Symplectic Maps from Cluster Algebras*

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Abstract. We consider nonlinear recurrences generated from the iteration of maps that arise from cluster algebras. More precisely, starting from a skew-symmetric integer matrix, or its corresponding quiver, one can define a set of mutation operations, as well as a set of associated cluster mutations that are applied to a set of affine coordinates (the cluster variables). Fordy and Marsh recently provided a complete classification of all such quivers that have a certain periodicity property under sequences of mutations. This periodicity implies that a suitable sequence of cluster mutations is precisely equivalent to iteration of a nonlinear recurrence relation. Here we explain briefly how to introduce a symplectic structure in this setting, which is preserved by a corresponding birational map (possibly on a space of lower dimension). We give examples of both integrable and non-integrable maps that arise from this construction. We use algebraic entropy as an approach to classifying integrable cases. The degrees of the iterates satisfy a tropical version of the map.

Key words: integrable maps; Poisson algebra; Laurent property; cluster algebra; algebraic entropy; tropical

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1 Introduction

The purpose of this short note is to make a preliminary announcement of our recent results on recurrence relations which arise in the context of cluster mutations. More details and many more examples will be published in a later paper [7].

The theory of cluster algebras was introduced by Fomin and Zelevinsky [4]. One can start with an $N \times N$, skew-symmetric integer matrix $B = (b_{jk}) \in \mathrm{Mat}_N(\mathbb{Z})$, which defines a quiver Q with N nodes having no 1-cycles or 2-cycles. This is a directed graph specified by the rule that $b_{jk} = -b_{kj}$ denotes the number of arrows from node j to node k (with an overall minus sign corresponding to reversing the direction of the arrows). Starting from the quiver Q and its associated matrix $B = B(Q)$ one can define a set of operations μ_k, called mutations, associated to the vertices of Q (labelled by $k = 1, \dots, N$), each of which produces a new quiver $\tilde{Q} = \mu_k Q$, together with its associated matrix $\tilde{B} = B(\tilde{Q})$, whose components are given by

$$\tilde{b}_{j\ell} = \begin{cases} -b_{j\ell}, & \text{if} \quad j = k \quad \text{or} \quad \ell = k, \\ b_{j\ell} + \frac{1}{2}(|b_{jk}|b_{k\ell} + b_{jk}|b_{k\ell}|), & \text{otherwise.} \end{cases}$$

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In this theory, an affine coordinate x_j, $j = 1, \ldots, N$, is associated to each vertex of the quiver Q, and the set of these N coordinates defines an initial cluster. Together with the quiver mutations (and the above equivalent set of matrix mutations), there is an associated set of cluster mutations, which replace the initial cluster $(x_1, \ldots, x_N)$ with a new cluster $(\tilde{x}_1, \ldots, \tilde{x}_N)$, where the new variables are related to the old ones by a simple birational transformation:

$$\tilde{x}_k x_k = \prod_{j=1}^{N} x_j^{[b_{k,j}]_+} + \prod_{j=1}^{N} x_j^{[-b_{k,j}]_+}, \qquad \tilde{x}_j = x_j, \qquad j \neq k,$$

where the square bracket notation is defined by $[b]_+ = \max(b, 0)$. This map is an involution, because applying the mutation μ_k once more restores the original cluster $\mathbf{x} = (x_j)$ and also sends $\tilde{B}$ back to B. (Note that in this paper we do not consider the most general version of cluster algebras, but only coefficient-free algebras for which the exchange matrix B is skew-symmetric.)

The full set of cluster variables, obtained by all possible sequences of mutations, generates the cluster algebra. However, in certain cases, which were classified recently by Fordy and Marsh [8], the quiver has a special periodicity property, which means that a particular sequence of mutations is equivalent to generating the cluster variables via iteration of a single recurrence relation of the form

$$x_{n+N} x_n = \prod_{j=1}^{N-1} x_{n+j}^{[b_{1,j+1}]_+} + \prod_{j=1}^{N-1} x_{n+j}^{[-b_{1,j+1}]_+}, \tag{1.1}$$

where the exponents $\pm b_{1,k}$ are the elements of the top row of the skew-symmetric integer matrix B. The results of [8] imply that in order for this periodicity property to hold, the vector $(b_{12}, \ldots, b_{1N})$ of exponents in the top row must be palindromic, but is otherwise arbitrary, and the rest of matrix B is completely determined from the these exponents.

By a theorem of Fomin and Zelevinsky [4], recurrences which are constructed in this way are guaranteed to have the Laurent property, meaning that the iterates are Laurent polynomials in the initial data $x_1, \ldots, x_N$ with integer coefficients (but see [5] for a more general treatment of the Laurent phenomenon). One of the earliest examples of a rational (nonlinear) recurrence having the Laurent property is known as Somos-5.

Example 1.1 (The Somos-5 recurrence)**.** The Somos-5 recurrence is

$$x_{n+5} x_n = x_{n+4} x_{n+1} + x_{n+3} x_{n+2}. \tag{1.2}$$

As shown in [8], this recurrence corresponds to the cluster exchange relation generated by the node labelled by $j = 1$ in a quiver Q with five nodes, that is equivalent to the skew-symmetric 5×5 integer matrix

$$B = \begin{pmatrix} 0 & -1 & 1 & 1 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ -1 & 2 & 0 & -2 & 1 \\ -1 & 0 & 2 & 0 & -1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix}. \tag{1.3}$$

The mutation μ_1 transforms this matrix to

$$\tilde{B} = \mu_1 B = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ -1 & 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & -2 & 0 \\ 1 & -1 & 2 & 0 & -2 \\ -1 & -1 & 0 & 2 & 0 \end{pmatrix},$$

which, in this case, turns out to be the permutation of indices $(1,2,3,4,5) \mapsto (5,1,2,3,4)$. In other words, $\tilde{B} = \rho B \rho^{-1}$, where

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

is the matrix representing this cyclic permutation. At the same time, the associated transformation of the cluster variables is $(x_1, x_2, x_3, x_4, x_5) \mapsto (\tilde{x}_1, x_2, x_3, x_4, x_5)$, where $\tilde{x}_1$ is defined by the exchange relation

$$\tilde{x}_1 x_1 = x_2 x_5 + x_3 x_4.$$

Because the first matrix mutation μ_1 just corresponds to a cyclic permutation of the indices, it follows that a subsequent mutation μ_2 of the new matrix $\tilde{B}$ produces the same formula for the exchange relation (up to relabelling the variables). It is this property which allows an infinite sequence of mutations to be considered as equivalent to the iteration of a single recurrence relation (in this case, the recurrence (1.2) above).

Remark 1.2 (Laurent phenomenon)**.** The Somos-5 recurrence is distinguished for several reasons. It is one of the first known examples of a rational recurrence whose iterates are Laurent polynomials in the initial values with integer coefficients, which for (1.2) means that

$$x_n \in \mathbb{Z}\big[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}, x_5^{\pm 1}\big]$$

for all n. This Laurent phenomenon was noted by Michael Somos, who saw that it explained his earlier observation that the recurrence (1.2) generates an integer sequence starting from the initial data $(x_1, \ldots, x_5) = (1,1,1,1,1)$, that is

$$1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, 6161, 22833, \ldots.$$

The latter sequence also appears in a problem of Diophantine geometry, and Elkies and others explained the connection with elliptic curves and theta functions (see [2]). Excellent accounts of the earliest results on the Laurent property appear in the articles of Gale [9]. However, a more complete picture did not begin to emerge until the work of Fomin and Zelevinksy, who were able to treat a wide variety of examples of the Laurent phenomenon on the same footing [5].

Another reason to highlight the Somos-5 recurrence is the fact that it is connected to one of the archetypal examples of a discrete integrable system, namely a member of the Quispel–Roberts–Thompson (QRT) family of maps of the plane [15]. This is described in Example 2.1 below.

The main motivation behind [7] (and the current announcement) is to explain the connection between discrete integrable systems and cluster algebras, and try to understand to what extent examples like Somos-5 are isolated rarities. However, in order to be able to talk about Liouville–Arnold integrability in the finite-dimensional setting, it is necessary to have a symplectic (or Poisson) structure to hand. In the next section we describe how this arises in the context of cluster algebras, and use Somos-5 for illustration.

A further example that reduces to a map of the plane is introduced and evidence given for its non-integrability. In the third section we consider recurrences of the form (1.1) from the viewpoint of their algebraic entropy (as defined in [1]), and suggest that this leads to a very sharp classification result for these systems.

The final section is reserved for some conclusions.

2 Poisson and symplectic structures

Cluster transformations were considered in the setting of Poisson geometry by Gekhtman, Shapiro and Vainshtein [10], who found Poisson structures of log-canonical type for the cluster variables:

$$\{x_j, x_k\} = c_{jk}\, x_j x_k, \tag{2.1}$$

for some constant skew-symmetric matrix $C = (c_{jk})$. Such Poisson brackets are compatible with cluster mutations, in the sense that the new cluster obtained by mutation also satisfies a log-canonical bracket:

$$\{\tilde{x}_j, \tilde{x}_k\} = \tilde{c}_{jk}\, \tilde{x}_j \tilde{x}_k.$$

However, in general $\tilde{C} \neq C$, so the mutation map from the variables $\mathbf{x}$ to $\tilde{\mathbf{x}}$ (which can be considered as an involution of N-dimensional affine space) does not preserve the original Poisson structure.

Nevertheless, the recurrences defined by the special sequence of mutations available in the case of mutation-periodic exchange matrices *do* correspond to maps with an invariant Poisson bracket of log-canonical form (perhaps on a lower-dimensional symplectic manifold). Theorem 2.4 below describes how this works.

Given the recurrence (1.1), we define a birational map $\varphi : \mathbb{C}^N \to \mathbb{C}^N$:

$$\varphi : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix} \longmapsto \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_N \\ x_{N+1} \end{pmatrix}, \qquad \text{where} \quad x_{N+1} = \frac{\prod_{j=1}^{N-1} x_{j+1}^{[b_{1,j+1}]_+} + \prod_{j=1}^{N-1} x_{j+1}^{[-b_{1,j+1}]_+}}{x_1}. \tag{2.2}$$

It is possible to seek a skew-symmetric matrix $C = (c_{jk})$ such that the Poisson bracket (2.1) is invariant. The nature of the map (being of the form $\tilde{x}_j = x_{j+1}$) means that this matrix has a banded structure, so only the top row needs to be determined.

Example 2.1 (Invariant Poisson bracket for the Somos-5 map)**.** The Somos-5 map is

$$\varphi : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \longmapsto \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ \dfrac{x_2 x_5 + x_3 x_4}{x_1} \end{pmatrix}, \tag{2.3}$$

which preserves the log-canonical Poisson bracket

$$\{x_j, x_k\} = (j-k) x_j x_k, \tag{2.4}$$

in the sense that $\{\varphi^* F, \varphi^* G\} = \varphi^*\{F, G\}$ for any pair of functions F, G. The corresponding matrix

$$C = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 \\ 1 & 0 & -1 & -2 & -3 \\ 2 & 1 & 0 & -1 & -2 \\ 3 & 2 & 1 & 0 & -1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

has rank 2, and the three independent null vectors

$$\mathbf{m}_1 = (1,-2,1,0,0)^T, \qquad \mathbf{m}_2 = (0,1,-2,1,0)^T, \qquad \mathbf{m}_3 = (0,0,1,-2,1)^T$$

provide three independent Casimir functions for the bracket, $f_j = \mathbf{x}^{\mathbf{m}_j}$ for $j = 1,2,3$:

$$f_1 = \frac{x_1 x_3}{x_2^2}, \qquad f_2 = \frac{x_2 x_4}{x_3^2}, \qquad f_3 = \frac{x_3 x_5}{x_4^2}.$$

However, under the action of φ, as in (2.3), the Casimirs transform as

$$\varphi^* f_1 = f_2, \qquad \varphi^* f_2 = f_3, \qquad \varphi^* f_3 = \frac{f_2 f_3 + 1}{f_1 f_2^2 f_3^2}.$$

This induced action of φ on the variables f_j was written in terms of a third-order recurrence in [12].

Remark 2.2 (Non-invariance of symplectic leaves)**.** We see that φ does not preserve the symplectic leaves of the bracket because the Casimirs are not invariant under the map. It follows that the log-canonical bracket on the cluster variables x_j is not relevant to the integrability of the map (1.2).

In fact, the variables

$$y_1 = \frac{x_1 x_4}{x_2 x_3}, \qquad y_2 = \frac{x_2 x_5}{x_3 x_4} \tag{2.5}$$

which are themselves Casimirs of this bracket, since $y_1 = f_1 f_2$ and $y_2 = f_2 f_3$, transform as a map of the plane, given by

$$\hat{\varphi} : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_2 \\ (y_2+1)/(y_1 y_2) \end{pmatrix}. \tag{2.6}$$

This is a particular case of the QRT map [15] and possesses an invariant Poisson bracket of log-canonical type:

$$\{y_1, y_2\} = y_1 y_2. \tag{2.7}$$

Moreover, the quantity

$$H = y_1 + y_2 + \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_1 y_2}$$

is an invariant of the map, which means that (2.6) is integrable in the Liouville–Arnold sense [17]. The smooth level curves of H have genus one, which explains the connection with elliptic curves and leads to the explicit solution of the initial value problem for (1.2) in terms of the Weierstrass sigma function [12].

2.1 Symplectic form from the B matrix

The above construction of the Poisson brackets (2.4) and (2.7) was presented in an ad hoc way. The matrix $C = (c_{jk})$ for the map of the coordinates x_j, as well as the bracket between the y_j, can be derived purely from the assumption that a log-canonical bracket exists for each of the maps φ and $\hat{\varphi}$. Furthermore, these calculations start from the *map* and have *no connection* with the cluster construction. We now explain how to derive the relevant Poisson brackets from the exchange matrix B.

We define the log-canonical two-form associated to B, by

$$\omega = \sum_{j<k} \frac{b_{jk}}{x_j x_k} \mathrm{d}x_j \wedge \mathrm{d}x_k; \tag{2.8}$$

this two-form was first introduced in [11], and was also considered in [3]. Let φ be the map (2.2), associated with the matrix mutation μ_1 of B. Then we have the following:

Lemma 2.3. *Let B be a skew-symmetric integer matrix. The following conditions are equivalent.*

1. *The matrix B defines a cluster mutation-periodic quiver with period* 1.
2. *The matrix elements B satisfy the relations*

$$b_{j,N} = b_{1,j+1}, \qquad j = 1, \dots, N-1, \tag{2.9}$$

and

$$b_{j+1,k+1} = b_{j,k} + b_{1,j+1}[-b_{1,k+1}]_+ - b_{1,k+1}[-b_{1,j+1}]_+, \tag{2.10}$$

for $1 \leq j, k \leq N-1$.

3. *The two-form ω is preserved by the map φ, i.e. $\varphi^*\omega = \omega$.*

The above result follows from a direct calculation, which will be presented in [7]; it can also be understood in the light of Theorem 2.1 in [11], which shows that the two-form (2.8) is *covariant* with respect to general cluster transformations. We emphasise that in the case of *period* 1 *quivers*, this two-form is *invariant*.

Note that the relations (2.9) and (2.10) mean that the vector $(b_{12}, \dots, b_{1N}) = (a_1, \dots, a_{N-1})$ must be palindromic, i.e. $a_j = a_{N-j}$ for all j, and the matrix B can be completely reconstructed from these exponents in the first row (see Theorem 6.1 of [8]).

The case where B is degenerate is more delicate, but the general situation is described by the following.

Theorem 2.4. *The map φ is symplectic whenever B is nondegenerate. In the degenerate case that $\operatorname{rank} B = 2K < N$, there is a rational map π and a symplectic birational map $\hat{\varphi}$ with symplectic form $\hat{\omega}$ such that the diagram*

$$\begin{array}{ccc} \mathbb{C}^N & \xrightarrow{\varphi} & \mathbb{C}^N \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{C}^{2K} & \xrightarrow{\hat{\varphi}} & \mathbb{C}^{2K} \end{array}$$

is commutative, and the symplectic form $\hat{\omega}$ on $\mathbb{C}^{2K}$ satisfies $\pi^*\hat{\omega} = \omega$.

The latter result can be viewed as a special case of the symplectic reduction of the form (2.8) presented in [11].

Example 2.5 (Somos-5 map)**.** We consider the 2-form (2.8) with matrix B given by (1.3). Written in terms of the variables $z_i = \log x_i$, we have

$$\omega = \sum_{j<k} b_{jk} dz_j \wedge dz_k.$$

The rank of matrix B is 2, since it has three independent null vectors

$$\mathbf{u}_1 = (1,1,1,1,1)^T, \qquad \mathbf{u}_2 = (1,2,3,4,5)^T, \qquad \mathbf{u}_3 = (1,-1,1,-1,1)^T.$$

$\operatorname{Im} B$ is spanned by

$$\mathbf{v}_1 = (1, -1, -1, 1, 0)^T, \qquad \mathbf{v}_2 = (0, 1, -1, -1, 1)^T,$$

and $\mathbf{u}_i \cdot \mathbf{v}_j = 0$, for all i, j. This means that

$$\omega = (dz_2 - dz_3 - dz_4 + dz_5) \wedge (dz_1 - dz_2 - dz_3 + dz_4),$$

which, in x-coordinates, gives

$$\omega = d\log\left(\frac{x_2x_5}{x_3x_4}\right) \wedge d\log\left(\frac{x_1x_4}{x_2x_3}\right) = \frac{dy_2 \wedge dy_1}{y_1y_2}, \qquad \text{where} \quad y_1 = \frac{x_1x_4}{x_2x_3}, \quad y_2 = \frac{x_2x_5}{x_3x_4}.$$

We see that on the two-dimensional space with coordinates y_1, y_2, this gives a form $\hat{\omega}$ which is non-degenerate, so defines an invariant symplectic form.

We are thus led directly to the variables (2.5), which satisfy the QRT map (2.6), and to the Poisson bracket (2.7), which is defined through the inverse of the symplectic form. In the general case, the analogues of the variables (2.5) correspond to particular choices of the so-called τ-coordinates introduced in [10].

Example 2.6 (A sixth-order recurrence)**.** The recurrence

$$x_{n+6}x_n = x_{n+5}^2 x_{n+3}^4 x_{n+1}^2 + (x_{n+4}x_{n+2})^6, \tag{2.11}$$

comes from the degenerate exchange matrix

$$B = \begin{pmatrix} 0 & -2 & 6 & -4 & 6 & -2 \\ 2 & 0 & -14 & 6 & -16 & 6 \\ -6 & 14 & 0 & 10 & 6 & -4 \\ 4 & -6 & -10 & 0 & -14 & 6 \\ -6 & 16 & -6 & 14 & 0 & -2 \\ 2 & -6 & 4 & -6 & 2 & 0 \end{pmatrix}. \tag{2.12}$$

Clearly, iteration of this recurrence is equivalent to iterating the map $\varphi : \mathbb{C}^6 \to \mathbb{C}^6$ given by

$$\varphi : \ (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_2, x_3, x_4, x_5, x_6, x_7), \qquad x_7 = \frac{x_6^2 x_4^4 x_2^2 + x_5^6 x_3^6}{x_1},$$

and this map preserves the degenerate two-form given by the expression (2.8) with coefficients b_{jk} as in (2.12).

Written in terms of the variables $z_i = \log x_i$, we have

$$\omega = \sum_{j<k} b_{jk} dz_j \wedge dz_k.$$

The rank of matrix B is 2, since it has four independent null vectors

$$\mathbf{u}_1 = (1, 0, -1, 0, 1, 0)^T, \qquad \mathbf{u}_2 = (0, 1, 0, -1, 0, 1)^T,$$
$$\mathbf{u}_3 = (3, 1, 0, 0, 0, -1)^T, \qquad \mathbf{u}_3 = (-1, 0, 0, 0, 1, 3)^T.$$

$\operatorname{Im} B$ is spanned by

$$\mathbf{v}_1 = (1, -3, 2 - 3, 1, 0)^T, \qquad \mathbf{v}_2 = (0, 1, -3, 2 - 3, 1)^T,$$

and $\mathbf{u}_i \cdot \mathbf{v}_j = 0$, for all i, j. This means that

$$\omega = 2(dz_2 - 3dz_3 + 2dz_4 - 3dz_5 + dz_6) \wedge (dz_1 - 3dz_2 + 2dz_3 - 3dz_4 + dz_5),$$

which, in x-coordinates, gives

$$\omega = 2d\log\left(\frac{x_2x_4^2x_6}{x_3^3x_5^3}\right)\wedge d\log\left(\frac{x_1x_3^2x_5}{x_2^3x_4^3}\right) = 2\frac{dy_2\wedge dy_1}{y_1y_2},$$

where

$$y_1 = \frac{x_1x_3^2x_5}{x_2^3x_4^3}, \qquad y_2 = \frac{x_2x_4^2x_6}{x_3^3x_5^3}.$$

We see that on the 2 dimensional space with coordinates y_1, y_2, ω is non-degenerate, so defines a symplectic form, which is invariant under the induced map

$$\hat{\varphi}:\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_2 \\ (y_2^2+1)/(y_1y_2^3) \end{pmatrix}. \tag{2.13}$$

However, in this case we know of no first integral and, furthermore, numerical plots indicate the presence of chaos.

In the next section we discuss algebraic entropy and show that the algebraic entropy of this particular map is non-zero.

3 Algebraic entropy

Bellon and Viallet [1] proposed a suitable measure of entropy for rational maps. This counts the degree d_n of the nth iterate of such a map in dimension N, defined as the maximum degree of the iterates, considered as rational functions of the N coordinates corresponding to the initial data. The algebraic entropy $\mathcal{E}$ of the map is defined as

$$\mathcal{E} = \lim_{n\to\infty}\frac{1}{n}\log d_n.$$

For a generic map of degree d, the entropy is $\log d > 0$, but for special maps cancellations of factors from the numerators and denominators of the rational functions can occur upon iteration, so that the entropy is smaller than expected.

It has been observed (see [1] for a discussion) that rational maps which are Liouville–Arnold integrable have zero algebraic entropy and it is *conjectured* that zero algebraic entropy implies integrability. In any case, it appears to be a reliable indicator of integrability. In the setting of this paper, Liouville–Arnold integrability would require the existence of K independent functions of the variables $y_1, \dots, y_{2K}$, which are invariant under the action of the map $\hat{\varphi}$ and in involution with respect to the Poisson bracket defined by $\hat{\omega}$.

In general it can be difficult to calculate the entropy of a rational map explicitly. Even numerical calculations become difficult if the dimension is large, since it is necessary to perform a symbolic computation of the rational functions generated by the map and count their degrees. However, for the family of maps of the form (2.2) (or equivalently the recurrences (1.1)), there is a considerable simplification. From the Laurent property of the associated cluster algebra [4], it follows that the iterates x_n are Laurent polynomials in the initial conditions $x_1, \dots, x_N$. Thus it is sufficient to count the degrees of the monomials that appear as the denominators of the iterates.

Upon writing each iterate in lowest terms as $x_n = D_n(\mathbf{x})/M_n(\mathbf{x})$ (where, as before, the vector $\mathbf{x} = (x_j)$ denotes the variables of the initial cluster, and $x_j \nmid D_n(\mathbf{x})$ for any $j = 1, \dots, N$), the denominator is a monomial $M_n = \mathbf{x}^{\mathbf{d}_n}$, with $\mathbf{d}_n = \left(d_n^{(1)}, \dots, d_n^{(N)}\right)^T$ being the vector of degrees. Substituting this form of the Laurent polynomials into (1.1) and comparing denominators on each side, it is apparent that (for sufficiently large n) the degrees $d_n^{(j)}$ satisfy the same recurrence for all j, namely

$$d_{n+N} + d_n = \max\left(\sum_{j=2}^{N}[b_{1,j}]_+ d_{n+j-1},\ \sum_{j=2}^{N}[-b_{1,j}]_+ d_{n+j-1}\right). \tag{3.1}$$

The above equation is precisely the ultra-discrete version of (1.1) (an example of tropical mathematics [16], where the ordinary field operations are replaced by the max-plus algebra). The entropy of these maps is measured by the growth of the degrees of the denominators, and generically the total degree, $\deg M_n = \sum_{j=1}^{N} d_n^{(j)}$, grows exponentially at the same rate as each component $d_n^{(j)}$, which is controlled by the tropical version of the map, as in (3.1).

Example 3.1 (Tropical Somos-5)**.** For the Somos-5 recurrence, the ultra-discrete analogue of the bilinear equation (1.2) is

$$d_{n+5} + d_n = \max\left(d_{n+4} + d_{n+1}, d_{n+3} + d_{n+2}\right). \tag{3.2}$$

The singularity analysis of ultra-discrete maps has been considered quite recently [13]. For this particular example, it is possible to describe the general solution quite explicitly. Indeed, the tropical analogue of the quantity y_n (for the recurrence version of the QRT map (2.6)) is

$$Y_n = d_{n+3} + d_n - d_{n+2} - d_{n+1},$$

which satisfies the second order relation

$$Y_{n+2} + Y_n = [Y_{n+1}]_+ - Y_{n+1}. \tag{3.3}$$

The latter is the tropical version of the QRT map defined in (2.6). Although in the context described above d_n counts the degree of a monomial, so that only non-negative integer values of d_n arise, which lead to Y_n taking integer values only, the recurrence (3.3) can be considered more generally as defining a piecewise linear map $\mathbb{R}^2 \to \mathbb{R}^2$. For this map it can be verified directly that all the orbits are periodic with period 7, from which it follows that d_n satisfies a linear recurrence of order 10 with constant coefficients, namely

$$\left(\mathcal{S}^7 - 1\right)\left(d_{n+3} + d_n - d_{n+2} - d_{n+1}\right) = 0, \tag{3.4}$$

where we have used $\mathcal{S}$ to denote the shift operator, defined by $\mathcal{S}f_n = f_{n+1}$ for any function f of $n \in \mathbb{Z}$. (Note that the periodicity of orbits in ultradiscrete QRT maps was proved by Nobe in [14].)

For the sequence of degrees of the tropical Somos-5 recurrence (3.2), in each variable x_j the denominator has four steps of degree zero, say $d_1 = d_2 = d_3 = d_4 = 0$, before the first non-zero degree $d_5 = 1$, and these five initial conditions generate a sequence beginning

$$0, 0, 0, 0, 1, 1, 1, 2, 3, 3, 4, 5, 6, 6, 8, 9, 10, 11, 13, 14, 15, 17, 19, 20, 22, 24, 26, 27, 30, \ldots,$$

which has quadratic growth with n, indicating that the algebraic entropy is zero. To see that $d_n = O(n^2)$, it is enough to note that the above sequence satisfies the tenth order linear recurrence (3.4), whose characteristic polynomial is $(\lambda^7 - 1)(\lambda^3 - \lambda^2 - \lambda + 1)$, and all of the roots of the latter have modulus 1, with $\lambda = 1$ being a triple root.

Example 3.2 (Tropical sixth-order recurrence)**.** The example of the recurrence (2.11) is quite different, in that the sequence of degrees, which begins

$$0, 0, 0, 0, 0, 1, 2, 6, 16, 42, 110, 287, 754, 1974, 5168, 13530, 35422, 92737, 242788, 635628, \dots,$$

grows exponentially with n. In this case, the degrees satisfy the tropical recurrence

$$d_{n+6} + d_n = \max\left(2d_{n+5} + 4d_{n+3} + 2d_{n+1}, 6(d_{n+4} + d_{n+2})\right),$$

while the analogue of (2.13) is the piecewise linear map on $\mathbb{R}^2$ given by

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \longmapsto \begin{pmatrix} Y_2 \\ 2\,[Y_2]_+ - 3Y_2 - Y_1 \end{pmatrix}, \tag{3.5}$$

where

$$Y_n = d_{n+4} + 2d_{n+2} + d_n - 3d_{n+3} - 3d_{n+1}.$$

The particular orbit of the map for Y_n corresponding to the degree sequence has initial data $(Y_1, Y_2) = (0, 1)$, and the sequence of Y_n begins $0, 1, -1, 2, -1, 1, 0, -1, 3, -2, 3, -1$; thereafter it is periodic with period 12, which implies that the degrees d_n satisfy a linear recurrence of order 16 with constant coefficients, that is

$$\left(\mathcal{S}^{12} - 1\right)\left(d_{n+4} + 2d_{n+2} + d_n - 3d_{n+3} - 3d_{n+1}\right) = 0.$$

(However, note that the general orbit of the map (3.5) is not periodic.) The right hand bracket above contributes a factor of $\lambda^4 - 3\lambda^3 + 2\lambda^2 - 3\lambda + 1 = (\lambda^2 + 1)(\lambda^2 - 3\lambda + 1)$ to the characteristic polynomial, and this contains the root of largest magnitude, namely $\lambda_{\max} = (3 + \sqrt{5})/2$. This implies that $\log d_n \sim n \log \lambda_{\max}$ as $n \to \infty$, giving a positive value to the algebraic entropy, $\mathcal{E} = \log \lambda_{\max}$.

Although we are not aware of any theorems that directly connect algebraic entropy to the Liouville–Arnold notion of integrability for maps, the fact that a map has $\mathcal{E} > 0$ is a strong indicator of non-integrability. For each of the recurrences (1.1), the degrees of each variable appearing in the denominators of the sequence of Laurent polynomials satisfy the same tropical recurrence (3.1), which means that any set of $N+1$ adjacent degrees always satisfies one of the two linear relations

$$d_{n+N} + d_n = \sum_{j=2}^{N} [\pm b_{1,j}]_+ d_{n+j-1}.$$

Empirically we have observed that for large enough n the degrees d_n actually switch periodically between the latter two linear recurrences. This suggests that the growth of the degrees is controlled by whichever of the two characteristic polynomials

$$P_\pm(\lambda) = \lambda^N + 1 - \sum_{j=1}^{N-1} [\pm b_{1,j+1}]_+ \lambda^j$$

has the root $\lambda_{\max}$ of largest magnitude. It is fairly easy to show that $|\lambda_{\max}| > 1$ provided that

$$\max\left(\sum_{j=2}^{N} [b_{1,j}]_+, \sum_{j=2}^{N} [-b_{1,j}]_+\right) \geq 3, \tag{3.6}$$

and in that case the entropy should be $\mathcal{E} = \log \lambda_{\max} > 0$. Thus we are led to the

Conjecture. *The condition* (3.6) *is sufficient for positive algebraic entropy.*

More detailed evidence for this conjecture will be described in [7].

4 Conclusions

In this paper we have briefly discussed the recurrences (1.1) (and corresponding maps (2.2)), which arise naturally in the context of cluster mutations of mutation-periodic quivers [8]. Our main interest is to understand the integrability or otherwise of the resulting maps. To this end, we have shown that the cluster exchange matrix B defines a 2-form (2.8), which is invariant under the action of the corresponding map. Furthermore, since, for an even number of nodes, the matrix B (corresponding to a generic, period 1 quiver) is non-singular, this 2-form is generically symplectic, so the inverse matrix defines a non-degenerate, invariant Poisson bracket of log-canonical type. However, for any quiver with an odd number of nodes and for some special (but important) examples with an even number, the 2-form (2.8) is degenerate. However, as we have seen in the two examples presented here, it is possible to reduce the map to a lower-dimensional manifold with an invariant symplectic form; this is the Weil–Petersson form introduced in [11]. Again, we emphasise that in our context, this symplectic form is *invariant*, not just *covariant*.

Finding the invariant symplectic form (and hence the Poisson bracket) is only part of the task. For K degrees of freedom we need K invariant functions which are in involution with one another. When $K = 1$ it is enough to find a single invariant function (as with the QRT map (2.6)). For $K > 1$ it can be much more difficult. The simplest period 1 quivers are the primitives (see [8] for the definition), which are linearisable. In this case it is possible [6] to construct a bi-Hamiltonian ladder to prove complete integrability. In [7] we present other linearisable examples which can also be shown to be completely integrable, although this requires a different construction.

We have used algebraic entropy arguments to show that complete integrability will be rare for the maps obtained through cluster mutation. A non-integrable example is provided by the map (2.11), which was reduced to a symplectic map with one degree of freedom, but this was shown to have positive entropy, and we have numerical results indicating chaotic behaviour.

The conjectured criterion (3.6) for the maps to have positive algebraic entropy leads to a sharp classification of the integrable maps that can arise from recurrences of the form (1.1). From this criterion it follows that systems with vanishing entropy must satisfy $S_\pm \leq 2$, where $S_\pm = \sum_{j=2}^{N} [\pm b_{1,j}]_+$ are the sums of the positive and negative entries, respectively. By the symmetry $B \to -B$, which does not change the recurrence, one can take $S_+ \geq S_-$ without loss of generality. It then follows that the sums of the positive and negative entries are restricted to take the values

$$(S_+, S_-) = (1, 0),\ (2, 0),\ (2, 1), (2, 2)$$

only. Each of these four cases can be subjected to further analysis, to verify that they correspond to symplectic maps (perhaps on a reduced manifold) that are integrable in the Liouville–Arnold sense (see [7] for further details).

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