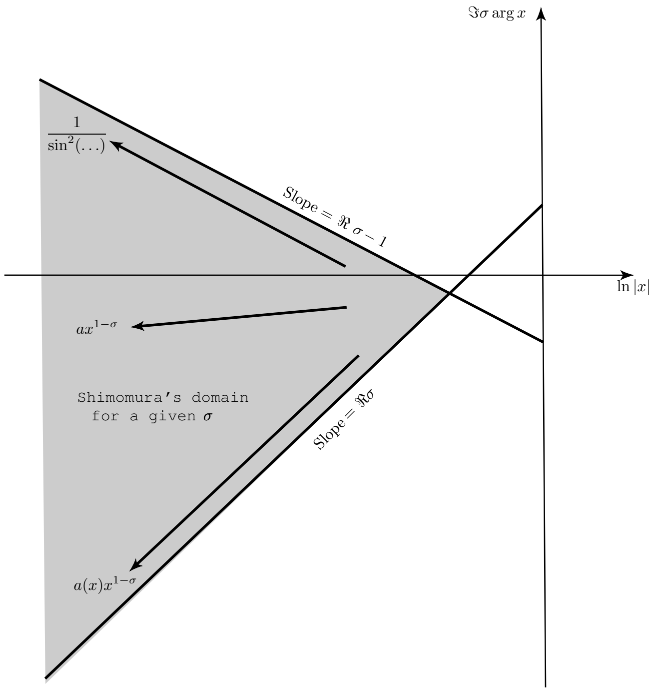

FIGURE 1. Critical behavior of $y(x;\sigma,k)$ along different lines in $\mathbf{D}_s(r;\sigma,k)$. The plane is the plane $(\ln|x|, \Im\sigma \arg x)$

transcendent (6), and we solved the connection problem. More precisely, we proved that the exponents of Shimomura's representation are given by (7), and k_i by an extension of (8), where $a^{(i)} = 4\exp\{-k_i\}$, $i = 0, 1, \infty$[5].

5. The Elliptic Representation

The elliptic representation was introduced by P. Painlevé in [**16**] and R. Fuchs in [**3**]. Let

$$\mathbf{L} := x(1-x)\frac{d^2}{dx^2} + (1-2x)\frac{d}{dx} - \frac{1}{4}.$$

be a linear differential operator and let $\wp(z;\omega_1,\omega_2)$ be the Weierstrass elliptic function of the independent variable $z \in \mathbf{P}^1$, with *half-periods* ω_1, ω_2. Let us consider the

[5] To be precise, in [**6**], the solution of the connection problem for Shimomura's solutions is done for the special choice $\beta, \gamma, \delta - 1/2 = 0$. Nevertheless, the procedure of [**7**] can be repeated for the Shimomura's solutions. Also, in [**7**], generic values of $\alpha, \beta, \gamma, \delta$ are considered. With more technical complications one can repeat the proofs for non-generic cases. One of them is precisely [**6**].

following independent solutions of the *hyper-geometric equation* $\mathbf{L}\omega = 0$:

$$\omega_1(x) := \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right), \qquad \omega_2(x) := i\frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; 1-x\right),$$

where $F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right)$ is the standard notation for the hyper-geometric function. Here x is in the universal covering of $\mathbf{P}^1\backslash\{0, 1, \infty\}$, so that at this stage we do not worry about the choice of branch-cuts. It is proved in [**3**] that PVI is equivalent to the following differential equation for a new function $u(x)$:

$$\begin{aligned}(23)\quad \mathbf{L}(u) = \frac{1}{2x(1-x)}\Bigg[&2\alpha \frac{\partial}{\partial u}\wp\left(\frac{u}{2}; \omega_1, \omega_2\right) - 2\beta \frac{\partial}{\partial u}\wp\left(\frac{u}{2} + \omega_2; \omega_1, \omega_2\right) + \\ &+ 2\gamma \frac{\partial}{\partial u}\wp\left(\frac{u}{2} + \omega_1; \omega_1, \omega_2\right) + (1-2\delta)\frac{\partial}{\partial u}\wp\left(\frac{u}{2} + \omega_1 + \omega_2; \omega_1, \omega_2\right)\Bigg]\end{aligned}$$

The connection between $u(x)$ and a solution $y(x)$ of PVI is the following:

$$y(x) = \wp\left(\frac{u(x)}{2}; \omega_1(x), \omega_2(x)\right) + \frac{1+x}{3}.$$

The algebraic-geometrical properties of the elliptic representations where studied in [**13**]. Nevertheless, to my knowledge, the analytic properties of the function $u(x)$ were not studied before [**7**] (and [**6**]), except for the special case $\alpha = \beta = \gamma = 1 - 2\delta = 0$, which was known to Picard [**17**]. Its critical behavior was studied in [**14**]. In [**7**], we studied the local analytic properties of $u(x)$ at $x = 0, 1, \infty$, for *any* value of $\alpha, \beta, \gamma, \delta$. Then, we solved the connection problem in elliptic representation, for generic values of $\alpha, \beta, \gamma, \delta$(6).

The general solution of $\mathbf{L}(u) = 0$, is $u_0(x) = 2\nu_1\omega_1(x) + 2\nu_2\omega_2(x)$, $\nu_1, \nu_2 \in \mathbf{C}$. Let us look for a solution $u(x) = 2\nu_1\omega_1(x) + 2\nu_2\omega_2(x) + 2v(x)$ of (23), where $v(x)$ is a perturbation of u_0. Let again $\mathbf{C}_0 := \mathbf{C}\backslash\{0\}$, $\widetilde{\mathbf{C}_0}$ the universal covering, and $0 < r < 1$. We define the domains

$$(24)\quad \mathbf{D}(r; \nu_1, \nu_2) := \left\{x \in \widetilde{\mathbf{C}_0} \ \text{ such that } |x| < r, \left|\frac{e^{-i\pi\nu_1}}{16^{1-\nu_2}} x^{1-\nu_2}\right| < r, \left|\frac{e^{i\pi\nu_1}}{16^{\nu_2}} x^{\nu_2}\right| < r\right\}$$

$$(25)\qquad \mathbf{D}_0(r) := \left\{x \in \widetilde{\mathbf{C}_0} \ \text{ such that } |x| < r\right\}.$$

---

(6)The condition defining the *generic case* is:

$$\nu_2^{(i)},\ \theta_0,\ \theta_x,\ \theta_1,\ \theta_\infty\ \notin \mathbf{Z}; \qquad \frac{\pm 1 \pm \nu_2^{(i)} \pm \theta_1 \pm \theta_\infty}{2},\ \frac{\pm 1 \pm \nu_2^{(i)} \pm \theta_0 \pm \theta_x}{2}\ \notin \mathbf{Z}.$$

This is a technical condition which can be abandoned (except for $\nu_2^{(i)} \notin \mathbf{Z}$) at the price of making the computations more complicated. For example, the non-generic case $\beta = \gamma = 1 - 2\delta = 0$ and α any complex number was analyzed in [**2, 14, 6**].

Let us introduce the following expansion:

$$(26)\quad v(x;\nu_1,\nu_2):=\sum_{n\geq 1}a_nx^n+\sum_{n\geq 0,m\geq 1}b_{nm}x^n\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right]^m+$$
$$+\sum_{n\geq 0,m\geq 1}c_{nm}x^n\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}\right]^m.$$

In [**7**] we proved the following:

Theorem 5.1. — *Let PVI be given, with no restriction on $\alpha,\beta,\gamma,\delta$.*

I) *For any $\nu_1,\nu_2\in\mathbf{C}$, such that $\Im\nu_2\neq 0$, there exist a positive number $r<1$ and a transcendent*

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2);\ \omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$$

such that $v(x;\nu_1,\nu_2)$ is holomorphic in the domain $\mathbf{D}(r;\nu_1,\nu_2)$ and it is given by the expansion (26), which is convergent in $\mathbf{D}(r;\nu_1,\nu_2)$. The coefficients a_n, b_{nm}, c_{nm}, $i=1,2$, are certain rational functions of ν_2. Moreover, there exists a positive constant $M(\nu_2)$ such that

$$(27)\qquad |v(x;\nu_1,\nu_2)|\leq M(\nu_2)\left(|x|+\left|e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right|+\left|e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}\right|\right),$$

in $\mathbf{D}(r;\nu_1,\nu_2)$.

II) *For any $\nu_1\in\mathbf{C}$ and real ν_2, with the constraint $0<\nu_2<1$ or $1<\nu_2<2$, there exists a positive $r<1$ and a transcendent*

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2);\ \omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3},$$

if $0<\nu_2<1$. Or,

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;-\nu_1,2-\nu_2);\ \omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3},$$

if $1<\nu_2<2$. The functions $v(x;\nu_1,\nu_2)$ and $v(x;-\nu_1,2-\nu_2)$ are holomorphic in $\mathbf{D}_0(r)$, with convergent expansion (26) and bound (27) (for $1<\nu_2<2$ substitute $\nu_1\mapsto-\nu_1$, $\nu_2\mapsto 2-\nu_2$).

Note that in the theorem, case **II)**, $\nu_2\neq 0,1$. If ν_2 is greater that 2 or less then 0, namely if $-2N<\nu_2<2-2N$, the formulae of case **II)** hold with the substitution $\nu_2\mapsto\nu_2+2N$.

If we expand in Fourier series the $\wp$-function w.r.t. ω_2, it is possible to compute the critical behavior when $x\to 0$, along the paths defined as follows. Let $\Im\nu_2\neq 0$ and $\nu^*\in\mathbf{C}$. We define the following family of paths joining a point $x_0\in\mathbf{D}(r;\nu_1,\nu_2)$ to $x=0$

$$(28)\qquad \arg x=\arg x_0+\frac{\Re\nu_2-\nu^*}{\Im\nu_2}\ln\frac{|x|}{|x_0|},\qquad 0\leq\nu^*\leq 1.$$

The paths are contained in $\mathbf{D}(r;\nu_1,\nu_2)$. If $\Im\nu_2=0$ any regular path contained in $\mathbf{D}_0(r)$ can be considered.

Theorem 5.2. — *Let ν_1, ν_2 be given.*

If $\Im\nu_2\neq 0$, the critical behavior of the transcendent $y(x)=\wp(\nu_1\omega_1+\nu_2\omega_2+v(x;\nu_1,\nu_2);\omega_1,\omega_2)+(1+x)/3$ when $x\to 0$ along the path (28) is:

For $0<\nu^*<1$:

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]\ x^{\nu_2}\ \left(1+O(|x^{\nu_2}|+|x^{1-\nu_2}|)\right). \tag{29}$$

For $\nu^*=0$:

(30)
$$y(x)=\left[\frac{x}{2}+\sin^{-2}\left(-i\frac{\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\sum_{m\geq 1}c_{0m}\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}\right]^m\right)\right]\ \left(1+O(x)\right).$$

For $\nu^*=1$:

(31)
$$y(x)=x\ \sin^2\left(i\frac{1-\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\sum_{m\geq 1}b_{0m}\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right]^m\right)\ (1+O(x)).$$

If ν_2 is real, we have two cases. For $0<\nu_2<1$, the transcendent $y(x)=\wp(\nu_1\omega_1+\nu_2\omega_2+v(x;\nu_1,\nu_2);\omega_1,\omega_2)+(1+x)/3$ defined in $\mathbf{D}_0(r)$ has behavior

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]\ x^{\nu_2}\ \left(1+O(|x^{\nu_2}|+|x^{1-\nu_2}|)\right),\quad 0<\nu_2<1. \tag{32}$$

For $1<\nu_2<2$, the transcendent $y(x)=\wp(\nu_1\omega_1+\nu_2\omega_2+v(x;-\nu_1,2-\nu_2);\omega_1,\omega_2)+(1+x)/3$ defined in $\mathbf{D}_0(r)$ has behavior

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1}\ x^{2-\nu_2}\ \left(1+O(|x^{2-\nu_2}|+|x^{\nu_2-1}|)\right),\quad 1<\nu_2<2. \tag{33}$$

Observe that, in general, $x\to 0$ along a spiral path. It is interesting to observe the oscillatory behavior (30), which neither vanishes nor diverges at $x=0$. We will return later to this point. Generically, anyway, the behavior is of the type (29). Namely, $y(x)=ax^{\nu_2}(1+\text{ higher orders in } x)$, where $e^{i\pi\nu_1}=-4a\ 16^{\nu_2-1}$. Similar results hold at $x=1,\infty$ (see Remark 2 and [**7**]). The behavior (29) extends that of Jimbo's paper to the domain $\mathbf{D}(r;\nu_1,\nu_2)$.

The connection problem was solved in [**7**] (and [**6**]) by the isomonodromy deformation method. We had to extend the techniques of [**9**] to the domains $\mathbf{D}(r;\nu_1,\nu_2)$, and similar domains at $x=1$ and $x=\infty$. We showed that a trascendent has three representations at $x=0,1,\infty$

$$\begin{aligned} y(x) &= \wp(\nu_1^{(0)}\omega_1^{(0)}+\nu_2^{(0)}\omega_2^{(0)}+v^{(0)})+\frac{1+x}{3}, \quad \omega_1^{(0)}:=\omega_1,\ \omega_2^{(0)}:=\omega_2;\\ &= \wp(\nu_1^{(1)}\omega_1^{(1)}+\nu_2^{(1)}\omega_2^{(1)}+v^{(1)})+\frac{1+x}{3}, \quad \omega_1^{(1)}:=\omega_2,\ \omega_2^{(1)}:=\omega_1;\\ &= \wp(\nu_1^{(\infty)}\omega_1^{(\infty)}+\nu_2^{(\infty)}\omega_2^{(\infty)}+v^{(\infty)})+\frac{1+x}{3}, \quad \omega_1^{(\infty)}:=\omega_1+\omega_2,\ \omega_2^{(\infty)}:=\omega_2. \end{aligned}$$

in suitable domains. The procedure to connect the three couples of parameters $(\nu_1^{(0)},\nu_2^{(0)})$, $(\nu_1^{(1)},\nu_2^{(1)})$, $(\nu_1^{(\infty)},\nu_2^{(\infty)})$, is explained in section 6 below.

The critical behavior at $x=1,\infty$ of the above transcendent is similar to the behavior at $x=0$ (in Remark 2 of section 2 we explained how $x=0,1,\infty$ can be interchanged): it may be oscillatory along special directions, like (30) and (31), but for a generic path, it is like (29). Namely:

$$y(x)=a^{(0)}x^{\nu_2^{(0)}}(1+\text{ higher orders in } x), \quad x\to 0 \tag{34}$$

$$y(x)=1-a^{(1)}(1-x)^{\nu_2^{(1)}}(1+\text{ higher orders in } (1-x)), \quad x\to 1 \tag{35}$$

$$y(x)=a^{(\infty)}x^{1-\nu_2^{(\infty)}}(1+\text{ higher orders in } x^{-1}), \quad x\to \infty \tag{36}$$

and the parameters $\nu_1^{(i)}$ are given by[7]

$$\begin{gathered} e^{i\pi\nu_1^{(0)}}=-4a^{(0)}\ 16^{\nu_2^{(0)}-1},\\ e^{-i\pi\nu_1^{(1)}}=-4a^{(1)}\ 16^{\nu_2^{(1)}-1}, \quad e^{i\pi\nu_1^{(\infty)}}=-4a^{(\infty)}\ 16^{\nu_2^{(\infty)}-1} \end{gathered} \tag{37}$$

So, we have obtained an extension of (1) (2) (3), if we identify the exponents $\sigma^{(i)}=1-\nu_2^{(i)}$, for $0\le\Re\nu_2^{(i)}\le 1$. The extension occurs when we let $\nu_2^{(i)}$ be any complex number (with the constraint $\nu_2^{(i)}\notin(-\infty,0]\cup\{1\}\cup[2,+\infty)$).

---

[7] If $\nu_2^{(i)}$ is real, the behavior is as above when $0<\nu_2^{(i)}<1$. Otherwise, when $1<\nu_2^{(i)}<2$, it is

$$\begin{gathered} y(x)=a^{(0)}x^{2-\nu_2^{(0)}}(1+\text{ higher orders in } x), \quad x\to 0\\ y(x)=1-a^{(1)}(1-x)^{2-\nu_2^{(1)}}(1+\text{ higher orders in } (1-x)), \quad x\to 1\\ y(x)=a^{(\infty)}x^{\nu_2^{(\infty)}-1}(1+\text{ higher orders in } x^{-1}), \quad x\to\infty \end{gathered}$$

with

$$\begin{gathered} e^{-i\pi\nu_1^{(0)}}=-4a^{(0)}\ 16^{1-\nu_2^{(0)}}, \quad e^{i\pi\nu_1^{(1)}}=-4a^{(1)}\ 16^{1-\nu_2^{(1)}},\\ e^{-i\pi\nu_1^{(\infty)}}=-4a^{(\infty)}\ 16^{1-\nu_2^{(\infty)}}. \end{gathered}$$

The three sets of parameters $(\nu_1^{(i)}, \nu_2^{(i)})$, $i = 0, 1, \infty$ are functions of the monodromy data θ_0, θ_x, θ_1, θ_∞, $\mathrm{tr}(M_0M_x)$, $\mathrm{tr}(M_0M_1)$, $\mathrm{tr}(M_1M_x)$. In [**7**] we showed that:

$$2\cos(\pi\nu_2^{(0)}) = -\mathrm{tr}(M_0M_x), \tag{38}$$

$$2\cos(\pi\nu_2^{(1)}) = -\mathrm{tr}(M_1M_x), \tag{39}$$

$$2\cos(\pi\nu_2^{(\infty)}) = -\mathrm{tr}(M_0M_1), \tag{40}$$

and,

$$\begin{aligned}\exp\{i\pi\nu_1^{(0)}\} = {} & -4\ 16^{\nu_2^{(0)}-1} \\ & \times a^{(0)}\bigl(1-\nu_2^{(0)}; \theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1), \mathrm{tr}(M_1M_x)\bigr).\end{aligned} \tag{41}$$

The function $a^{(0)}$ is given in (10), while $\nu_1^{(1)}$, $\nu_1^{(\infty)}$ are computed from (37), where the functions $a^{(1)}$ is obtained from $a^{(0)}$ with the substitutions $\nu_2^{(0)} \mapsto \nu_2^{(1)}$ and (13) (14) (15); $a^{(\infty)}$ is obtained from $a^{(0)}$ with the substitutions $\nu_2^{(0)} \mapsto \nu_2^{(\infty)}$ and (16) (17) (18).

This concludes the discussion of problem **1**: the critical behavior of (6) is known and the connection problem is solved for almost all the monodromy data, except for

$$\mathrm{tr}(M_iM_j) = -2$$

We recall that we required that $\nu_2^{(i)} \neq 0, 1$ (and 2). The condition $\nu_2^{(i)} \neq 1$ is equivalent to $\mathrm{tr}(M_iM_j) \neq 2$. Nevertheless, this case is solved in Jimbo's paper (case $\sigma^{(i)} = 0$). The condition $\nu_2^{(i)} \neq 0$ (and 2), is more serious. It implies that we can not give the critical behaviors (and the elliptic representation) of (6) at $x = 0$ for $\mathrm{tr}(M_0M_x) = -2$; at $x = 1$ for $\mathrm{tr}(M_1M_x) = -2$; at $x = \infty$ for $\mathrm{tr}(M_0M_1) = -2$. To our knowledge, these cases have not yet been studied in the literature, except for the special case of [**14**].

We now turn to problem **2**. For simplicity, let us consider the local behavior at $x = 0$, and let us write again ω_i and ν_i instead of $\omega_i^{(0)}$ and $\nu_i^{(0)}$.

Let us first investigate the effect of $\sigma^{(0)} \mapsto \sigma^{(0)} - 2N$, $N \in \mathbf{Z}$. It corresponds to $\nu_2 \mapsto \nu_2 + 2N$. Here, we are considering non-real ν_2, otherwise no translation is allowed. Is is a consequence of the results of our first theorem that, for any $N \in \mathbf{Z}$ and for any complex ν_1, ν_2 such that $\Im\nu_2 \neq 0$, there exists $r_N < 1$ and a transcendent $y(x) = \wp\Bigl(\nu_1\omega_1(x) + [\nu_2 + 2N]\omega_2(x) + v(x; \nu_1, \nu_2 + 2N);\ \omega_1(x), \omega_2(x)\Bigr) + \frac{1+x}{3}$ in $\mathbf{D}(r; \nu_1, \nu_2 + 2N)$. By periodicity of the $\wp$-function the above is equal to:

$$y(x) = \wp\Bigl(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x; \nu_1, \nu_2 + 2N);\ \omega_1(x), \omega_2(x)\Bigr) + \frac{1+x}{3} \tag{42}$$

in $\mathbf{D}(r; \nu_1, \nu_2 + 2N)$. It is natural to ask the question if a transcendent

$$y(x) = \wp\Bigl(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x; \nu_1, \nu_2);\ \omega_1(x), \omega_2(x)\Bigr) + \frac{1+x}{3} \tag{43}$$

defined in in $\mathbf{D}(r, \nu_1, \nu_2)$ for some ν_1, ν_2, $\Im\nu_2 \neq 0$, can be represented in $\mathbf{D}(r; \nu_1, \nu_2 + 2N)$ in the form (42). The answer is yes, provided that we replace ν_1 with a new

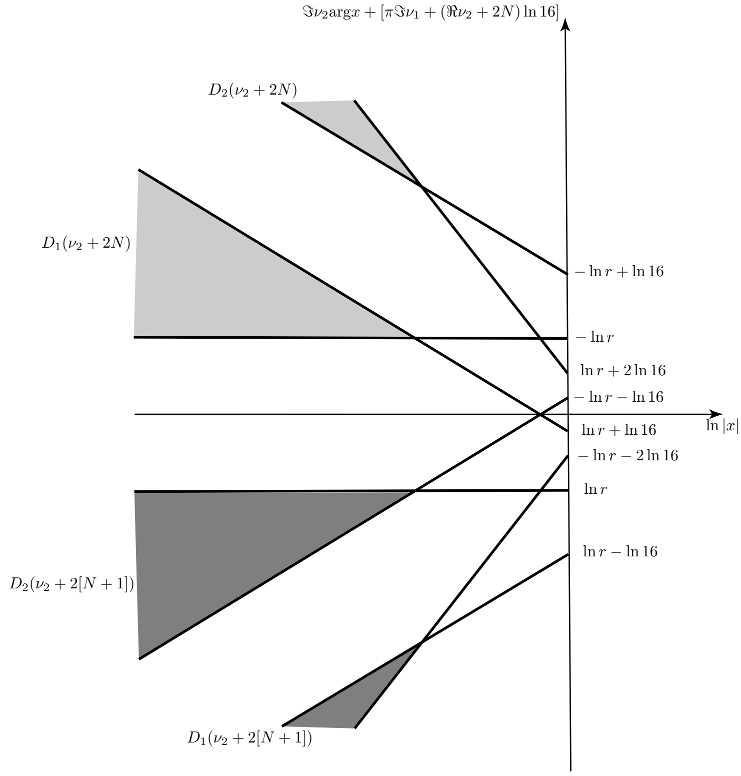

FIGURE 2. The domains $\mathbf{D}_1(r;\nu_1,\nu_2+2N) := \mathbf{D}(r;\nu_1,\nu_2+2N)$, $\mathbf{D}_2(r;\nu_1,\nu_2+2N) := \mathbf{D}(r;-\nu_1,2-\nu_2-2N)$ and $\mathbf{D}_1(r;\nu_1,\nu_2+2[N+1])$, $\mathbf{D}_2(r;\nu_1,\nu_2+2[N+1])$ for arbitrarily fixed values of ν_1, ν_2, N.

value ν_1'. Namely, for any integer N there exists $\nu_1' = \nu_1'(\nu_1,\nu_2,N)$ such that (43) has representation

$$y(x) = \wp\Big(\nu_1'\omega_1(x) + \nu_2\omega_2(x) + v(x;\nu_1',\nu_2+2N);\ \omega_1(x),\omega_2(x)\Big) + \frac{1+x}{3} \tag{44}$$

in $\mathbf{D}(r_N,\nu_1',\nu_2+2N)$, for sufficiently small r_N. This result is a consequence of the one to one correspondence of both (43) and (44) with (6). The explicit form of $\nu_1' = \nu_1'(\nu_1,\nu_2,N)$ is computed by (41), with $\nu_2 \mapsto \nu_2 + 2N$.

We consider now $\sigma^{(0)} \mapsto -\sigma^{(0)}$, which corresponds to $\nu_2 \mapsto 2-\nu_2$. By (41) and (10), we can see that the effect on ν_1 is: $\nu_1 \mapsto -\nu_1$. Namely, the transcendent (6) has representation (43) in $\mathbf{D}(r,\nu_1,\nu_2)$ if and only if it has representation $y(x) = \wp\Big(-\nu_1\omega_1(x)+[2-\nu_2]\,\omega_2(x)+v(x;-\nu_1,2-\nu_2);\ \omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$ in $\mathbf{D}(r;-\nu_1,2-\nu_2)$.

Due to the parity and periodicity of $\wp$, this last is equivalent to

$$y(x) = \wp\Big(\nu_1\omega_1(x) + \nu_2\omega_2(x) - v(x; -\nu_1, 2-\nu_2);\ \omega_1(x), \omega_2(x)\Big) + \frac{1+x}{3}, \tag{45}$$

We have therefore proved that a transcendent (6) has the elliptic representations (43) in $\mathbf{D}(r, \nu_1, \nu_2)$, (44) in $\mathbf{D}(r, \nu_1', \nu_2 + 2N)$, and (45) in $\mathbf{D}(r; -\nu_1, 2-\nu_2)$. In other words, we have found different behaviors of (6) in different domains, corresponding to the freedom in the choice of "exponents"

$$\nu_2 \mapsto \pm\nu_2 + 2N, \quad N \in \mathbf{Z},$$

namely, $\sigma^{(0)} \mapsto \pm\sigma^{(0)} \pm 2N$. The same arguments can be repeated at $x = 1, \infty$. This is exactly the solution of our problem **2**.

Figure 2 is a picture of the union of the domains $\mathbf{D}(r_N; \pm\nu_1'(N), \pm\nu_2 + 2N)$, in the $(\ln|x|, \Im\nu_2 \arg x)$-plane, for $\Im\nu_2 \neq 0$ (if ν_2 is real, the domain $\mathbf{D}_0$ is the left half-plane $\ln|x| < \ln r < 0$). The union of the domain is the largest domain where the elliptic representation of a given transcendent (6) is known. Note that, in general, not all the left half-plane is covered by the union. Actually, we do not know what happens in the strips between two domains. Movable poles may exist there. Qualitatively speaking, the oscillatory behaviors (30) depend on the vicinity of such poles [**7**].

6. Appendix on the Connection Problem

We already mentioned that the connection between monodromy data and critical behavior, is given by (38), (39), (40); by (41) (and (10)); by (13), (14), (15), (16), (17), (18).

When the critical behavior is given at, say, $x = 0$, we know $(\nu_1^{(0)}, \nu_2^{(0)})$. How can we compute $(\nu_1^{(1)}, \nu_2^{(1)})$ and $(\nu_1^{(\infty)}, \nu_2^{(\infty)})$? We give here the procedure to do that.

First, we have to compute the traces of the monodromy matrices. As for M_0M_x, we have $2\cos(\pi\nu_2^{(0)}) = -\mathrm{tr}(M_0M_x)$. As for the other two products, it is possible to write explicitly the formulae as follows. Consider three auxiliary matrices

$$A := \begin{pmatrix} \dfrac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} & \dfrac{\Gamma(c-a-b)\Gamma(2-c)}{\Gamma(1-a)\Gamma(1-b)} \\[2ex] \dfrac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)} & \dfrac{\Gamma(a+b-c)\Gamma(2-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} \end{pmatrix}$$

$$\text{where } \begin{cases} a = \dfrac{\theta_\infty + \theta_1 + 1 - \nu_2^{(0)}}{2} \\ b = 1 + \dfrac{-\theta_\infty + \theta_1 + 1 - \nu_2^{(0)}}{2} \\ c = 2 - \nu_2^{(0)} \end{cases}$$

$$B=\begin{pmatrix} \frac{\Gamma(1+\alpha_0-\beta_0)\Gamma(1-\gamma_0)}{\Gamma(1-\beta_0)\Gamma(1+\alpha_0-\gamma_0)}\, e^{-i\pi\alpha_0} & \frac{\Gamma(1+\beta_0-\alpha_0)\Gamma(1-\gamma_0)}{\Gamma(1-\alpha_0)\Gamma(1+\beta_0-\gamma_0)}\, e^{-i\pi\beta_0} \\ \frac{\Gamma(1+\alpha_0-\beta_0)\Gamma(\gamma_0-1)}{\Gamma(\alpha_0)\Gamma(\gamma_0-\beta_0)}\, e^{i\pi(\gamma_0-1-\alpha_0)} & \frac{\Gamma(1+\beta_0-\alpha_0)\Gamma(\gamma_0-1)}{\Gamma(\beta_0)\Gamma(\gamma_0-\alpha_0)}\, e^{i\pi(\gamma_0-1-\beta_0)} \end{pmatrix}$$

$$C=\begin{pmatrix} \frac{\Gamma(\gamma_0-\alpha_0-\beta_0)\Gamma(1+\alpha_0-\beta_0)}{\Gamma(1-\beta_0)\Gamma(\gamma_0-\beta_0)} & \frac{\Gamma(\gamma_0-\alpha_0-\beta_0)\Gamma(1+\beta_0-\alpha_0)}{\Gamma(1-\alpha_0)\Gamma(\gamma_0-\alpha_0)} \\ \frac{\Gamma(\alpha_0+\beta_0-\gamma_0)\Gamma(1+\alpha_0-\beta_0)}{\Gamma(1+\alpha_0-\gamma_0)\Gamma(\alpha_0)}\, e^{i\pi(\gamma_0-\alpha_0-\beta_0)} & \frac{\Gamma(\alpha_0+\beta_0-\gamma_0)\Gamma(1+\beta_0-\alpha_0)}{\Gamma(1+\beta_0-\gamma_0)\Gamma(\beta_0)}\, e^{i\pi(\gamma_0-\alpha_0-\beta_0)} \end{pmatrix}$$

$$\text{where } \begin{cases} \alpha_0 &= \dfrac{\nu_2^{(0)}-1+\theta_0+\theta_x}{2} \\ \beta_0 &= 1+\frac{1-\nu_2^{(0)}+\theta_0+\theta_x}{2} \\ \gamma_0 &= 1+\theta_0 \end{cases}.$$

Let s be a non-zero complex number. We consider the products

$$m_1 := A^{-1}\, e^{2\pi i\ \mathrm{diag}(\frac{\theta_1}{2},-\frac{\theta_1}{2})}\ A \tag{46}$$

$$m_0 := \left[\left\{B\begin{pmatrix}1 & 0\\ 0 & \frac{s}{2-\nu_2^{(0)}}\end{pmatrix}\right\}^{-1} e^{2\pi i\ \mathrm{diag}(\frac{\theta_0}{2},-\frac{\theta_0}{2})}\ \left\{B\begin{pmatrix}1 & 0\\ 0 & \frac{s}{2-\nu_2^{(0)}}\end{pmatrix}\right\}\right] \tag{47}$$

$$m_x := \left[\left\{C\begin{pmatrix}1 & 0\\ 0 & \frac{s}{2-\nu_2^{(0)}}\end{pmatrix}\right\}^{-1} e^{2\pi i\ \mathrm{diag}(\frac{\theta_x}{2},-\frac{\theta_x}{2})}\ \left\{C\begin{pmatrix}1 & 0\\ 0 & \frac{s}{2-\nu_2^{(0)}}\end{pmatrix}\right\}\right] \tag{48}$$

Let us now choose

$$s=-\frac{1}{4}\ \frac{16^{\nu_2^{(0)}-1}e^{-i\pi\nu_1^{(0)}}}{(1-\nu_2^{(0)})^3}\ \left[\theta_0+\theta_x+1-\nu_2^{(0)}\right] \times\left[-\theta_0+\theta_x+1-\nu_2^{(0)}\right]\left[\theta_0+\theta_x-1+\nu_2^{(0)}\right]\left[\theta_0-\theta_x+1-\nu_2^{(0)}\right].$$

The traces of the products of the monodromy matrices are obtained by

$$\mathrm{tr}(M_1M_x)=\mathrm{tr}(m_1m_x),\quad \mathrm{tr}(M_0M_1)=\mathrm{tr}(m_0m_1),$$
$$\text{and}\quad \mathrm{tr}(M_0M_x)=\mathrm{tr}(m_0m_x)=-2\cos(\pi\nu_2^{(0)}).$$

Once the traces are computed, it is possible to compute $\nu_2^{(1)}$ and $\nu_2^{(\infty)}$, by (39), (40). Finally, we can compute $\nu_1^{(1)}$ and $\nu_1^{(\infty)}$, by formulae (37), where the functions

$a^{(1)}$ is obtained from $a^{(0)}$ with the substitutions $\nu_2^{(0)} \mapsto \nu_2^{(1)}$ and (13) (14) (15); $a^{(\infty)}$ is obtained from $a^{(0)}$ with the substitutions $\nu_2^{(0)} \mapsto \nu_2^{(\infty)}$ and (16) (17) (18).

The construction of the above procedure is explained in [**7**]. I apologize that I do not write here $\operatorname{tr}(m_1 m_x)$, $\operatorname{tr}(m_0 m_1)$ explicitly, because they are very long expressions that would take up too much space.

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