Two-Dimensional Supersymmetric Quantum Mechanics: Two Fixed Centers of Force*

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Abstract. The problem of building supersymmetry in the quantum mechanics of two Coulombian centers of force is analyzed. It is shown that there are essentially two ways of proceeding. The spectral problems of the SUSY (scalar) Hamiltonians are quite similar and become tantamount to solving entangled families of Razavy and Whittaker–Hill equations in the first approach. When the two centers have the same strength, the Whittaker–Hill equations reduce to Mathieu equations. In the second approach, the spectral problems are much more difficult to solve but one can still find the zero-energy ground states.

Key words: supersymmetry; integrability; quantum mechanics; two Coulombian centers

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1 Introduction

Supersymmetry is a bold idea which arose in the seventies in string and field theory. It was immediately realized that mechanisms of spontaneous supersymmetry breaking should be investigated searching for explanations of the apparent lack of supersymmetry in nature. In a series of papers [1, 2, 3] Witten proposed to analyze this phenomenon in the simplest possible setting: supersymmetric quantum mechanics. A new area of research in quantum mechanics was born with far-reaching consequences both in mathematics and physics.

Of course, there were antecedents in ordinary quantum mechanics (*nihil novum sub sole*), and indeed even before. The track can be followed back to some work by Clifford on the Laplacian operator, see [4], quoted in [5]. Recast in modern SUSY language, the Clifford supercharge is:

$$Q = \begin{pmatrix} 0 & i\nabla_1 + j\nabla_2 + k\nabla_3 \\ 0 & 0 \end{pmatrix}, \qquad i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k \ \text{ciclyc},$$

where i, j, k are the imaginary unit quaternions and

$$\nabla_1 = \frac{\partial}{\partial x_1} + A_1(\vec{x}), \qquad \nabla_2 = \frac{\partial}{\partial x_2} + A_2(\vec{x}), \qquad \nabla_3 = \frac{\partial}{\partial x_3} + A_3(\vec{x})$$

are the components of the gradient modified by the components of the electromagnetic vector potential. The SUSY Hamiltonian is:

$$Q^\dagger Q + QQ^\dagger = \begin{pmatrix} -\triangle + iB_1(\vec{x}) + jB_2(\vec{x}) + kB_3(\vec{x}) & 0 \\ 0 & -\triangle - iB_1(\vec{x}) - jB_2(\vec{x}) - kB_3(\vec{x}) \end{pmatrix},$$

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the Laplacian plus Pauli terms. Needless to say, an identical construction relates the Dirac operator in electromagnetic and/or gravitational fields backgrounds with the Klein–Gordon operator. The factorization method of identifying the spectra of Schrodinger operators, see [6] for a review, is another antecedent of supersymmetric quantum mechanics that can also be traced back to the 19th century through the Darboux theorem.

In its modern version, supersymmetric quantum mechanics prompted the study of many one-dimensional systems from a physical point of view. A good deal of this work can be found in [7, 8, 9, 10]. A frequent starting point in this framework is the following problem: given a non-SUSY one-dimensional quantum Hamiltonian, is it possible to build a supersymmetric extension? The answer to this question is positive when one finds a solution to the Riccati equation

$$\frac{1}{2}\frac{dW}{dx}\frac{dW}{dx}+\frac{\hbar}{2}\frac{d^2W}{dx^2}=V(x),$$

that identifies the – a priori unknown – "superpotential" $W(x)$ from the – given – potential $V(x)$. Several examples of this strategy have been worked out in [11].

The formalism of physical supersymmetric systems with more than one degree of freedom was first developed by Andrianov, Ioffe and coworkers in a series of papers [12, 13], published in the eighties. The same authors, almost simultaneously, considered higher than one-dimensional SUSY quantum mechanics from the point of view of the factorization of N-dimensional quantum systems [14, 15]. Factorability, even though essential in N-dimensional SUSY quantum mechanics, is not so effective as compared with the one-dimensional situation. Some degree of separability is also necessary to achieve analytical results. For this reason we started a program of research in the two-dimensional supersymmetric classical mechanics of Liouville systems [16]; i.e., those separable in elliptic, polar, parabolic, or Cartesian coordinates, see the papers [17] and [18]. We followed this path in the quantum domain for Type I Liouville models in [19], whereas Ioffe et al. also studied the interplay between supersymmetry and integrability in quantum and classical settings in other type of models in [20, 21]. In these papers, a new structure was introduced: second-order supercharges provided intertwined scalar Hamiltonians even in the two-dimensional case, see [23] for a review. This higher-order SUSY algebra allows for new forms of non-conventional separability in two dimensions. There are two possibilities: (1) a similarity transformation performs separation of variables in the supercharges and some eigenfunctions (partial solvability) can be found, see [24, 25]. (2) One of the two intertwined Hamiltonian allows for exact separability: the spectrum of the other is consequently known [26, 27].

The second difficulty with the jump in dimensions is the identification of the superpotential. Instead of the Riccati equation one must solve the PDE:

$$\frac{1}{2}\vec{\nabla}W(\vec{x})\cdot\vec{\nabla}W(\vec{x})+\frac{\hbar}{2}\nabla^2W(\vec{x})=V(\vec{x}).$$

In our case, we look for solutions of this PDE when $V(\vec{x})$ is the potential energy of the two Coulombian centers. We do not know how to solve it in general, but two different strategies should help us. First, following the work in [38] and [39] on the supersymmetric Coulomb problem, we shall choose the superpotential as the solution of the Poisson equation:

$$\frac{\hbar}{2}\nabla^2W(\vec{x})=V(\vec{x}).$$

The superpotential will be the solution of *another* Riccati-like PDE where a classical piece must be added to the potential of the two centers. Second, as in [41, 34] the selection of superpotential requires the solution of the Hamilton–Jacobi equation:

$$\frac{1}{2}\vec{\nabla}W(\vec{x})\cdot\vec{\nabla}W(\vec{x})=V(\vec{x}).$$

Again the superpotential must solve a third Riccati-like PDE, where now a quantum piece must be added to the potential of the two centers.

The organization of the paper is as follows: We start by briefly recalling the non-SUSY classical, Newtonian [40], and quantal, Coulombian [42, 33], two-center problem. We shall constrain the particle to move in one plane containing the two centers. The third coordinate is cyclic and it would be easy to extend our results to three dimensions. In Section 2 the formalism of two-dimensional SUSY quantum mechanics is developed, and the superpotential of the first Type is identified for two Coulombian centers. Bosonic zero-energy ground states are also found. Sections 3 and 4 are devoted to formulating the SUSY system in elliptic coordinates where the problem is separable in order to find fermionic zero-energy ground states. It is also shown that the spectral problem is tantamount to families of two ODE's of Razavy [43], and Whittaker–Hill type [44], see also [28]. Since these systems are quasi-exactly solvable, several eigenvalues are found following the work in [32]. In Section 5 two centers of the same strength are studied and some eigenfunctions are also found. Section 6 is fully devoted to the analysis of the Manton–Heumann approach applied to the two Coulombian centers. Finally, a summary is offered in Section 7.

1.1 The classical problem of two Newtonian/Coulombian centers

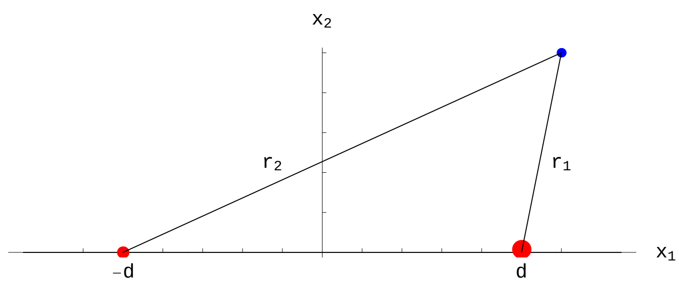


Figure 1. Location of the two centers and distances to the particle from the centers.

The classical action for a system of a light particle moving in a plane around two heavy bodies which are sources of static Newtonian/Coulombian forces is:

$$\tilde{S} = \int dt \left\{ \frac{1}{2} m \left(\frac{dx_1}{dt}\frac{dx_1}{dt} + \frac{dx_2}{dt}\frac{dx_2}{dt} \right) - \frac{\alpha_1}{r_1} - \frac{\alpha_2}{r_2} \right\}.$$

The centers are located at the points $(x_1 = -d,\ x_2 = 0)$, $(x_1 = d,\ x_2 = 0)$, their strengths are $\alpha_1 = \alpha \geq \alpha_2 = \delta\alpha > 0$, $\delta \in (0, 1]$, and

$$r_1 = \sqrt{(x_1 - d)^2 + x_2^2}, \qquad r_2 = \sqrt{(x_1 + d)^2 + x_2^2}$$

are the distances from the particle to the centers. In the following formulas we show the dimensions of the coupling constants and parameters and define non-dimensional variables:

$$[\alpha_1] = [\alpha_2] = [\alpha] = ML^3T^{-2}, \qquad [d] = L, \qquad [\delta] = 1,$$
$$x_1 \to d\,x_1, \qquad x_2 \to d\,x_2, \qquad t \to \sqrt{\frac{d^3 m}{\alpha}}\,t,$$
$$r_1 \to d\,r_1 = d\sqrt{(x_1 - 1)^2 + x_2^2}, \qquad r_2 \to d\,r_2 = d\sqrt{(x_1 + 1)^2 + x_2^2}.$$

In the rest of the paper we shall use non-dimensional variables. From the non-dimensional action

$$\tilde{S} = \sqrt{md\alpha}\,S = \sqrt{md\alpha}\int dt \left\{ \frac{1}{2}\left(\frac{dx_1}{dt}\frac{dx_1}{dt} + \frac{dx_2}{dt}\frac{dx_2}{dt} \right) - \frac{1}{r_1} - \frac{\delta}{r_2} \right\},$$

the linear momenta and Hamiltonian are defined:

$$p_1 = \frac{\partial L}{\partial \dot{x_1}} = \frac{dx_1}{dt}, \qquad p_2 = \frac{\partial L}{\partial \dot{x_2}} = \frac{dx_2}{dt},$$
$$\tilde{H} = \frac{\alpha}{d} H, \qquad H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{r_1} + \frac{\delta}{r_2}.$$

This system is completely integrable because there exists a "second invariant" in involution with the Hamiltonian:

$$\tilde{I}_2 = (md\alpha) I_2, \qquad I_2 = \frac{1}{2}(l^2 - p_2^2) + x_1 \left(\frac{\delta}{r_2} - \frac{1}{r_1} \right), \qquad l^2 = (x_1 p_2 - x_2 p_1)^2.$$

1.2 The quantum problem of two Coulombian centers of force

If $\sqrt{m\alpha d}$ is of the order of the Planck constant $\hbar$, the system is of quantum nature. Canonical quantization in terms of the non-dimensional $\bar{\hbar}$ constant,

$$p_i \rightarrow \hat{p}_i = -i\bar{\hbar}\frac{\partial}{\partial x_i}, \qquad x_i \rightarrow \hat{x}_i = x_i,$$
$$[\hat{x}_i, \hat{p}_j] = i\bar{\hbar}\delta_{ij}, \qquad \bar{\hbar} = \frac{\hbar}{\sqrt{md\alpha}},$$

converts the dynamical variables into operators. The quantum Hamiltonian, $\hat{\tilde{H}} = \frac{\alpha}{d}\hat{H}$, and the quantum symmetry operator, $\hat{\tilde{I}}_2 = (md\alpha)\hat{I}_2$, are mutually commuting operators:

$$\hat{H} = -\frac{\bar{\hbar}}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + \frac{1}{r_1} + \frac{\delta}{r_2}, \qquad [\hat{H}, \hat{I}_2] = \hat{H}\hat{I}_2 - \hat{I}_2\hat{H} = 0,$$
$$\hat{I}_2 = -\frac{\bar{\hbar}^2}{2}\left((x_1^2 - 1)\frac{\partial^2}{\partial x_2^2} + x_2^2\frac{\partial^2}{\partial x_1^2} - 2x_1x_2\frac{\partial^2}{\partial x_1 \partial x_2} - x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} \right) + x_1\left(\frac{\delta}{r_2} - \frac{1}{r_1} \right).$$

2 Two-dimensional $\mathcal{N} = 2$ SUSY quantum mechanics

We now describe how to build a non-specific system in two-dimensional $\mathcal{N} = 2$ SUSY quantum mechanics. Besides commuting – non-commuting – operators there are anti-commuting – non-anti-commuting operators to be referred respectively as "bosonic" and "fermionic" by analogy with QFT. The Fermi operators are represented on Euclidean spinors in $\mathbb{R}^4$ by the Hermitian 4×4 gamma matrices:

$$\psi_1^j = \frac{i}{\sqrt{2}}\gamma^j, \qquad \psi_2^j = -\frac{i}{\sqrt{2}}\gamma^{2+j}, \qquad (\gamma^j)^\dagger = \gamma^j, \qquad (\gamma^{2+j})^\dagger = \gamma^{2+j},$$
$$\{\gamma^j, \gamma^k\} = 2\delta^{jk} = \{\gamma^{2+j}, \gamma^{2+k}\}, \qquad \{\gamma^j, \gamma^{2+k}\} = 0, \qquad j, k = 1, 2.$$

The building blocks of the SUSY system are the two ($\mathcal{N} = 2$) quantum Hermitian supercharges: $\hat{Q}_1^\dagger = \hat{Q}_1$, $\hat{Q}_2^\dagger = \hat{Q}_2$,

$$\hat{Q}_1 = \sqrt{\bar{\hbar}}\sum_{j=1}^{2}\left(-i\bar{\hbar}\frac{\partial}{\partial x_j}\psi_1^j - \frac{\partial W}{\partial x_j}\psi_2^j \right), \qquad \hat{Q}_2 = \sqrt{\bar{\hbar}}\sum_{j=1}^{2}\left(-i\bar{\hbar}\frac{\partial}{\partial x_j}\psi_2^j + \frac{\partial W}{\partial x_j}\psi_1^j \right).$$

It is convenient to define the non-Hermitian supercharges $\hat{Q}_\pm = \hat{Q}_1 \pm i\hat{Q}_2$,

$$\hat{Q}_+ = i\sqrt{\bar{\hbar}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ \bar{\hbar}\frac{\partial}{\partial x_1} - \frac{\partial W}{\partial x_1} & 0 & 0 & 0 \\ \bar{\hbar}\frac{\partial}{\partial x_2} - \frac{\partial W}{\partial x_2} & 0 & 0 & 0 \\ 0 & -\bar{\hbar}\frac{\partial}{\partial x_2} + \frac{\partial W}{\partial x_2} & \bar{\hbar}\frac{\partial}{\partial x_1} - \frac{\partial W}{\partial x_1} & 0 \end{pmatrix},$$

$$\hat{Q}_- = i\sqrt{\bar{\hbar}} \begin{pmatrix} 0 & \bar{\hbar}\frac{\partial}{\partial x_1} + \frac{\partial W}{\partial x_1} & \bar{\hbar}\frac{\partial}{\partial x_2} + \frac{\partial W}{\partial x_2} & 0 \\ 0 & 0 & 0 & -\bar{\hbar}\frac{\partial}{\partial x_2} - \frac{\partial W}{\partial x_2} \\ 0 & 0 & 0 & \bar{\hbar}\frac{\partial}{\partial x_1} + \frac{\partial W}{\partial x_1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

because their anti-commutator determines the Hamiltonian $\hat{H}_S$ of the supersymmetric system:

$$\{\hat{Q}_+, \hat{Q}_-\} = 2\bar{\hbar}\hat{H}_S, \qquad [\hat{Q}_+, \hat{H}_S] = [\hat{Q}_-, \hat{H}_S] = 0.$$

The explicit form of the quantum SUSY Hamiltonian is enlightened by the "Fermi" number $F = \sum_{j=1}^{2} \psi_+^j \psi_-^j$ operator:

$$\hat{H}_S = \begin{pmatrix} \hat{h}^{(0)} & 0 & 0 & 0 \\ 0 & \hat{h}^{(1)}_{11} & \hat{h}^{(1)}_{12} & 0 \\ 0 & \hat{h}^{(1)}_{21} & \hat{h}^{(1)}_{22} & 0 \\ 0 & 0 & 0 & \hat{h}^{(2)} \end{pmatrix}, \qquad F = \sum_{j=1}^{2} \psi_+^j \psi_-^j = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

It has a block diagonal structure acting on the sub-spaces of the Hilbert space of Fermi numbers 0, 1, and 2. In the sub-spaces of Fermi numbers even $\hat{H}_S$ acts as ordinary differential Schrödinger operators. The scalar Hamiltonians are:

$$2\hat{h}^{(f=0)} = -\bar{\hbar}^2\nabla^2 + \vec{\nabla}W\vec{\nabla}W + \bar{\hbar}\nabla^2 W = -\bar{\hbar}^2\nabla^2 + 2\hat{V}^{(0)},$$
$$2\hat{h}^{(f=2)} = -\bar{\hbar}^2\nabla^2 + \vec{\nabla}W\vec{\nabla}W - \bar{\hbar}\nabla^2 W = -\bar{\hbar}^2\nabla^2 + 2\hat{V}^{(2)}.$$

In the sub-space of Fermi number 1, however, $\hat{H}_S$ is a matrix of differential operators, the 2×2 matrix Hamiltonian:

$$2\hat{h}^{(f=1)} = \begin{pmatrix} -\bar{\hbar}^2\nabla^2 + \vec{\nabla}W\vec{\nabla}W - \bar{\hbar}\Box^2 W & -2\bar{\hbar}\frac{\partial^2 W}{\partial x_1 \partial x_2} \\ -2\bar{\hbar}\frac{\partial^2 W}{\partial x_1 \partial x_2} & -\bar{\hbar}^2\nabla^2 + \vec{\nabla}W\vec{\nabla}W + \bar{\hbar}\Box^2 W \end{pmatrix},$$

$$\vec{\nabla} = \frac{\partial}{\partial x_1}\cdot\vec{e}_1 + \frac{\partial}{\partial x_2}\cdot\vec{e}_2, \qquad \nabla^2 = \frac{\partial^2}{\partial x_1 \partial x_1} + \frac{\partial^2}{\partial x_2 \partial x_2}, \qquad \Box^2 = \frac{\partial^2}{\partial x_1 \partial x_1} - \frac{\partial^2}{\partial x_2 \partial x_2}.$$

This is exactly the structure unveiled in [12, 13]. All the interactions expressed in $\hat{H}_S$ come from the as yet unspecified function $W(x_1, x_2)$, which is thus called the superpotential.

2.1 The superpotential I for the two-center problem

To build a supersymmetric system containing the interactions due to two Coulombian centers of force, we must start by identifying the superpotential. One possible choice, inspired by [38], is having the two-center potential energy in $\hat{h}^{(0)}$ in the term proportional to $\bar{\hbar}$. We must therefore solve the Poisson equation to find the superpotential I:

$$\frac{\bar{\hbar}}{2}\nabla^2\hat{W} = -\frac{1}{r_1} - \frac{\delta}{r_2}, \qquad \hat{W}(x_1, x_2) = -\frac{2r_1}{\bar{\hbar}} - \frac{2\delta r_2}{\bar{\hbar}}. \tag{1}$$

Note that the anticommutator between the supercharges induces a $\bar{\hbar}$ factor in front of the Laplacian. This fact, in turns, forces the singularity of the superpotential (henceforth, also of the potential) at the classical limit $\bar{\hbar} = 0$. The same singularity arises in the bound state spectra

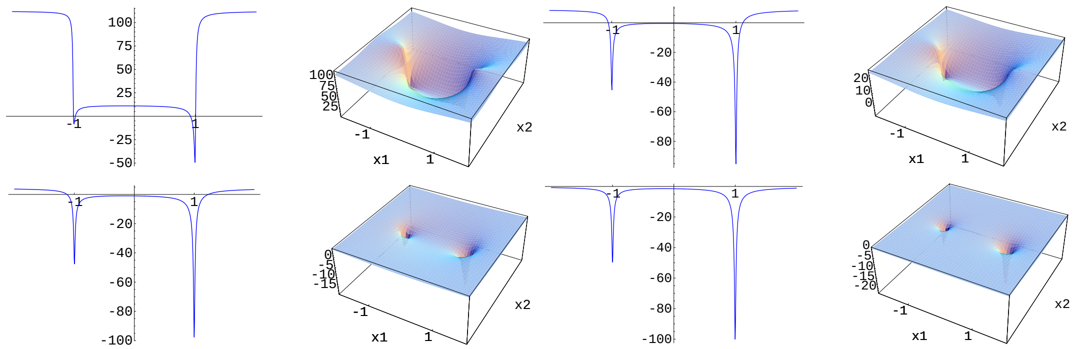


Figure 2. Cross section ($x_2 = 0$) and 3D graphics of the quantum potential $\hat{V}^{(0)}$ for $\delta = 1/2$. Cases: Upper row: (a) $\bar{\hbar} = 0.2$, (b) $\bar{\hbar} = 0.4$. Lower row: (a) $\bar{\hbar} = 1$ and (b) $\bar{\hbar} = 10$. Increasing $\bar{\hbar}$ the centers become more and more attractive.

of atoms, e.g., in the energy levels of the hydrogen atom. The potential energies in the scalar sectors are accordingly:

$$\hat{V}_I^{\binom{(0)}{(2)}} = \frac{2}{\bar{\hbar}^2}\left[1+\delta^2+\delta\left(\frac{r_1}{r_2}+\frac{r_2}{r_1}-\frac{4}{r_1 r_2}\right)\right] \mp \left(\frac{1}{r_1}+\frac{\delta}{r_2}\right).$$

We stress that the superpotential I is a solution of the Riccati-like PDE's:

$$\vec{\nabla}\hat{W}\vec{\nabla}\hat{W} \pm \bar{\hbar}\nabla^2\hat{W} = 2\hat{V}_I^{\binom{(0)}{(2)}}$$

and the scalar Hamiltonians for two SUSY Coulombian centers read:

$$\hat{h}^{\binom{(0)}{(2)}} = -\frac{\bar{\hbar}^2}{2}\nabla^2 + \frac{2}{\bar{\hbar}^2}\left[1+\delta^2+\delta\left(\frac{r_1}{r_2}+\frac{r_2}{r_1}-\frac{4}{r_1 r_2}\right)\right] \mp \left(\frac{1}{r_1}+\frac{\delta}{r_2}\right).$$

2.2 Bosonic zero modes I

The bosonic zero modes

$$\hat{Q}_\pm \Psi_0^{(0)}(x_1,x_2) = 0, \qquad \hat{Q}_\mp \Psi_0^{(2)}(x_1,x_2) = 0,$$

if normalizable, are the bosonic ground states of the system:

$$\Psi_0^{(0)}(x_1,x_2) = \begin{pmatrix} \exp[\frac{(-2r_1-2\delta r_2)}{\bar{\hbar}^2}] \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \Psi_0^{(2)}(x_1,x_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \exp[\frac{(2r_1+2\delta r_2)}{\bar{\hbar}^2}] \end{pmatrix}.$$

The norm of the true bosonic ground state $\Psi_0^{(0)}$ is finite and given in terms of Bessel functions:

$$N(\bar{\hbar}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_1 dx_2\, e^{\frac{2}{\bar{\hbar}}\hat{W}(x_1,x_2)} = 2\int_0^\infty dx_1 \int_0^\infty dx_2\, e^{\frac{2}{\bar{\hbar}^2}(-2r_1-2\delta r_2)},$$

$$N(\bar{\hbar}) = 2\pi\left[\frac{\bar{\hbar}^2}{4(1+\delta)} I_0\left(\frac{4}{\bar{\hbar}^2}(1-\delta)\right) K_1\left(\frac{4}{\bar{\hbar}^2}(1+\delta)\right) + \frac{\bar{\hbar}^2}{4(1-\delta)} I_1\left(\frac{4}{\bar{\hbar}^2}(1-\delta)\right) K_0\left(\frac{4}{\bar{\hbar}^2}(1+\delta)\right)\right].$$

In Fig. 3 plots of the zero energy bosonic ground state probability density of finding the particle in some area of the plane are shown for several values of $\bar{\hbar}$. The drawings reveal the physical meaning of the $\bar{h} = 0$ singularity: for $\bar{h} = 0.2$ we see the particle probability density peaked around the center on the right with a very small probability. Exactly at $\bar{h} = 0$, $e^{-\frac{4r_1}{\bar{h}^2}}$ is finite only for $r_1 = 0$ and zero otherwise, giving probability of finding the particle exactly in the center. $e^{-\frac{4\delta r_2}{\bar{h}^2}}$, however, is zero $\forall\, r_2$ meaning that the probability of this state is zero at the classical limit; in classical mechanics there are no isolated discrete states.

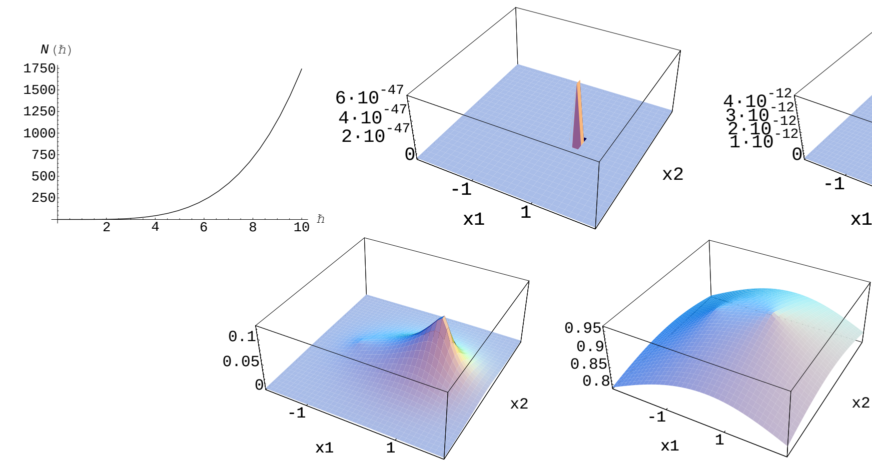


Figure 3. Graphics of the norm as a function of $\bar{h}$ and the ground state probability density, $|\Psi_0^{(0)}(x_1, x_2)|^2$, for $\delta = 1/2$ and the values of $\bar{h} = 0.2$, 0.4, 1 and 10. Note the extreme smallness for $\bar{h} = 0.2$. The norms for these four cases are: $N(0.2) = 3.5806 \cdot 10^{-47}$, $N(0.4) = 2.0576 \cdot 10^{-13}$, $N(1) = 0.00942$ and $N(10) = 1743.94$.

3 Two-dimensional $\mathcal{N} = 2$ SUSY quantum mechanics in elliptic coordinates

The search for more eigenfunctions of the SUSY Hamiltonian requires the use of the separability in elliptic coordinates of the problem at hand. This, in turn, needs the translation of our two-dimensional $\mathcal{N} = 2$ SUSY system to elliptic coordinates. A general reference (in Russian) where SUSY quantum mechanics is formulated in curvilinear coordinates is [36], see also [37] to find a more geometric version of SUSY QM on Riemannian manifolds.

The change from Cartesian to elliptic coordinates,

$$x_1 = uv \in (-\infty, +\infty), \qquad x_2 = \pm\sqrt{(u^2-1)(1-v^2)} \in (-\infty, +\infty),$$
$$u = \frac{1}{2}(r_1 + r_2) \in (1, +\infty), \qquad v = \frac{1}{2}(r_2 - r_1) \in (-1, 1),$$

induces a map from the plane to the infinite elliptic strip: $\mathbb{R}^2 \equiv (-\infty, +\infty) \times (-\infty, +\infty) \Longrightarrow \mathbb{E}^2 \equiv (-1, 1) \times (1, +\infty)$. This map also induces a non-Euclidean (but flat) metric in $\mathbb{E}^2$:

$$g(u, v) = \begin{pmatrix} g_{uu} = \frac{u^2-v^2}{u^2-1} & g_{uv} = 0 \\ g_{vu} = 0 & g_{vv} = \frac{u^2-v^2}{1-v^2} \end{pmatrix},$$

with Christoffel symbols:

$$\Gamma^u_{uu} = \frac{-u(1-v^2)}{(u^2-v^2)(u^2-1)}, \qquad \Gamma^v_{vv} = \frac{v(u^2-1)}{(u^2-v^2)(1-v^2)}, \qquad \Gamma^u_{uv} = \Gamma^u_{vu} = \frac{-v}{u^2-v^2},$$

$$\Gamma^v_{uu}=\frac{v(1-v^2)}{(u^2-v^2)(u^2-1)},\qquad \Gamma^u_{vv}=\frac{-u(u^2-1)}{(u^2-v^2)(1-v^2)},\qquad \Gamma^v_{uv}=\Gamma^v_{vu}=\frac{u}{u^2-v^2}.$$

Using the zweig-bein chosen in this form,

$$g^{uu}(u,v)=\sum_{j=1}^{2}e^u_j(u,v)e^u_j(u,v),\qquad g^{vv}(u,v)=\sum_{j=1}^{2}e^v_j(u,v)e^v_j(u,v),$$
$$e^u_1(u,v)=\left(\frac{u^2-1}{u^2-v^2}\right)^{\frac{1}{2}},\qquad e^v_2(u,v)=\left(\frac{1-v^2}{u^2-v^2}\right)^{\frac{1}{2}}$$

we now define "elliptic" spinors, "elliptic" Fermi operators, and "elliptic" supercharges:

$$\psi^u_\pm(u,v)=e^u_1(u,v)\psi^1_\pm,\qquad \psi^v_\pm(u,v)=e^v_2(u,v)\psi^2_\pm,$$
$$\hat{C}_+=-i\sqrt{\hbar}\begin{pmatrix}0&0&0&0\\ e^u_1\nabla^-_u&0&0&0\\ e^v_2\nabla^-_v&0&0&0\\ 0&-e^v_2\left(\nabla^-_v-\frac{\hbar v}{u^2-v^2}\right)&e^u_1\left(\nabla^-_u+\frac{\hbar u}{u^2-v^2}\right)&0\end{pmatrix},\qquad \nabla^\mp_u=\hbar\frac{\partial}{\partial u}\mp\frac{d\hat{F}}{du},$$
$$\hat{C}_-=-i\sqrt{\hbar}\begin{pmatrix}0&e^u_1\left(\nabla^+_u+\frac{\hbar u}{u^2-v^2}\right)&e^v_2\left(\nabla^+_v-\frac{\hbar v}{u^2-v^2}\right)&0\\ 0&0&0&-e^v_2\nabla^+_v\\ 0&0&0&e^u_1\nabla^+_u\\ 0&0&0&0\end{pmatrix},\qquad \nabla^\mp_v=\hbar\frac{\partial}{\partial v}\mp\frac{d\hat{G}}{dv},$$

where $\hat{W}(u,v)=\hat{F}(u)+\hat{G}(v)$.

To obtain the supercharges in Cartesian coordinates from the supercharges in elliptic coordinates, besides expressing u and v in terms of x_1 and x_2, one needs to act by conjugation with the idempotent, Hermitian matrix:

$$\mathcal{S}=\begin{pmatrix}1&0&0&0\\ 0&-ve^u_1(u,v)&-ue^v_2(u,v)&0\\ 0&-ue^v_2(u,v)&ve^u_1(u,v)&0\\ 0&0&0&-1\end{pmatrix},\qquad \mathcal{S}\hat{C}_+\mathcal{S}=\hat{Q}_+,\qquad \mathcal{S}\hat{C}_-\mathcal{S}=\hat{Q}_-.$$

Equation (1) in elliptic coordinates,

$$\frac{\hbar}{2}\left[\frac{u^2-1}{u^2-v^2}\left(\frac{d^2\hat{F}}{du^2}+\frac{u}{u^2-1}\frac{d\hat{F}}{du}\right)+\frac{1-v^2}{u^2-v^2}\left(\frac{d^2\hat{G}}{dv^2}-\frac{v}{1-v^2}\frac{d\hat{G}}{dv}\right)\right]$$
$$=-\frac{(1+\delta)u}{u^2-v^2}+\frac{(\delta-1)v}{u^2-v^2}\qquad (2)$$

is separable:

$$(u^2-1)\frac{d^2\hat{F}}{du^2}+u\frac{d\hat{F}}{du}+\frac{2(1+\delta)u}{\hbar}=\kappa=-(1-v^2)\frac{d^2\hat{G}}{dv^2}+v\frac{d\hat{G}}{dv}+\frac{2(\delta-1)v}{\hbar},$$

with separation constant κ. The general solution of equation (2) depends on two integration constants (besides an unimportant additive constant):

$$\hat{W}(u,v;\kappa,C_1,C_2)=-2\frac{(1+\delta)}{\hbar}u+\frac{C_1}{\hbar}\ln\left(u+\sqrt{u^2-1}\right)+\frac{\kappa}{2\hbar}\left(\ln\left(u+\sqrt{u^2-1}\right)\right)^2$$
$$+2\frac{(1-\delta)}{\hbar}v+\frac{C_2}{\hbar}\arcsin v-\frac{\kappa}{2\hbar}\left(\arcsin v\right)^2.$$

We find thus a three-parametric family of supersymmetric models for which the potentials in the scalar sectors are:

$$\hat{V}_I^{\binom{(0)}{(2)}}(x_1,x_2;\kappa,C_1,C_2) = \frac{2}{\hbar^2}\left[1+\delta^2+\delta\left(\frac{r_1}{r_2}+\frac{r_2}{r_1}-\frac{4}{r_1r_2}\right)\right]$$
$$+\frac{1}{2\hbar^2 r_1 r_2}\left[\left(C_1+\kappa\ln\frac{r_1+r_2+\sqrt{(r_1+r_2)^2-4}}{2}\right)^2+\left(C_2-\kappa\arcsin\frac{r_2-r_1}{2}\right)^2\right.$$
$$-2(1+\delta)\sqrt{(r_1+r_2)^2-4}\left(C_1+\kappa\ln\frac{r_1+r_2+\sqrt{(r_1+r_2)^2-4}}{2}\right)$$
$$\left.+2(1-\delta)\sqrt{4-(r_2-r_1)^2}\left(C_2-\kappa\arcsin\frac{r_2-r_1}{2}\right)\right]\mp\left(\frac{1}{r_1}+\frac{\delta}{r_2}\right).$$

We shall restrict ourselves in the sequel (as before) to the simplest choice $\kappa=C_1=C_2=0$ such that we shall work with the "elliptic" superpotential I

$$\hat{W}(u,v)=-\frac{2(1+\delta)u}{\hbar}+\frac{2(1-\delta)v}{\hbar},\qquad \hat{W}(x_1,x_2)=-\frac{2r_1}{\hbar}-\frac{2\delta r_2}{\hbar},$$

because this election is significative and contains enough complexity.

Considering families of superpotentials related to the same physical system in our 2D framework differs from a similar analysis on the 1D SUSY oscillator, see e.g. [29], in two aspects: (a) Because we solve the Poisson equation, not the Riccati equation, the family of superpotentials induces different families of potentials in both $\hat{h}^{(0)}$ and $\hat{h}^{(2)}$. (b) $\hat{h}^{(0)}$ and $\hat{h}^{(2)}$ are not iso-spectral because are not directly intertwined. The spectrum of $\hat{h}^{(1)}$ is the union of the spectra of $\hat{h}^{(0)}$ and $\hat{h}^{(2)}$.

3.1 Fermionic zero modes I

The fermionic zero modes

$$\hat{C}_+\Psi_0^{(1)}(u,v)=0\quad,\quad \hat{C}_-\Psi_0^{(1)}(u,v)=0,\qquad \Psi_0^{(1)}(u,v)=\begin{pmatrix}0\\ \psi_0^{(1)1}(u,v)\\ \psi_0^{(1)2}(u,v)\\ 0\end{pmatrix},$$

$$\Psi_0^{(1)}(u,v)=\begin{pmatrix}0\\ \psi_0^{(1)1}(u,v)\\ \psi_0^{(1)2}(u,v)\\ 0\end{pmatrix}=\frac{A_1}{\sqrt{u^2-v^2}}\begin{pmatrix}0\\ e^{\frac{-\hat{F}(u)+\hat{G}(v)}{\hbar}}\\ 0\\ 0\end{pmatrix}+\frac{A_2}{\sqrt{u^2-v^2}}\begin{pmatrix}0\\ 0\\ e^{\frac{\hat{F}(u)-\hat{G}(v)}{\hbar}}\\ 0\end{pmatrix}$$

are fermionic ground states if normalizable. Because the norm is:

$$N(\hbar)=2\int_{-1}^{1}dv\int_1^\infty du\left(\frac{A_1^2}{\sqrt{(u^2-1)(1-v^2)}}\,e^{-2\frac{\hat{F}(u)-\hat{G}(v)}{\hbar}}+\frac{A_2^2}{\sqrt{(u^2-1)(1-v^2)}}\,e^{2\frac{\hat{F}(u)-\hat{G}(v)}{\hbar}}\right),$$

it is finite if either $A_1=0$ or $A_2=0$. With our choice of sign in $F(u)$ the fermionic ground state is:

$$\Psi_0^{(1)}(u,v)=\frac{1}{\sqrt{u^2-v^2}}\begin{pmatrix}0\\ 0\\ e^{-\frac{2(1+\delta)u+2(1-\delta)v}{\hbar^2}}\\ 0\end{pmatrix}$$

and the norm is also given in terms of Bessel functions:

$$N(\bar\hbar) = 2\int_1^\infty du \int_{-1}^1 d\bar v \frac{e^{-\frac{4}{\bar\hbar^2}(1+\delta)u}}{\sqrt{u^2-1}} \frac{e^{-\frac{4}{\bar\hbar^2}(1-\delta)v}}{\sqrt{1-v^2}} = 2\pi K_0\left(\frac{4}{\bar\hbar^2}(1+\delta)\right) I_0\left(\frac{4c}{\bar\hbar^2}(1-\delta)\right).$$

It is also possible to give the fermionic ground state in $\mathbb{R}^2$ using the $\mathbb{S}$ matrix:

$$\Psi_0^{(1)}(x_1,x_2) = \mathcal{S}\Psi_0^{(1)}(r_1,r_2) = \begin{pmatrix} 0 \\ -\frac{(r_1+r_2)}{4\sqrt{r_1 r_2}}\sqrt{\frac{4}{r_1 r_2} - \frac{r_1}{r_2} - \frac{r_2}{r_1} + 2} \\ i\frac{(r_2-r_1)}{4\sqrt{r_1 r_2}}\sqrt{\frac{4}{r_1 r_2} - \frac{r_1}{r_2} - \frac{r_2}{r_1} - 2} \\ 0 \end{pmatrix} e^{-\frac{2(\delta r_1 + r_2)}{\bar\hbar^2}}.$$

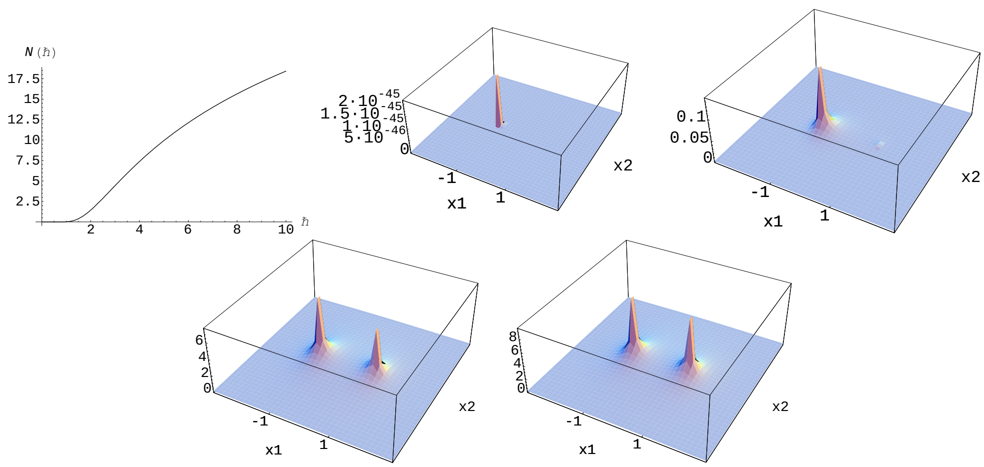


Figure 4. Graphics of $N(\bar\hbar)$ for: $\delta = 1/2$, $\kappa = 0$. $|\Psi_0^{(1)}(x_1,x_2)|^2$ for $\delta = 1/2$, $\bar\hbar = 0.2$, 1, 4 and 10. Norms: $N(0.2) = 1.3518 \cdot 10^{-45}$, $N(1) = 0.0178$, $N(4) = 7.3881$, $N(10) = 18.4297$.

4 The bosonic spectral problem I

The spectral problem for the scalar Hamiltonians is also separable in elliptic coordinates. Plugging in the separation ansatz

$$\hat h^{\binom{(0)}{(2)}} \psi_E^{\binom{(0)}{(2)}}(u,v) = E\psi_E^{\binom{(0)}{(2)}}(u,v), \qquad \psi_E^{\binom{(0)}{(2)}}(u,v) = \eta_E^{\binom{(0)}{(2)}}(u)\xi_E^{\binom{(0)}{(2)}}(v)$$

in the above spectral equation we find:

$$\left[-\bar\hbar^2(u^2-1)\frac{d^2}{du^2} - \bar\hbar^2 u\frac{d}{du} + \left(4\frac{(1+\delta)^2}{\bar\hbar^2}(u^2-1) \mp 2(1+\delta)u - 2Eu^2\right)\right]\eta_E^{\binom{(0)}{(2)}}(u)$$
$$= I\eta_E^{\binom{(0)}{(2)}}(u),$$
$$\left[-\bar\hbar^2(1-v^2)\frac{d^2}{dv^2} + \bar\hbar^2 v\frac{d}{dv} + \left(4\frac{(1-\delta)^2}{\bar\hbar^2}(1-v^2) \mp 2(1-\delta)v + 2Ev^2\right)\right]\xi_E^{\binom{(0)}{(2)}}(v)$$
$$= -I\xi_E^{\binom{(0)}{(2)}}(v),$$

where I is the eigenvalue of the symmetry operator $\hat I = -\{\hat h^{\binom{(0)}{(2)}} + \hat I_2^{\binom{(0)}{(2)}}\}$.

Research on the solution of these two ODE's by power series expansions will be published elsewhere. Here, we shall describe how another change of variables transmutes the first ODE into Razavy equation [43, 28]:

$$-\frac{d^2\eta_\pm(x)}{dx^2} + \left(\zeta_\pm \cosh 2x - M_\pm\right)^2 \eta_\pm(x) = \lambda_\pm\ \eta_\pm(x), \qquad x = \frac{1}{2}\text{arccosh}\, u$$

with parameters:

$$\zeta_\pm = \pm\frac{2}{\bar{\hbar}}\sqrt{\frac{4}{\bar{\hbar}^2}(1+\delta)^2 - 2E_\pm}, \qquad \lambda_\pm = M_\pm^2 + \frac{4}{\bar{\hbar}^2}\left(I_\pm + 4\frac{(1+\delta)^2}{\bar{\hbar}^2}\right),$$
$$M_\pm^2 = \frac{2(1+\delta)^2}{2(1+\delta)^2 - \bar{\hbar}^2 E_\pm}$$

Simili modo, another change of variables leads from the second ODE to the Whittaker–Hill or Razavy trigonometric [44, 28], equation

$$\frac{d^2\xi_\pm(y)}{dy^2} + \left(\beta_\pm \text{cos}2y - N_\pm\right)^2\ \xi_\pm(y) = \mu_\pm\ \xi_\pm(y), \qquad y = \frac{1}{2}\arccos v \in \left[0, \tfrac{\pi}{2}\right]$$

with parameters:

$$\beta_\pm = \mp\frac{2}{\bar{\hbar}}\sqrt{\frac{4}{\bar{\hbar}^2}(1-\delta)^2 - 2E_\pm}, \qquad N_\pm^2 = \frac{2(1-\delta)^2}{2(1-\delta)^2 - \bar{\hbar}^2 E_\pm},$$
$$\mu_\pm = N_\pm^2 + \frac{4}{\bar{\hbar}^2}\left(I_\pm + \frac{4}{\bar{\hbar}^2}(1-\delta)^2\right).$$

Therefore, the spectral problem in the scalar sectors is tantamount to the solving of two entangled sets – one per each pair (E,I) – of Razavy and Whittaker–Hill equations. If $M_\pm = n_1^\pm + 1$, $n_1^\pm \in \mathbb{N}^+$, the Razavy equation is QES; i.e., there are known $n+1$ finite eigenfunctions with an eigenvalue, see [32]:

$$E_\pm = E_{n_1^\pm} = 2\frac{(1+\delta)^2}{\bar{\hbar}^2}\left(1 - \frac{1}{(n_1^\pm + 1)^2}\right).$$

This means that for those values of $E_\pm$ one expects bound eigenstates of the SUSY Hamiltonian, although the v-dependence cannot be identified. If $N_\pm = n_2^\pm + 1$, $n_2^\pm \in \mathbb{N}^+$, there exist finite eigenfunctions of the Whittaker–Hill equation, with eigenvalues:

$$E_\pm = E_{n_2^\pm} = 2\frac{(1-\delta)^2}{\bar{\hbar}^2}\left(1 - \frac{1}{(n_2^\pm + 1)^2}\right).$$

Again, one expects these values of $E_\pm$ to be eigenvalues of the SUSY Hamiltonian, although their eigenfunctions are expected to be non-normalizable (except the $n_2^+ = 0$ case) because the u-dependent part of the eigenfunction is non finite and u is a non-compact variable. In any case, $n_1^+ = n_2^+ = 0$ gives the bosonic zero mode.

4.1 The fermionic spectrum I

The eigenfunctions of the matrix Hamiltonian,

$$\hat{h}_{11}^{(1)} = -\frac{1}{2}\bar{\hbar}^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \frac{2}{\bar{\hbar}^2}\left[1 + \delta^2 + \delta\left(\frac{r_1}{r_2} + \frac{r_2}{r_1} - \frac{4}{r_1 r_2}\right)\right]$$

$$
\begin{aligned}
&\qquad -\left(\frac{(x_1-1)^2-x_2^2}{r_1^3}+\delta\frac{(x_1+1)^2-x_2^2}{r_2^3}\right),\\
&\hat{h}_{12}^{(1)}=\hat{h}_{21}^{(1)}=-2\left(\frac{x_2(x_1-1)}{r_1^3}+\delta\frac{x_2(x_1+1)}{r_2^3}\right),\\
&\hat{h}_{22}^{(1)}=-\frac{1}{2}\bar{\hbar}^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)+\frac{2}{\bar{\hbar}^2}\left[1+\delta^2+\delta\left(\frac{r_1}{r_2}+\frac{r_2}{r_1}-\frac{4}{r_1r_2}\right)\right]\\
&\qquad +\left(\frac{(x_1-1)^2-x_2^2}{r_1^3}+\delta\frac{(x_1+1)^2-x_2^2}{r_2^3}\right),
\end{aligned}
$$

except the fermionic ground states, follows easily from the SUSY algebra. The fermionic eigenfunctions in elliptic coordinates,

$$
\Psi_{E_+}^{(1)}(u,v)=\hat{C}_+\Psi_{E_+}^{(0)}(u,v)=\begin{pmatrix}0\\ \psi_{E_+}^{(1)1}(u,v)\\ \psi_{E_+}^{(1)2}(u,v)\\ 0\end{pmatrix}=-i\sqrt{\bar{\hbar}}\begin{pmatrix}0\\ e_1^u\nabla_u^-\psi_{E_+}^{(0)}(u,v)\\ e_2^v\nabla_v^-\psi_{E_+}^{(0)}(u,v)\\ 0\end{pmatrix},\quad E_+=\begin{cases}E_{n_1^+},\\ E_{n_2^+},\end{cases}
$$

$$
\Psi_{E_-}^{(1)}(u,v)=\hat{C}_-\Psi_{E_-}^{(2)}(u,v)=\begin{pmatrix}0\\ \psi_{E_-}^{(1)1}(u,v)\\ \psi_{E_-}^{(1)2}(u,v)\\ 0\end{pmatrix}=-i\sqrt{\bar{\hbar}}\begin{pmatrix}0\\ -e_2^v\nabla_v^+\psi_{E_-}^{(2)}(u,v)\\ e_1^u\nabla_u^+\psi_{E_-}^{(2)}(u,v)\\ 0\end{pmatrix},\quad E_-=\begin{cases}E_{n_1^-},\\ E_{n_2^-},\end{cases}
$$

in Cartesian coordinates $\Psi_{E_\pm}^{(1)}(x_1,x_2)=\mathcal{S}\Psi_{E_\pm}^{(1)}(u,v)$ read:

$$
\Psi_{E_+}^{(1)}(x_1,x_2)=\hat{Q}_+\Psi_{E_+}^{(0)}(x_1,x_2)=\begin{pmatrix}0\\ \psi_{E_+}^{(1)1}(x_1,x_2)\\ \psi_{E_+}^{(1)2}(x_1,x_2)\\ 0\end{pmatrix}=-i\sqrt{\bar{\hbar}}\begin{pmatrix}0\\ (\bar{\hbar}\frac{\partial}{\partial x_1}-\frac{\partial W}{\partial x_1})\psi_{E_+}^{(0)}(x_1,x_2)\\ (\bar{\hbar}\frac{\partial}{\partial x_2}-\frac{\partial W}{\partial x_2})\psi_{E_+}^{(0)}(x_1,x_2)\\ 0\end{pmatrix},
$$

$$
\Psi_{E_-}^{(1)}(x_1,x_2)=\hat{Q}_-\Psi_{E_-}^{(2)}(x_1,x_2)=\begin{pmatrix}0\\ \psi_{E_-}^{(1)1}(x_1,x_2)\\ \psi_{E_-}^{(1)2}(x_1,x_2)\\ 0\end{pmatrix}=-i\sqrt{\bar{\hbar}}\begin{pmatrix}0\\ (-\bar{\hbar}\frac{\partial}{\partial x_2}-\frac{\partial W}{\partial x_2})\psi_{E_-}^{(2)}(x_1,x_2)\\ (\bar{\hbar}\frac{\partial}{\partial x_1}+\frac{\partial W}{\partial x_1})\psi_{E_-}^{(2)}(x_1,x_2)\\ 0\end{pmatrix}.
$$

5 Two centers of the same strength

If the two centers have the same strength, $\delta=1$, the spectral problem in the scalar sectors becomes tantamount to the two ODE's:

$$
\begin{aligned}
&\left[-\bar{\hbar}^2(u^2-1)\frac{d^2}{du^2}-\bar{\hbar}^2u\frac{d}{du}+\left(\frac{16}{\bar{\hbar}^2}(u^2-1)\mp 4u-2Eu^2\right)\right]\eta_E^{\binom{(0)}{(2)}}(u)=I\eta_E^{\binom{(0)}{(2)}}(u),\\
&\left[-\bar{\hbar}^2(1-v^2)\frac{d^2}{dv^2}+\bar{\hbar}^2v\frac{d}{dv}+2Ev^2\right]\xi_E^{\binom{(0)}{(2)}}(v)=-I\xi_E^{\binom{(0)}{(2)}}(v).
\end{aligned}
$$

Identical changes of variables as those performed in the $0\leq\delta\leq 1$ cases now lead to the Razavy and Mathieu equations [35], with parameters:

$$
-\frac{d^2\eta_\pm(x)}{dx^2}+(\zeta_\pm\cosh 2x-M_\pm)^2\,\eta_\pm(x)=\lambda_\pm\;\eta_\pm(x),
$$

$$\zeta_\pm = \pm\frac{2}{\bar\hbar}\sqrt{\frac{16}{\bar\hbar^2} - 2E_\pm}, \qquad M_\pm^2 = \frac{8}{8-\bar\hbar^2 E_\pm}, \qquad \lambda_\pm = M_\pm^2 + \frac{4}{\bar\hbar^2}\left(I_\pm + \frac{16}{\bar\hbar^2}\right),$$
$$-\frac{d^2\xi_\pm(y)}{dy^2} + (\alpha_\pm \cos 4y + \sigma_\pm)\,\xi_\pm(y) = 0, \qquad \alpha_\pm = \frac{4E_\pm}{\bar\hbar^2}, \qquad \sigma_\pm = \frac{4}{\bar\hbar^2}(I_\pm + E).$$

We now select the three lowest energy levels from finite solutions of the Razavy equation:

$$E_0 = 0, \qquad E_1 = \frac{6}{\bar\hbar^2}, \qquad E_2 = \frac{64}{9\bar\hbar^2}.$$

The corresponding eigenfunctions of the Razavy Hamiltonians for $n_1^\pm = 0, 1, 2$ are:

$$\eta_\pm^{01}(u) = e^{\mp\frac{4u}{\bar\hbar^2}}, \qquad \eta_\pm^{11}(u) = e^{\mp\frac{2u}{\bar\hbar^2}}\sqrt{2(u+1)}, \qquad \eta_\pm^{12}(u) = -e^{\mp\frac{2u}{\bar\hbar^2}}\sqrt{2(u-1)},$$
$$\eta_\pm^{21}(u) = -2e^{\mp\frac{4u}{3\bar\hbar^2}}\sqrt{u^2-1}, \qquad \eta_\pm^{22}(u) = \pm\frac{3\bar\hbar^2}{8}e^{\mp\frac{4u}{3\bar\hbar^2}}\left[\pm\frac{16}{3\bar\hbar^2}u - 1 + \sqrt{1+\frac{256}{9\bar\hbar^4}}\right],$$
$$\eta_\pm^{23}(u) = \pm\frac{3\bar\hbar^2}{8}e^{\mp\frac{4u}{3\bar\hbar^2}}\left[\pm\frac{16}{3\bar\hbar^2}u - 1 - \sqrt{1+\frac{256}{9\bar\hbar^4}}\right].$$

The energy degeneracy is labeled by the eigenvalues of the symmetry operator

$$I_\pm^{nm} = \frac{\bar\hbar^2}{4}(\lambda_\pm^{nm} - (n+1)^2) - \frac{16}{\bar\hbar^2}, \qquad m = 1, 2, \ldots, n+1,$$
$$I_\pm^{01} = 0, \qquad I_\pm^{11} = -\frac{\bar\hbar^2}{4} - \frac{12}{\bar\hbar^2} \mp 2, \qquad I_\pm^{12} = -\frac{\bar\hbar^2}{4} - \frac{12}{\bar\hbar^2} \pm 2,$$
$$I_\pm^{21} = -\bar\hbar^2 - \frac{128}{9\bar\hbar^2}, \qquad I_\pm^{22} = -\frac{\bar\hbar^2}{2} - \frac{128}{9\bar\hbar^2} - \frac{1}{6}\sqrt{256+9\bar\hbar^4},$$
$$I_\pm^{23} = -\frac{\bar\hbar^2}{2} - \frac{128}{9\bar\hbar^2} + \frac{1}{6}\sqrt{256+9\bar\hbar^4}.$$

In this rotationally non-invariant system the symmetry operator $\hat{I}$ replaces the orbital angular momentum in providing a basis of common eigenfunctions with $\bar{H}$ in each degenerate in energy sub-space of the Hilbert space in such a way that a quantum number reminiscent of the orbital angular momentum arises.

Next we next consider even/odd in v solutions of the Mathieu equations

$$\xi_{\pm\text{even}}^{nm}(v) = \frac{c_1}{2}\left(C\left[a_\pm^{nm}, q_n, \arccos(v)\right] + C\left[a_\pm^{nm}, q_n, \arccos(-v)\right]\right)$$
$$+ \frac{c_2}{2}\left(S\left[a_\pm^{nm}, q_n, \arccos(v)\right] + S\left[a_\pm^{nm}, q_n, \arccos(-v)\right]\right),$$
$$\xi_{\pm\text{odd}}^{nm}(v) = \frac{d_1}{2}\left(C\left[a_\pm^{nm}, q_n, \arccos(v)\right] - C\left[a_\pm^{nm}, q_n, \arccos(-v)\right]\right)$$
$$+ \frac{d_2}{2}\left(S\left[a_\pm^{nm}, q_n, \arccos(v)\right] - S\left[a_\pm^{nm}, q_n, \arccos(-v)\right]\right).$$

The reason is that the invariance of the problem under $r_1 \leftrightarrow r_2 \equiv v \leftrightarrow -v$ – the exchange between the two centers – forces even or odd eigenfunctions in v. The parameters of the Mathieu equations determined by the spectral problem are:

$$a_\pm^{nm} = -\frac{\sigma_\pm^{nm}}{4} = -\frac{E_n + I_\pm^{nm}}{\bar\hbar^2}, \qquad q_n = \frac{\alpha_n}{8} = \frac{E_n}{2\bar\hbar^2}.$$

To fit in with the parameters of the Razavy Hamiltonians set by $n_1^\pm = 0, 1, 2$ we must choose:

$$q_0 = 0, \qquad a_\pm^{01} = 0,$$

$$q_1 = \frac{3}{\bar{\hbar}^4}, \qquad a_\pm^{11} = \frac{6}{\bar{\hbar}^4} \pm \frac{2}{\bar{\hbar}^2} + \frac{1}{4}, \qquad a_\pm^{12} = \frac{6}{\bar{\hbar}^4} \mp \frac{2}{\bar{\hbar}^2} + \frac{1}{4},$$
$$q_2 = \frac{32}{9\bar{\hbar}^4}, \qquad a_\pm^{21} = 1 + \frac{64}{9\bar{\hbar}^4}, \qquad a_\pm^{22} = \frac{1}{2} + \frac{64}{9\bar{\hbar}^4} + \frac{1}{6\bar{\hbar}^2}\sqrt{256 + 9\bar{\hbar}^4},$$
$$a_\pm^{23} = \frac{1}{2} + \frac{64}{9\bar{\hbar}^4} - \frac{1}{6\bar{\hbar}^2}\sqrt{256 + 9\bar{\hbar}^4}.$$

Therefore,

$$\psi^{(0)}_{nm\ \mathrm{even/odd}}(u,v) = \eta_+^{nm}(u)\xi^{nm}_{+\mathrm{even/odd}}(v), \qquad n \geq 0, \qquad m = 1, 2, \dots, m+1,$$

is a set of bound states of non-zero energy of the scalar Hamiltonian of two SUSY Coulombian centers of the same strength. The paired fermionic eigenstates are obtained through the action of the appropriate supercharge. Since every non-zero-energy state come in bosonic-fermionic pairs, the criterion for spontaneous supersymmetry breaking is the existence of a Fermi–Bose pair ground state of positive energy connected one with each other by one of the supercharges. Our SUSY Hamiltonian has both bosonic and fermionic zero modes as single ground states; consequently, supersymmetry is not spontaneously broken in this system.

In the next figures we show several graphics of bosonic SUSY eigenfunctions for the choice: $c_2 = d_2 = 0$, $c_1 = d_1 = 1$.

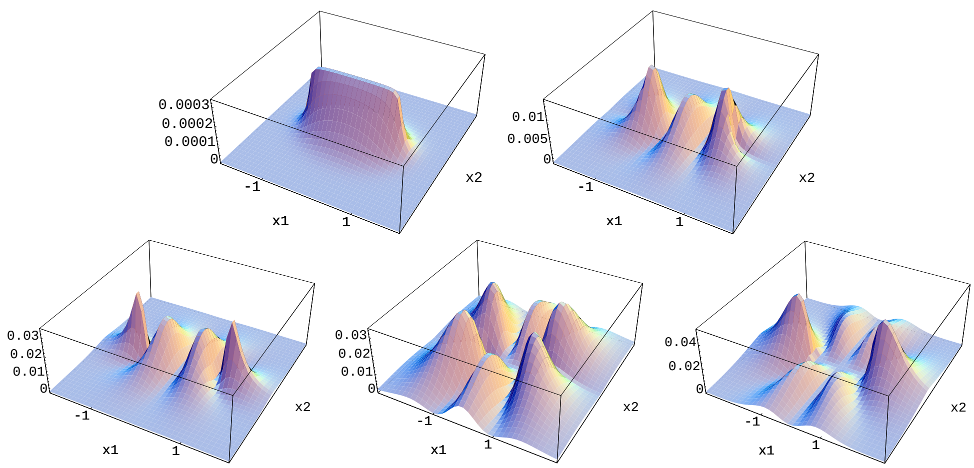


Figure 5. Graphics with ($\bar{\hbar} = 1$): $|\psi^{01}_{+\mathrm{even}}(x_1,x_2)|^2$; $|\psi^{11}_{+\mathrm{even}}(x_1,x_2)|^2$ and $|\psi^{11}_{+\mathrm{odd}}(x_1,x_2)|^2$; $|\psi^{21}_{+\mathrm{even}}(x_1,x_2)|^2$ and $|\psi^{21}_{+\mathrm{odd}}(x_1,x_2)|^2$.

5.1 Bosonic and fermionic ground states for two centers of the same strength

For comparison, we also plot the bosonic and fermionic ground states for several values of $\bar{\hbar}$.

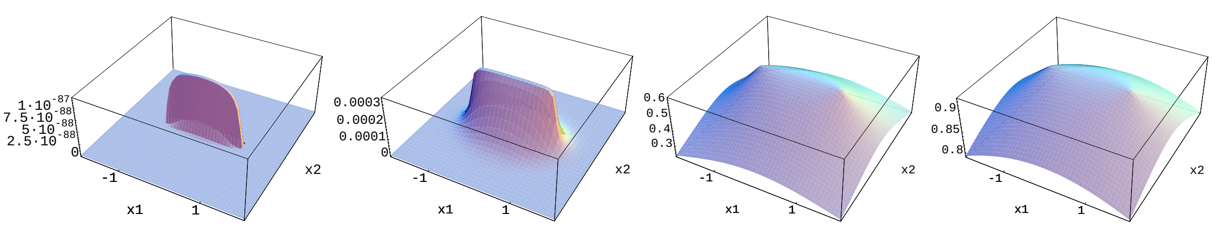


Figure 6. Graphics of the probability density $|\Psi_0^{(0)}(x_1,x_2)|^2$ for $\delta = 1$, and the values of $\bar{\hbar} = 0.2$, 1, 4 and 10.

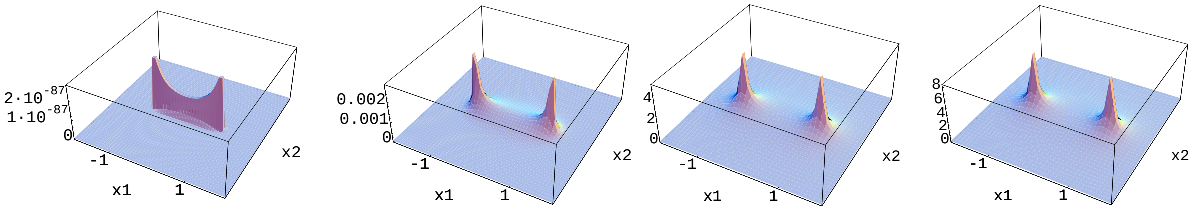


Figure 7. Graphics of $|\Psi_0^{(1)}(x_1,x_2)|^2$ for $\delta=1$, and $\bar{h}=0.2$, 1, 4 and 10. The norms are: $N(0.2)=7.7012\cdot 10^{-88}$, $N(1)=0.0009$, $N(4)=5.8083$, $N(10)=16.6347$.

6 The superpotential II for the two-center problem

There is another possibility to find the potential energy of two Coulombian centers in a 2D $\mathcal{N}=2$ SUSY Hamiltonian. The superpotential must satisfy the Hamilton–Jacobi equation rather than the Poisson equation (1):

$$\frac{1}{r_1}+\frac{\delta}{r_2}=\frac{1}{2}\left(\frac{\partial W}{\partial x_1}\right)^2+\frac{1}{2}\left(\frac{\partial W}{\partial x_2}\right)^2. \tag{3}$$

Note that in this case the two centers must be repulsive to guarantee a real W. This point of view, which follows the path shown in [41] and [34] for the Coulomb problem, amounts to the quantization of a classical supersymmetric system, only semi-positive definite for repulsive potentials.

Again using elliptic coordinates, the Hamilton–Jacobi equation separates

$$\kappa=-(u^2-1)\left(\frac{dF}{du}\right)^2+2(1+\delta)u,\qquad \kappa=(1-v^2)\left(\frac{dG}{dv}\right)^2+2(\delta-1)v$$

by plugging in (3) the ansatz:

$$W(u,v;\kappa)=F_a(u;\kappa)+G_b(v;\kappa),\qquad a,b=0,1.$$

The quadratures

$$F_a(u;\kappa)=(-1)^a\int_1^u\frac{\sqrt{2(1+\delta)u-\kappa}}{\sqrt{u^2-1}}\,du,\qquad G_b(v;\kappa)=(-1)^b\int_{-1}^v\frac{\sqrt{2(1-\delta)v+\kappa}}{\sqrt{1-v^2}}\,dv$$

show that the separation constant κ is constrained in order to find real $F_a(u)$ and $G_b(v)$: $2(1-\delta)\leq\kappa\leq 2(1+\delta)$. Note that there are two different possibilities: $a=b$ and $a\neq b$. A different global sign only exchanges the (0) with the (2) and the (1)1 with the (1)2 sectors. The superpotential II is thus given in terms of incomplete and complete elliptic integrals of the first and second type, see [30, 31]:

$$\begin{aligned}F_a(u;\kappa)&=(-1)^a2i\sqrt{\kappa+2(1+\delta)}\left(E\left[\sin^{-1}\sqrt{\frac{\kappa-2(1+\delta)u}{\kappa-2(1+\delta)}},\frac{\kappa-2(1+\delta)}{\kappa+2(1+\delta)}\right]\right.\\&\left.-E\left[\frac{\pi}{2},\frac{\kappa-2(1+\delta)}{\kappa+2(1+\delta)}\right]-F\left[\sin^{-1}\sqrt{\frac{\kappa-2(1+\delta)u}{\kappa-2(1+\delta)}},\frac{\kappa-2(1+\delta)}{\kappa+2(1+\delta)}\right]+F\left[\frac{\pi}{2},\frac{\kappa-2(1+\delta)}{\kappa+2(1+\delta)}\right]\right),\\G_b(v;\kappa)&=(-1)^b2i\sqrt{\kappa-2(1-\delta)}\\&\times\left(-E\left[\sin^{-1}\sqrt{\frac{\kappa+2(1-\delta)v}{\kappa+2(1-\delta)}},\frac{\kappa+2(1-\delta)}{\kappa-2(1-\delta)}\right]+E\left[\sin^{-1}\sqrt{\frac{\kappa-2(1-\delta)}{\kappa+2(1-\delta)}},\frac{\kappa+2(1-\delta)}{\kappa-2(1-\delta)}\right]\right.\end{aligned}$$

$$+ F\left[\sin^{-1}\sqrt{\frac{\kappa+2(1-\delta)v}{\kappa+2(1-\delta)}},\frac{\kappa+2(1-\delta)}{\kappa-2(1-\delta)}\right] - F\left[\sin^{-1}\sqrt{\frac{\kappa-2(1-\delta)}{\kappa+2(1-\delta)}},\frac{\kappa+2(1-\delta)}{\kappa-2(1-\delta)}\right]\Bigg).$$

The scalar Hamiltonians, both in elliptic and Cartesian coordinates, read:

$$\hat{h}^{\binom{(0)}{(2)}} = \frac{1}{2(u^2-v^2)}\Bigg\{ -\bar{\hbar}^2\left((u^2-1)\frac{d^2}{du^2}+u\frac{d}{du}+(1-v^2)\frac{d^2}{dv^2}-v\frac{d}{dv}\right)$$
$$+ 2(1+\delta)u + 2(1-\delta)v$$
$$\pm\bar{\hbar}\left[(-1)^a(1+\delta)\sqrt{\frac{u^2-1}{2(1+\delta)u-\kappa}}+(-1)^b(1-\delta)\sqrt{\frac{1-v^2}{2(1-\delta)v+\kappa}}\right]\Bigg\},$$

$$\hat{h}^{\binom{(0)}{(2)}} = -\frac{\bar{\hbar}^2}{2}\nabla^2+\frac{1}{r_1}+\frac{\delta}{r_2}$$
$$\pm\frac{\bar{\hbar}}{4r_1r_2}\left\{\frac{(-1)^a(1+\delta)\sqrt{(r_1+r_2)^2-4}}{\sqrt{(1+\delta)(r_1+r_2)-\kappa}}+\frac{(-1)^b(1-\delta)\sqrt{4-(r_1-r_2)^2}}{\sqrt{\kappa-(1-\delta)(r_1-r_2)}}\right\}.$$

6.1 Type IIa and Type IIb two-center SUSY quantum mechanics

We now present the graphics of the scalar potential for $a = b$, a system that we shall call Type IIa $\mathcal{N} = 2$ SUSY two Coulombian centers. By the same token, the system arising from the superpotential I will be called Type I $\mathcal{N} = 2$ SUSY two Coulombian centers.

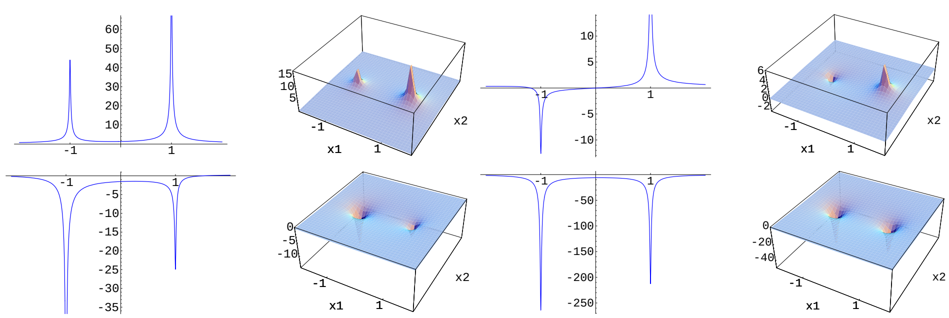


Figure 8. 3D graphics of the quantum potential $\hat{V}^{(0)}$ for $\mathbf{a} = \mathbf{b} = \mathbf{1}$ (or $\hat{V}^{(2)}$ for $\mathbf{a} = \mathbf{b} = \mathbf{0}$). We choose $\delta = 1/2$, $\kappa = 3$. Cases: (a) $\bar{\hbar} = 0.2$, $\bar{\hbar} = 2$. Observe that when $\bar{\hbar}$ is increased this provides a reduction of the strength of the repulsive centers and the left center becomes attractive whereas the right center is still repulsive. (b) $\bar{\hbar} = 4$ and $\bar{\hbar} = 10$, both centers are attractive. With increasing $\bar{\hbar}$ the centers become more and more attractive.

If $a \neq b$, we shall call the system Type IIb $\mathcal{N} = 2$ SUSY two Coulombian centers. The scalar potential is drawn in the graphics below for several values of $\bar{\hbar}$.

The differences between the Type IIa and Type IIb potentials, increasing with $\bar{\hbar}$, are shown in the next figure.

Below we shall discuss the coincidences and differences of the three distinct points of view.

6.2 The spectral problem

Both for the Type IIa and Type IIb systems the spectral problem in the scalar sectors is separable in elliptic coordinates:

$$\hat{h}^{\binom{(0)}{(2)}}\psi_E^{\binom{(0)}{(2)}}(u,v) = E\psi_E^{\binom{(0)}{(2)}}(u,v), \qquad \psi_E^{\binom{(0)}{(2)}}(u,v) = \eta_E^{\binom{(0)}{(2)}}(u)\zeta_E^{\binom{(0)}{(2)}}(v),$$

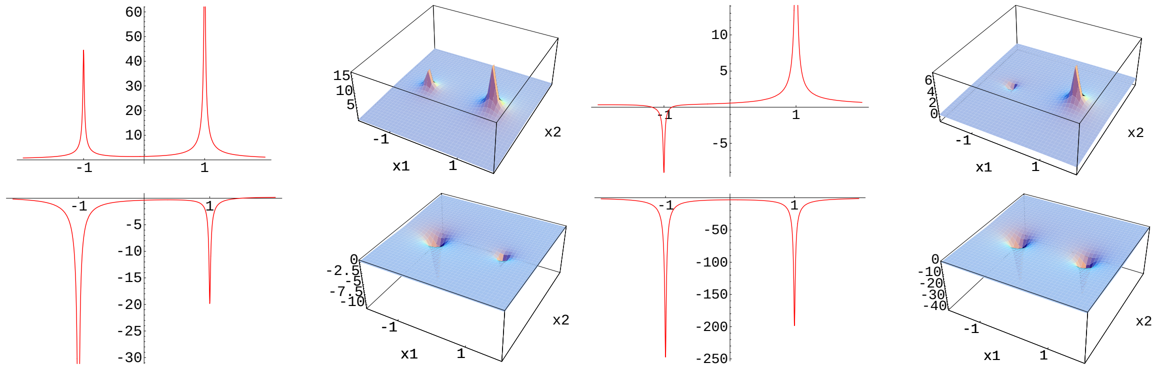


Figure 9. 3D graphics of the quantum potential $\hat{V}^{(0)}$ for $\mathbf{a}=\mathbf{1}$, $\mathbf{b}=\mathbf{0}$ (or $\hat{V}^{(2)}$ for $\mathbf{a}=\mathbf{0}$, $\mathbf{b}=\mathbf{1}$). We choose $\delta=1/2$, $\kappa=3$. Cases: (a) $\bar{\hbar}=0.2$, $\bar{\hbar}=2$. Observe that once again increasing $\bar{\hbar}$ provides a reduction of the strength of the repulsive centers and the left center becomes attractive whereas the right center is still repulsive. (b) $\bar{\hbar}=4$ and $\bar{\hbar}=10$, both centers are attractive.

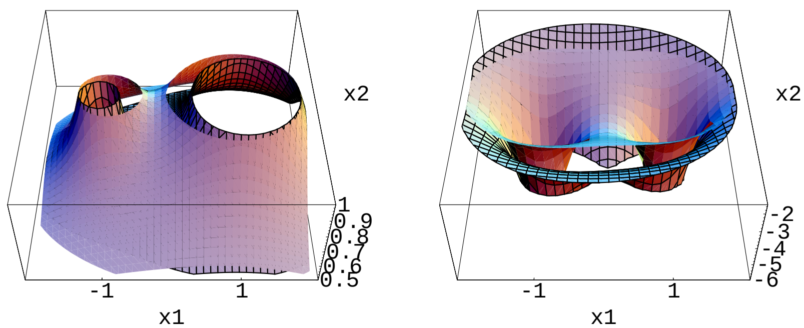


Figure 10. 3D graphics of the quantum potential $\hat{V}^{(0)}$ for $\mathbf{a}=\mathbf{1}=\mathbf{b}$ and $\mathbf{a}=\mathbf{1}$, $\mathbf{b}=\mathbf{0}$ in the cases $\bar{\hbar}=1$, $\bar{\hbar}=10$. In this range we note the differences between Type IIa and IIb quantum potentials.

$$\left[-\bar{\hbar}^2(u^2-1)\frac{d^2}{du^2}-\bar{\hbar}^2u\frac{d}{du}+2(1+\delta)u-2Eu^2\pm\bar{\hbar}(-1)^a(1+\delta)\sqrt{\frac{u^2-1}{2(1+\delta)u-\kappa}}\right]\eta_E^{\binom{(0)}{(2)}}(u)$$
$$=I\eta_E^{\binom{(0)}{(2)}}(u),$$
$$\left[-\bar{\hbar}^2(1-v^2)\frac{d^2}{dv^2}+\bar{\hbar}^2v\frac{d}{dv}+2(1-\delta)v+2Ev^2\pm\bar{\hbar}(-1)^b(1-\delta)\sqrt{\frac{1-v^2}{2(1-\delta)v-\kappa}}\right]\zeta_E^{\binom{(0)}{(2)}}(v)$$
$$=-I\zeta_E^{\binom{(0)}{(2)}}(v).$$

The separated ODE's are, however, much more difficult (non-linear) than in the Type I system and there is no hope of finding explicit eigenvalues and eigenfunctions.

6.3 Bosonic ground states

We shall therefore concentrate on searching for the ground states. First, the bosonic zero modes:

$$\hat{C}_+\Psi_0^{(0)}(u,v)=0, \qquad \hat{C}_-\Psi_0^{(2)}(u,v)=0.$$

The separation ansatz

$$\psi_0^{\binom{(0)}{(2)}}(u,v)=\eta_0^{\binom{(0)}{(2)}}(u)\zeta_0^{\binom{(0)}{(2)}}(v)$$

makes these equations equivalent to:

$$e_1^u \nabla_u^- \eta_0^{(0)}(u) = 0, \qquad e_1^u \nabla_u^+ \eta_0^{(2)}(u) = 0, \qquad e_2^v \nabla_v^- \zeta_0^{(0)}(v) = 0, \qquad -e_2^v \nabla_v^+ \zeta_0^{(2)}(v) = 0,$$

or,

$$\left(\bar{\hbar}\frac{d}{du} \mp (-1)^a \sqrt{\frac{2(1+\delta)u-\kappa}{u^2-1}}\right)\eta_0^{\binom{(0)}{(2)}}(u) = 0,$$
$$\left(\bar{\hbar}\frac{d}{dv} \mp (-1)^b \sqrt{\frac{2(1-\delta)v+\kappa}{1-v^2}}\right)\zeta_0^{\binom{(0)}{(2)}}(v) = 0.$$

The solutions, i.e. the bosonic zero modes, are:

$$\eta_0^{\binom{(0)}{(2)}}(v) = \exp\left[\pm\frac{F_a(u;\kappa)}{\bar{\hbar}}\right], \qquad \zeta_0^{\binom{(0)}{(2)}}(v) = \exp\left[\pm\frac{G_b(v;\kappa)}{\bar{\hbar}}\right].$$

It is not possible to calculate analytically the norm in these cases, but we offer a numerical integration of the $F=0$ bosonic ground states:

$$N(\bar{\hbar};\kappa,a,b) = 2\int_{-1}^{1} dv \int_{1}^{\infty} du\, \frac{u^2-v^2}{\sqrt{u^2-1}\,\sqrt{1-v^2}} \exp\left[\frac{2\,F_a(u;\kappa)}{\bar{\hbar}}\right] \exp\left[\frac{2\,G_b(v;\kappa)}{\bar{\hbar}}\right]$$

in the next figures. We observe that there is one normalizable bosonic ground state of zero energy of Type IIa and one of Type IIb once the value of a is set to be one. Remarkably, the Type IIb zero mode disappears (the norm becomes infinity) at the classical limit.

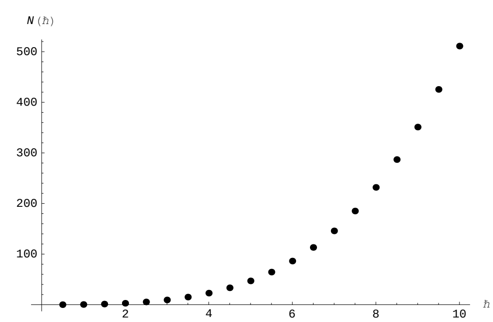


Figure 11. Numerical plot of $N(\bar{\hbar};3,1,1)$ as function of $\bar{\hbar}$ for $\delta = 1/2$.

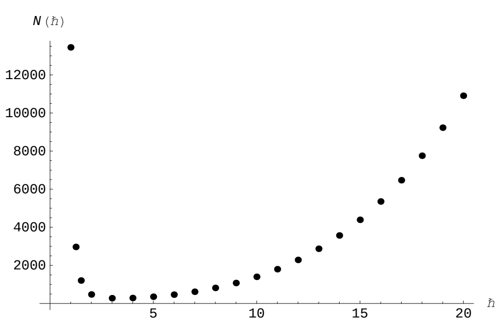


Figure 12. Numerical plot of $N(\bar{\hbar};3,1,0)$ as function of $\bar{\hbar}$ for $\delta = 1/2$.

Some 3D plots of the Type IIa and Type IIa bosonic zero modes for several values of $\bar{\hbar}$ are shown in the next figures:

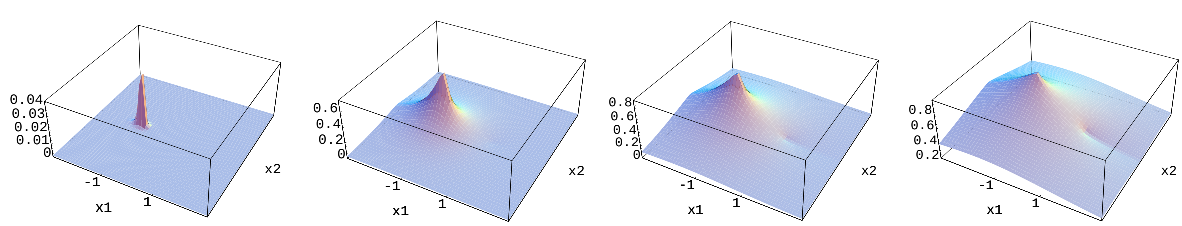


Figure 13. 3D graphics of the ground state probability density $|\Psi^{(0)}(x_1,x_2)|^2$, for $\delta = 1/2$, $\kappa = 3$, and $\mathbf{a} = \mathbf{b} = \mathbf{1}$ (or $|\Psi^{(2)}(x_1,x_2)|^2$ for $\mathbf{a} = \mathbf{b} = \mathbf{0}$). Cases: $\bar{\hbar} = 0.2$, $\bar{\hbar} = 2$, $\bar{\hbar} = 4$ and $\bar{\hbar} = 10$. The norms are $N(0.2) = 0.004914$, $N(2) = 2.83912$, $N(4) = 22.8914$ and $N(10) = 511.092$.

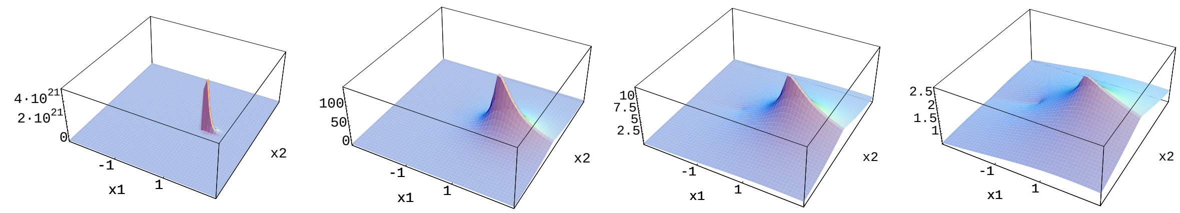


Figure 14. 3D graphics of the ground state probability density $|\Psi^{(0)}(x_1,x_2)|^2$, for $\delta = 1/2$, $\kappa = 3$, and $\mathbf{a} = \mathbf{1}$, $\mathbf{b} = \mathbf{0}$ (or $|\Psi^{(2)}(x_1,x_2)|^2$ for $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{1}$). Cases: $\bar{\hbar} = 0.2$, $\bar{\hbar} = 2$, $\bar{\hbar} = 4$ and $\bar{\hbar} = 10$. And the norms are $N(0.2) = 9.61622 \cdot 10^{20}$, $N(2) = 473.903$, $N(4) = 287.687$ and $N(10) = 1399.71$.

6.4 Fermionic ground states

Second, the fermionic ground states of zero energy:

$$\hat{C}_+\Psi_0^{(1)}(u,v) = 0, \qquad \hat{C}_-\Psi_0^{(1)}(u,v) = 0.$$

Unlike the bosonic case where the logic is *or* instead of *and*, note that both equations must be satisfied by the fermionic zero modes. The separation ansatz

$$\psi_0^{(0)1}(u,v) = \eta_0^{(0)1}(u)\zeta_0^{(0)1}(v), \qquad \psi_0^{(0)2}(u,v) = \eta_0^{(0)2}(u)\zeta_0^{(0)2}(v)$$

makes these equations tantamount to

$$e_1^u\left(\nabla_u^+ + \frac{\bar{\hbar}u}{u^2-v^2}\right)\psi_0^{(1)1}(u,v) + e_2^v\left(\nabla_v^+ - \frac{\bar{\hbar}v}{u^2-v^2}\right)\psi_0^{(1)2}(u,v) = 0,$$
$$-e_2^v\left(\nabla_v^- - \frac{\bar{\hbar}v}{u^2-v^2}\right)\psi_0^{(1)1}(u,v) + e_1^u\left(\nabla_u^- + \frac{\bar{\hbar}u}{u^2-v^2}\right)\psi_0^{(1)2}(u) = 0,$$

or,

$$\bar{\hbar}\frac{d\eta_0^{(1)1}}{du} + \left(\frac{dF_a}{du} + \frac{\bar{\hbar}u}{u^2-v^2}\right)\eta_0^{(1)1}(u) = 0, \qquad \bar{\hbar}\frac{d\eta_0^{(1)2}}{du} - \left(\frac{dF_a}{du} - \frac{\bar{\hbar}u}{u^2-v^2}\right)\eta_0^{(1)2}(u) = 0,$$
$$\bar{\hbar}\frac{d\zeta_0^{(1)1}}{dv} - \left(\frac{dG_a}{dv} + \frac{\bar{\hbar}v}{u^2-v^2}\right)\zeta_0^{(1)1}(v) = 0, \qquad \bar{\hbar}\frac{d\zeta_0^{(1)2}}{dv} + \left(\frac{dG_a}{dv} - \frac{\bar{\hbar}v}{u^2-v^2}\right)\zeta_0^{(1)2}(v) = 0.$$

The fermionic zero modes have the form of the linear combination:

$$\Psi_0^{(1)}(u,v) = \frac{1}{\sqrt{u^2-v^2}}\left\{A_1\begin{pmatrix}0\\ \exp[-\frac{(F_a(u;\kappa)-G_b(v;\kappa)}{\bar{\hbar}}]\\ 0\\ 0\end{pmatrix} + A_2\begin{pmatrix}0\\ 0\\ \exp[\frac{F_a(u;\kappa)-G_b(v;\kappa)}{\bar{\hbar}}]\\ 0\end{pmatrix}\right\}.$$

Because the norm is

$$N(\bar{\hbar};\kappa,a,b) = 2\int_1^\infty\int_{-1}^1 \frac{dudv}{\sqrt{(u^2-1)(1-v^2)}}\left(A_1^2 e^{-2\frac{F_a(u;\kappa)-G_b(v;\kappa)}{\bar{\hbar}}} + A_2^2 e^{2\frac{F_a(u;\kappa)-G_b(v;\kappa)}{\bar{\hbar}}}\right).$$

only $A_1 = 0$ or $A_2 = 0$ are normalizable, depending on the choice of a and b.

Again, these integrals cannot be computed analytically but the outcome of numerical calculations is shown in the next figures for several values of $\bar{\hbar}$.

In this case we see that the fermionic zero mode of Type IIa does not have a classical limit whereas the Type IIb fermionic ground state behaves smoothly near $\bar{\hbar} = 0$.

3D plots of fermionic zero modes, of both Type IIa and IIb, are shown in the last figures.

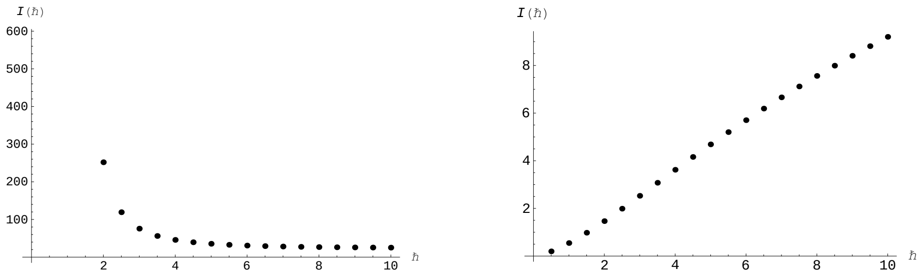


Figure 15. Numerical plots of $N(\bar{\hbar};3,1,1)$ and $N(\hbar;\bar{3},1,0)$ for $A_1=1$, $A_2=0$ and $\delta=1/2$ ($N(\bar{\hbar};3,0,0)$ and $N(\hbar;\bar{3},0,1)$ for $A_1=0$, $A_2=1$).

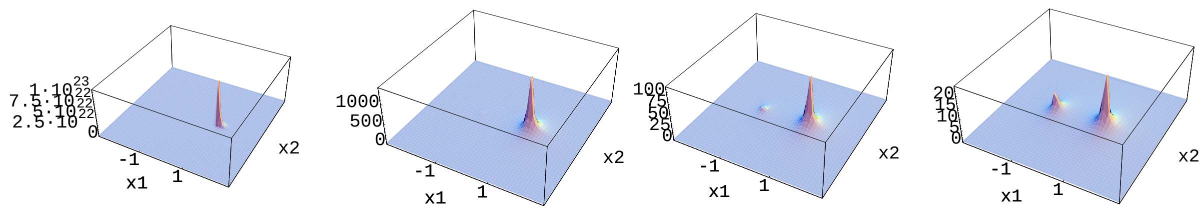


Figure 16. Graphics of $|\Psi_0^{(1)(1)}(x_1,x_2)|^2$ for $A_1=1$, $A_2=0$, $\delta=1/2$, $\kappa=3$, and the sign combination: $\mathbf{a}=\mathbf{b}=\mathbf{1}$. Cases: $\bar{\hbar}=0.2,2,4$ and 10. These graphics also represent $|\Psi_0^{(1)(2)}(x_1,x_2)|$ with $A_1=0$, $A_2=1$, $\delta=1/2$, $\kappa=3$, and $\mathbf{a}=\mathbf{b}=\mathbf{0}$. The norms are $N(0.2)=1.03792\cdot 10^{22}$, $N(2)=251.908$, $N(4)=45.7495$ and $N(10)=25.207$.

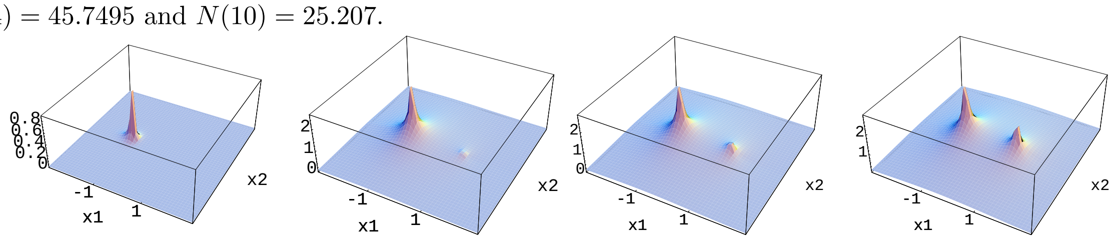


Figure 17. Graphics of the function $|\Psi^{(1)(1)}(x_1,x_2)|^2$ for $A_1=1$, $A_2=0$, $\delta=1/2$, $\kappa=3$, and the sign combination: $\mathbf{a}=\mathbf{1}$, $\mathbf{b}=\mathbf{0}$. Cases: $\bar{\hbar}=0.2,2,4$ and 10. These graphics also represent $|\Psi_0^{(1)(2)}(x_1,x_2)|^2$ for $A_1=0$, $A_2=1$ and $\mathbf{a}=\mathbf{0}$, $\mathbf{b}=\mathbf{1}$. The norms are $N(0.2)=0.05046$, $N(2)=1.46657$, $N(4)=3.62192$ and $N(10)=9.20207$.

7 Summary

In this paper we have built and studied two types of supersymmetric quantum mechanical systems starting from two Coulombian centers of force. Our theoretical analysis could be of interest in molecular physics seeking supersymmetric spectra closely related to the spectra of homonuclear or heteronuclear diatomic molecular ions, e.g., the hydrogen molecular ion or the same system with a proton replaced by a deuteron. In H_2^+ for instance, the first case, the value of the non-dimensional quantization parameter is checked to be $\bar{\hbar}=0.7$:

$$\hbar = 1.05\cdot 10^{-34}\ \mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1}, \qquad \sqrt{md\alpha} = 1.493\cdot 10^{-34}\ \mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-1},$$

using the international system of units (SI).

The first type is defined from a superpotential that solves the Poisson equation with the potential of the two centers as the source. In this case, the spectral problem is shown to be equivalent to entangled families of Razavy and Whittaker–Hill equations. Using the property of the quasi-exact solvability of these systems, many eigenvalues of the SUSY system corresponding to bound states have been identified when the strengths of the two centers are different. If the strengths are equal things become easier and some bound states are also found. In summary, for our simplest choice of Type I superpotential the main features of the spectrum are the following:

- There is an infinite set of discrete energy eigenvalues in the $F = 0$ Bose sub-space of the Hilbert space:

$$E_n = 2\frac{m\alpha^2}{\hbar^2}(1+\delta)^2\left(1 - \frac{1}{(n+1)^2}\right).$$

  The ionization energy is: $E_\infty = 2\frac{m\alpha^2}{\hbar^2}(1+\delta)^2$, the threshold of the continuous spectrum.

- There is a sub-space of dimension $n+1$ of degenerate eigenfunctions with energy E_n. The eigenvalues of the symmetry operator $\hat{I}$

$$I_{nm} = \frac{\hbar^2}{4}\left((\lambda_{nm} - (n+1)^2\right), \qquad m = 1, 2, \dots, n+1,$$

  where λ_{nm} are the $n+1$ roots of the polynomial that solves the Razavy equation, label a basis of eigefunctions in each energy sub-space.

- In the case $\delta = 1$ the eigenfunctions can be explicitly found. Moreover, the system enjoys a discrete symmetry under center exchange: $v \leftrightarrow -v$ ($r_1 \leftrightarrow r_2$). The energy eigenfunctions come in even/odd pairs of functions of v with respect to this reflection. This fact suggests that the purely bosonic two fixed centers Hamiltonian

$$H = -\frac{1}{2m}\nabla^2 + \frac{\alpha}{r_1} + \frac{\alpha}{r_2} + C$$

  enjoys a hidden supersymmetry (if the constant C is greater than $8\frac{m\alpha^2}{\hbar^2}$) of the kind recently unveiled in [45] for classy one-dimensional systems. This hidden supersymmetry is spontaneously broken because the even/odd ground states have positive energy: $E_\pm = C - 8\frac{m\alpha^2}{\hbar^2} > 0$.

- There are eigenfunctions in the $F = 1$ Fermi sub-space for the same eigenvalues E_n, $n > 0$. Analytically, the Fermi eigenfunctions are obtained from the Bose eigenfunctions through the action of $\hat{Q}_+$.

The second type starts from superpotentials solving the Hamilton–Jacobi equation. There are two non-equivalent sign combinations giving two sub-classes. Both Type IIa and Type IIb superpotentials are defined in terms of incomplete and complete elliptic integrals of the first and second kind. The superpotential in this approach is no more than the Hamilton's characteristic function for zero energy and flipped potential. The separability of the HJ equation in elliptic coordinates means that we can find a "complete" solution of this equation. The spectral problem is, however, hopeless for this Type.

All the zero-energy ground states, bosonic and fermionic, Type I and Type IIa/IIb, different strengths and equal strengths, have been obtained. The cross section ($x_2 = 0$) of the probability density of some of the ground states are shown, for the sake of comparison, in these final Tables. It is remarkable that, despite of being analytically very different, Type I and Type II zero modes show similar patterns.

Acknowledgements

We are grateful to Mikhail Ioffe for many enlightening lessons and conversations on SUSY quantum mechanics as well as letting us know about reference [28]. JMG thanks Nigel Hitchin for sending him his unpublished lecture notes on the Dirac operator.

Finally, we recognize financial support from the Spanish DGICYT and the Junta de Castilla y León under contracts: FIS2006-09417, VAO13C05.

Table 1. $\bar{\hbar} = 1$, $\delta = 1/2$.

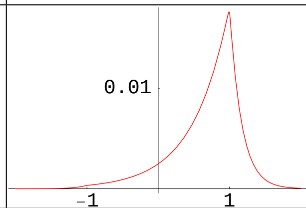
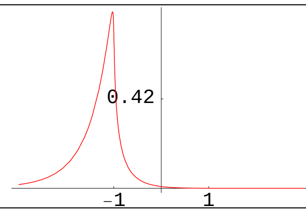
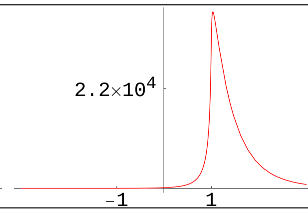
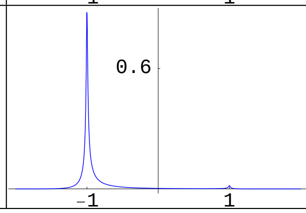
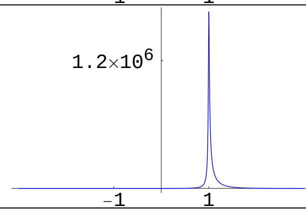
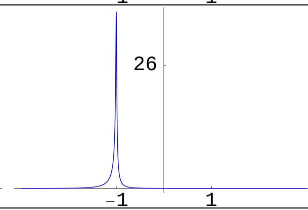
$\lvert\Psi_0^{(0/1)}(x_1, x_2)\rvert^2$	Type I	Type IIa $\kappa = 3$	Type IIb $\kappa = 3$
Bosonic zero mode			
Fermionic zero mode			

Table 2. $\bar{\hbar} = 2$, $\delta = 1/2$.

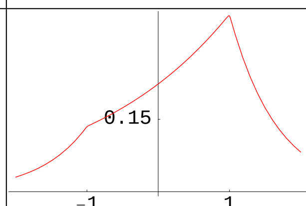
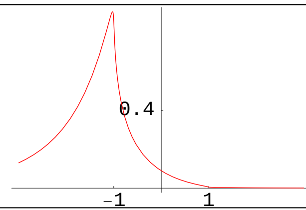
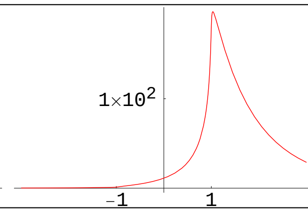
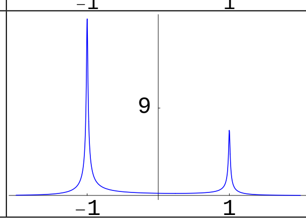
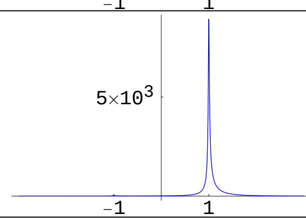
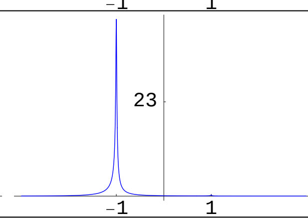
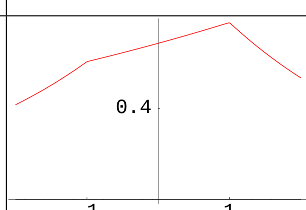
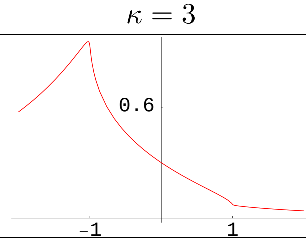
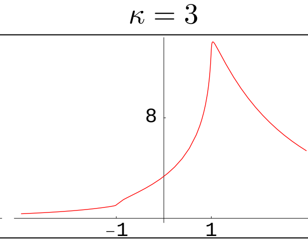
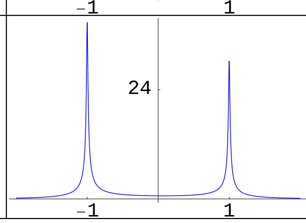
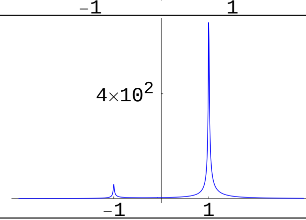
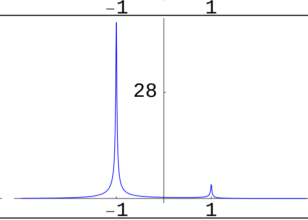
$\lvert\Psi_0^{(0/1)}(x_1, x_2)\rvert^2$	Type I	Type IIa $\kappa = 3$	Type IIb $\kappa = 3$
Bosonic zero mode			
Fermionic zero mode			

Table 3. $\bar{\hbar} = 4$, $\delta = 1/2$.

$\lvert\Psi_0^{(0/1)}(x_1, x_2)\rvert^2$	Type I	Type IIa $\kappa = 3$	Type IIb $\kappa = 3$
Bosonic zero mode			
Fermionic zero mode			

References

[1] Witten E., Dynamical breaking of supersymmetry, *Nuclear Phys. B* **188** (1981), 513–554.

[2] Witten E., Constraints on supersymmetry breaking, *Nuclear Phys. B* **202** (1982), 253–316.

[3] Witten E., Supersymmetry and Morse theory, *J. Differential Geom.* **17** (1982), 661–692.

[4] Clifford W.K., Mathematical papers, Editor R. Tucker, Macmillan & Co, London, 1882.

[5] Hitchin N., The Dirac operator, Oxford University Press, Oxford, 2002.

[6] Hull T., Infeld L., The factorization method, *Rev. Modern Phys.* **23** (1951), 21–68.

[7] Casalbuoni R., The classical mechanics for Bose–Fermi systems, *Nuovo Cim. A* **33** (1976), 389–431.

[8] Berezin F.A., Marinov M., Particle spin dynamics as the Grassmann variant of classical mechanics, *Ann. Phys.* **104** (1977), 336–362.

[9] Cooper F., Khare A., Sukhatme U., Supersymmetry and quantum mechanics, *Phys. Rep.* **251** (1995), 267–385, hep-th/9405029.

[10] Junker G., Supersymmetric methods in quantum and statistical physics, Springer, Berlin, 1996.

[11] González León M.A., Mateos Guilarte J., de la Torre M., From $N = 2$ supersymmetric classical to quantum mechanics and back: the SUSY WKB approximation, *Monografías de la Real Academia de Ciencias de Zaragoza* **29** (2006), 113–122, hep-th/0603225.

[12] Andrianov A.A., Borisov N., Eides M., Ioffe M.V., Supersymmetric origin of equivalent quantum systems, *Phys. Lett. A* **109** (1985), 143–148.

[13] Andrianov A.A., Borisov. N., Eides M., Ioffe M.V., Supersymmetric mechanics: a new look at the equivalence of quantum systems, *Theor. Math. Phys.* **61** (1984), 963–927.

[14] Andrianov A.A., Borisov N., Ioffe M.V., The factorization method and quantum systems with equivalent energy spectra, *Phys. Lett. A* **105** (1984), 19–22.

[15] Andrianov A.A., Borisov N., Ioffe M.V., Factorization method and the Darboux transformation for multidimensional Hamiltonians, *Theor. Math. Phys.* **61** (1984), 1078–1088.

[16] Perelomov A., Integrable systems of classical mechanics and Lie algebras, Birkhauser, 1992.

[17] Alonso Izquierdo A., González León M.A., Mateos Guilarte J., Invariants in supersymmetric classical mechanics, *Monografías de la Real Sociedad Española de Matemáticas* **2** (2001), 15–28, hep-th/0004053.

[18] Alonso Izquierdo A., González León M.A., Mateos Guilarte J., de la Torre M., Supersymmetry versus integrability in two-dimensional classical mechanics, *Ann. Phys.* **308** (2003), 664–691, hep-th/0307123.

[19] Alonso Izquierdo A., González León M.A., Mateos Guilarte J., de la Torre Mayado M., On two-dimensional superpotentials: from classical Hamilton–Jacobi theory to 2D supersymmetric quantum mechanics, *J. Phys. A: Math. Gen.* **37** (2004), 10323–10338, hep-th/0401054.

[20] Andrianov A.A., Ioffe M.V., Nishnianidze D.N., Classical integrable 2-dim models inspired by SUSY quantum mechanics, *J. Phys. A: Math. Gen.* **32** (1999), 4641–4654, solv-int/9810006.

[21] Cannata F., Ioffe M.V., Nishnianidze D.N., New methods for two-dimensional Schrödinger equation: SUSY-separation of variables and shape-invariance, *J. Phys. A: Math. Gen.* **35** (2002), 1389–1404, hep-th/0201080.

[22] Cannata F., Ioffe M.V., Nishnianidze D.N., Two-dimensional SUSY-pseudo-hermiticity without separation of variables, *Phys. Lett. A* **310** (2003), 344–352, hep-th/0302003.

[23] Ioffe M.V., SUSY-approach for investigation of two-dimensional quantum mechanical systems, *J. Phys. A: Math. Gen.* **37** (2004), 10363–10374, hep-th/0405241.

[24] Ioffe M.V., Mateos Guilarte J., Valinevich P., Two-dimensional supersymmetry: from SUSY quantum mechanics to integrable classical models, *Ann. Phys.* **321** (2006), 2552–2565, hep-th/0603006.

[25] Ioffe M.V., Mateos Guilarte J., Valinevich P., A class of partially solvable two-dimensional quantum models with periodic potentials, *Nuclear Phys. B* **790** (2008), 414–431, arXiv:0706.1344.

[26] Cannata F., Ioffe M.V., Nishnianidze D.N., Exactly solvable two-dimensional complex model with real spectrum, *Theor. Math. Phys.* **148** (2006), 960–967, hep-th/0512110.

[27] Ioffe M.V., Nishnianidze D.N., Exact solvability of two-dimensional real singular Morse potential, *Phys. Rev. A* **76** (2007), 052114, 5 pages, arXiv:0709.2960.

[28] Bondar D., Hnatich M., Lazur V., Two-dimensional problem of two Coulomb centers at small intercenter distances, *Theor. Math. Phys.* **148** (2006), 1100–1116.

[29] Mielnik B., Factorization method and new potentials with the oscillator spectrum, *J. Math. Phys.* **25** (1984), 3387–3389.

[30] Abramowitz M., Stegun I.A., Handbook of mathematical functions, Dover Publ., 1972.

[31] Byrd P.F., Friedman M.D., Handbook of elliptic integrals for engineers and scientists, Springer-Verlag, 1971.

[32] Finkel F., González-López A., Rodríguez M.A., On the families of orthogonal polynomials associated to the Razavy potential, *J. Phys. A: Math. Gen.* **32** (1999), 6821–6835, math-ph/9905020.

[33] Greiner W., Relativistic quantum mechanics. Wave equations, Springer-Verlag, 1997.

[34] Heumann R., The classical supersymmetric Coulomb problem, *J. Phys. A: Math. Phys.* **35** (2002), 7437–7460, hep-th/0205232.

[35] Ince E.L., Ordinary differential equations, Dover, 1956.

[36] Andrianov A.A., Ioffe M.V., Tsu C.-P., Factorization method in curvilinear coordinates and levels pairing for matrix potentials, *Vestnik Leningradskogo Universiteta (Ser. 4 Fiz. Chim.)* **25** (1988), 3–9 (in Russian).

[37] González-López A., Kamran N., The multidimensional Darboux transformation, *J. Geom. Phys.* **26** (1998), 202–226, hep-th/9612100.

[38] Kirchberg A., Länge J.D., Pisani P.A.G., Wipf A., Algebraic solution of the supersymmetric hydrogen atom in d dimensions, *Ann. Phys.* **303** (2003), 359–388, hep-th/0208228.

[39] Wipf A., Kirchberg A., Länge J.D., Algebraic solution of the supersymmetric hydrogen atom, in Proceedings of the 4th International Symposium "Quantum Theory and Symmetries" (QTS-4) (August 15–21, 2005, Varna, Bulgaria), Editor V.K. Dobrev, *Bulgar. J. Phys.* **33** (2006), 206–216, hep-th/0511231.

[40] Landau L.D., Lifshitz E.M., Mechanics, Pergamon Press, 1985.

[41] Manton N., Heumann R., Classical supersymmetric mechanics, *Ann. Phys.* **284** (2000), 52–88, hep-th/0001155.

[42] Pauling L., Bright Wilson E., Introduction to quantum mechanics with applications to chemistry, Dover, 1963.

[43] Razavy M., An exactly soluble Schrödinger equation with a bistable potential, *Amer. J. Phys.* **48** (1980), 285–288.

[44] Razavy M., A potential model for torsional vibrations of molecules, *Phys. Lett. A* **82** (1981), 7–9.

[45] Correa F., Plyushchay M., Hidden supersymmetry in quantum bosonic systems, *Ann. Phys.* **322** (2007), 2493–2500, hep-th/0605104.