

Orthogonal Polynomials Associated with Complementary Chain Sequences*

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Abstract. Using the minimal parameter sequence of a given chain sequence, we introduce the concept of complementary chain sequences, which we view as perturbations of chain sequences. Using the relation between these complementary chain sequences and the corresponding Verblunsky coefficients, the para-orthogonal polynomials and the associated Szegő polynomials are analyzed. Two illustrations, one involving Gaussian hypergeometric functions and the other involving Carathéodory functions are also provided. A connection between these two illustrations by means of complementary chain sequences is also observed.

Key words: chain sequences; orthogonal polynomials; recurrence relation; Verblunsky coefficients; continued fractions; Carathéodory functions; hypergeometric functions

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1 Preliminaries on Szegő polynomials

The Szegő polynomials $\{\Phi_n\}$, also referred to as orthogonal polynomials on the unit circle (OPUC), enjoy the orthogonality property

$$\int_{\partial\mathbb{D}} (\bar{z})^j \Phi_n(z) d\mu(z) = \int_{\partial\mathbb{D}} (z)^{-j} \Phi_n(z) d\mu(z) = 0 \quad \text{for } j = 0, 1, \dots, n-1, \quad n \geq 1.$$

Here $\mu(z) = \mu(e^{i\theta})$ is a nontrivial measure defined on the unit circle $\partial\mathbb{D} = \{z = e^{i\theta} : 0 \leq \theta \leq 2\pi\}$. Denoting the orthonormal Szegő polynomials by $\phi_n(z) = \chi_n \Phi_n(z)$, we also have the equivalent definition

$$\int_{\partial\mathbb{D}} \phi_n(z) \overline{\phi_m(z)} d\mu(z) = \delta_{m,n}.$$

Further, defining the moments $\mu_n = \int_{\partial\mathbb{D}} e^{-in\theta} d\mu(\theta)$, $n = 0, \pm 1, \dots$, where $\mu_{-n} = \bar{\mu}_n$, we have

$$\int_{\partial\mathbb{D}} (\bar{z})^n \Phi_n(z) d\mu(z) = \frac{\Delta_n}{\Delta_{n-1}} \neq 0, \quad n = 0, 1, \dots$$

Here $\Delta_n = \det\{\mu_{i-j}\}_{i,j=0}^n$ are the associated Toeplitz matrices with $\Delta_{-1} = 1$.

The monic Szegő polynomials satisfy the first order recurrence relations

$$\Phi_n(z) = z\Phi_{n-1}(z) - \bar{\alpha}_{n-1}\Phi_{n-1}^*(z), \quad \Phi_n^*(z) = -\alpha_{n-1}z\Phi_{n-1}(z) + \Phi_{n-1}^*(z), \quad n \geq 1,$$

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where $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})}$. The complex numbers $\alpha_{n-1} = -\overline{\Phi_n(0)}$ are called the Verblunsky coefficients [22]. The Verblunsky coefficients completely characterize the Szegő polynomials in the sense that any sequence $\{\alpha_{n-1}\}_{n=1}^\infty$ lying within the unit circle gives rise to a unique probability measure $\mu(z)$ which leads to a unique sequence of Szegő polynomials. The above result, called the Verblunsky theorem in [22], is the analogue of Favard's theorem on the real line. Conversely, algorithms exist in the literature that extracts these coefficients from any given Szegő system of orthogonal polynomials. Notable among them are the Schur algorithm, the Levinson algorithm and their modified versions given in [7, 14, 26].

The Szegő polynomials also satisfy the three term recurrence relation

$$\Phi_{n+1}(z) = \left(\frac{\Phi_{n+1}(0)}{\Phi_n(0)} + z \right) \Phi_n(z) - \frac{(1 - |\Phi_n(0)|^2)\Phi_{n+1}(0)}{\Phi_n(0)} z \Phi_{n-1}(z), \quad n \geq 1, \quad (1.1)$$

with $\Phi_0(z) = 1$ and $\Phi_1(z) = z + \Phi_1(0)$. Note that if $\Phi_n(0) = 0$, $n \geq 1$, then the three term recurrence relation ceases to exist. In such a case, $\Phi_n(z) = z^n$, which is given as the free case in [22, p. 85]. Denoting

$$\eta_{n+1} = \frac{\Phi_{n+1}(0)}{\Phi_n(0)} \quad \text{and} \quad \rho_{n+1} = \frac{(1 - |\Phi_n(0)|^2)\Phi_{n+1}(0)}{\Phi_n(0)}, \quad n \geq 1,$$

the following expressions are easily obtained from (1.1):

$$\Phi_{n+1}(0) = \frac{\Phi_n(0)}{1 - |\Phi_n(0)|^2} \frac{\int_{\partial\mathbb{D}} z \Phi_n(z) d\mu(z)}{\int_{\partial\mathbb{D}} z \Phi_{n-1}(z) d\mu(z)}, \quad 1 - |\Phi_n(0)|^2 = \frac{\int_{\partial\mathbb{D}} z^{-n} \Phi_n(z) d\mu(z)}{\int_{\partial\mathbb{D}} z^{-(n-1)} \Phi_{n-1}(z) d\mu(z)}$$

and

$$\chi_n^{-2} = \frac{\rho_2 \rho_3 \cdots \rho_{n+1}}{\eta_2 \eta_3 \cdots \eta_{n+1}} \mu_0 = \mu_0 (1 - |\Phi_1(0)|^2) (1 - |\Phi_2(0)|^2) \cdots (1 - |\Phi_n(0)|^2). \quad (1.2)$$

For early developments on the subject, we refer to the monographs [8, 10, 25]. For a compendium of modern research in the area as well as historical notes, we refer to [22, 23].

In order to develop a quadrature formula on the unit circle, Jones et al. [15] introduced the para-orthogonal polynomials which vanish only on the unit circle and, for $z, \omega_n \in \mathbb{C}$ with $|\omega_n| = 1$, have the representation

$$\mathcal{X}_n(z, \omega_n) = \Phi_n(z) + \omega_n \Phi_n^*(z), \quad n \geq 1.$$

The para-orthogonal polynomials satisfy the properties

$$\langle \mathcal{X}_n, z^m \rangle = 0, \quad m = 1, 2, \dots, n-1, \quad \langle \mathcal{X}_n, 1 \rangle \neq 0, \quad \langle \mathcal{X}_n, z^n \rangle \neq 0,$$

which are termed as deficiency in the orthogonality of these para-orthogonal polynomials. In recent years, these para-orthogonal polynomials have been linked to kernel polynomials $K_n(z, \omega)$, see [6, 11, 27]. The kernel polynomials $K_n(z, \omega)$ satisfy the Christoffel–Darboux formula

$$K_n(z, \omega) = \sum_{k=0}^n \phi_k(z) \overline{\phi_k(\omega)} = \frac{\phi_{n+1}^*(z) \overline{\phi_{n+1}^*(\omega)} - \phi_{n+1}(z) \overline{\phi_{n+1}(\omega)}}{1 - z\bar{\omega}}.$$

Denoting $\tau_n(\omega) = \Phi_n(\omega)/\Phi_n^*(\omega)$, for $n \geq 1$, the monic kernel polynomials related to the Szegő polynomials are given by

$$P_n(\omega; z) = \frac{z \Phi_n(z) - \omega \tau_n(\omega) \Phi_n^*(z)}{z - \omega}, \quad n \geq 1, \quad (1.3)$$

and are shown in [6] to satisfy a three term recurrence relation of the form

$$P_{n+1}(\omega; z) = [z + b_{n+1}(\omega)]P_n(\omega; z) - a_{n+1}(\omega)zP_{n-1}(\omega; z), \quad n \geq 1,$$

where

$$b_n(\omega) = \frac{\tau_n(\omega)}{\tau_{n-1}(\omega)}, \quad a_{n+1} = [1 + \tau_n(\omega)\alpha_{n-1}][1 - \overline{\omega\tau_n(\omega)\alpha_n}]\omega, \quad n \geq 1.$$

The polynomials $P_n(\omega; z)$ are $\overline{\tau_n(\omega)}$ -invariant sequences of polynomials which can be easily verified from (1.3). Note that a sequence of polynomials $\{\mathcal{Y}_n\}$ is called τ_n -invariant if [15]

$$\mathcal{Y}_n^*(z) = \tau_n \mathcal{Y}_n(z), \quad n \geq 1.$$

An important concept that is used in the sequel is the theory of chain sequences. We give a brief introduction to chain sequences and then illustrate the role played by them in the theory of OPUC.

A sequence $\{d_n\}_{n=1}^{\infty}$ which satisfies

$$d_n = (1 - g_{n-1})g_n, \quad n \geq 1,$$

is called a positive chain sequence [5] (see also [12, Section 7.2]). Here $\{g_n\}_{n=0}^{\infty}$, called the parameter sequence is such that $0 \leq g_0 < 1$, $0 < g_n < 1$ for $n \geq 1$. This is a stronger condition than the one used in [26], in which d_n is also allowed to be zero. The parameter sequence $\{g_n\}_{n=0}^{\infty}$ is called a minimal parameter sequence and denoted by $\{m_n\}_{n=0}^{\infty}$ if $m_0 = 0$. Every chain sequence has a minimal parameter sequence [5, pp. 91–92]. Further, for a fixed chain sequence $\{d_n\}_{n \geq 1}$, let \mathcal{G} be the set of all parameter sequences $\{g_k\}$ of $\{d_n\}_{n \geq 1}$. Let the sequence $\{M_n\}_{n=0}^{\infty}$ be defined by

$$M_n = \inf\{g_n, \text{ for each } n, \{g_k\} \in \mathcal{G}\}, \quad n \geq 0,$$

where inf is infimum of the set. Then, $\{M_n\}$ is called the maximal parameter sequence of $\{d_n\}$.

The role of chain sequences in the study of orthogonal polynomials on the real line is well known. Similarly, a positive chain sequence $\{d_n\}$ appears in the three term recurrence relation for the polynomials $R_n(z)$, that turn out to be scaled versions of the (kernel) polynomials $P_n(1, z)$, namely,

$$R_{n+1}(z) = [(1 + ic_{n+1})z + (1 - ic_{n+1})]R_n(z) - 4d_{n+1}zR_{n-1}(z), \quad n \geq 1, \quad (1.4)$$

with $R_0(z) = 1$ and $R_1(z) = (1 + ic_1)z + (1 - ic_1)$. It is indeed shown in [6], that

$$R_n(z) = \frac{\prod_{j=0}^{n-1} [1 - \tau_j \alpha_j]}{\prod_{j=0}^{n-1} [1 - \operatorname{Re}(\tau_j \alpha_j)]} P_n(1; z),$$

where

$$\tau_j = \tau_j(1) = \prod_{k=1}^j \frac{1 - ic_k}{1 + ic_k}, \quad j \geq 1,$$

on condition that

$$c_n = \frac{-\operatorname{Im}(\tau_{n-1} \alpha_{n-1})}{1 - \operatorname{Re}(\tau_{n-1} \alpha_{n-1})} \quad \text{and} \quad d_{n+1} = (1 - g_n)g_{n+1}, \quad n \geq 1,$$

is a chain sequence with parameter sequence

$$g_n = \frac{1}{2} \frac{|1 - \tau_{n-1}\alpha_{n-1}|^2}{[1 - \operatorname{Re}(\tau_{n-1}\alpha_{n-1})]}, \quad n \geq 1.$$

It is also not difficult to verify that in this case $R_n(z)$ has $r_{n,n} = \prod_{k=1}^n (1 + ic_k)$ as the leading coefficient and $r_{n,0} = \bar{r}_{n,n} = \prod_{k=1}^n (1 - ic_k)$ as the constant term.

It is known that $\{R_n(z)\}$ can be used to obtain a sequence of OPUC [2, 3, 6], with respect to the measure $\mu(z)$ and having the shifted sequence $\{\alpha_{n-1}\}_{n=1}^\infty$ as the Verblunsky coefficients. A further interesting fact is that the above parameter sequence $\{g_{n+1}\}_{n=0}^\infty$ is such that $g_1 = (1 - \epsilon)M_1$, ($0 \leq \epsilon < 1$), where $\{M_{n+1}\}_{n=0}^\infty$ is the maximal parameter sequence of $\{d_{n+1}\}_{n=1}^\infty$ and that ϵ , is the size of the pure point at $z = 1$ in the probability measure $\mu(z)$ associated with the Verblunsky coefficients $\{\alpha_{n-1}\}_{n=1}^\infty$. This means, if the measure does not have a pure point at $z = 1$ then $\{g_{n+1}\}_{n=0}^\infty$ is the maximal parameter sequence of $\{d_{n+1}\}_{n=1}^\infty$.

Consider now the Uvarov transformation of the measure $\mu(z)$, [6, p. 11],

$$\int_{\partial\mathbb{D}} f(z) d\mu^{(t)}(z) = \frac{(1-t)}{1-\epsilon} \int_{\partial\mathbb{D}} f(z) d\mu(z) + \frac{t-\epsilon}{1-\epsilon} f(1),$$

so that $\mu^{(t)}(z)$ has a jump t , $0 \leq t < 1$, at $z = 1$. These measures $\mu^{(t)}(z)$ are associated with the positive chain sequence $\{d_n\}_{n=1}^\infty$ obtained from $\{d_{n+1}\}_{n=1}^\infty$ by including the additional term $d_1 = (1-t)M_1$. We also denote the generalized sequence of Verblunsky coefficients associated with $\mu^{(t)}(z)$ by $\{\alpha_{n-1}^{(t)}\}_{n=1}^\infty$.

Note 1.1. Since the measure $\mu^{(t)}(z)$ has a parameter ‘ t ’, the notations for the polynomials $R_n(z)$ and the sequences $\{c_n\}$, $\{d_n\}$ should have involved a ‘ t ’. However, it has been proved [6, p. 7], that the kernel polynomials $P_n(1; z)$ and hence $R_n(z)$ as well as the sequences $\{c_n\}$ and $\{d_n\}$ are independent of ‘ t ’ and so their notations are devoid of ‘ t ’. But the minimal parameters depend on d_1 and this has been reflected in the notation $m_n^{(t)}$.

As shown in [6, Theorem 1.1], $\mu^{(t)}(z)$ can also be given by

$$\int_{\partial\mathbb{D}} f(z) d\mu^{(t)}(z) = (1-t) \int_{\partial\mathbb{D}} f(z) d\mu^{(0)}(z) + t f(1). \quad (1.5)$$

As is obvious from the notation, $\mu^{(0)}(z)$ are the measures arising when $t = 0$. This is the case when $d_1 = M_1$, so that both the minimal and maximal parameter sequences coincide. This equality can also be interpreted as the measure having zero jump.

Further, the Verblunsky coefficients $\alpha_{n-1}^{(t)}$ have the representation

$$\alpha_{n-1}^{(t)} = \bar{\tau}_n \left[\frac{1 - 2m_n^{(t)} - ic_n}{1 + ic_n} \right], \quad n \geq 1. \quad (1.6)$$

where $\{m_n^{(t)}\}$ is the minimal parameter sequence of the positive chain sequence $\{d_n\}_{n=1}^\infty$. The Szegő polynomials corresponding to (1.6) are [3, Theorem 5.2]

$$\Phi_n^{(t)}(z) = \frac{R_n(z) - 2(1 - m_n^{(t)})R_{n-1}(z)}{\prod_{k=1}^n (1 + ic_k)}, \quad n \geq 1. \quad (1.7)$$

It can be verified from (1.4) that if $c_k = 0$, $k \geq 0$, α_{n-1} , $n \geq 1$, are all real. The $R_n(z)$ are then the singular predictor polynomials of the second kind given in [7]. Indeed, if $c_n = 0$, $n \geq 1$, it can be easily shown from (1.7) that

$$(z - 1)R_n(z) = z\Phi_n^{(t)}(z) - (\Phi_n^{(t)})^*(z).$$

We would like to mention here that the Szegő polynomials, Verblunsky coefficients and the related measure have also been obtained for the para-orthogonal polynomials that are an extension of singular predictor polynomials of first kind. See [2] for the details of these extensions and also for a survey of recent developments in the theory connecting chain sequences and OPUC.

The purpose of the present manuscript is to introduce a particular perturbation in the chain sequence $\{d_n\}$, called the complementary chain sequence, and study its effect on the Verblunsky coefficients of the corresponding Szegő polynomials. The motivation for this follows from the fact that (1.6) guarantees an explicit relation between the Verblunsky coefficients and the minimal parameter sequence $\{m_n^{(t)}\}$ of $\{d_n\}$.

This manuscript is organized as follows. In Section 2 the concept of complementary chain sequences using the minimal parameter sequences is introduced. Using this concept, perturbations of Verblunsky coefficients are studied. As an illustration of this concept, in Section 3, the Szegő polynomials which characterizes the positive Perron–Carathéodory (PPC) fractions from a particular chain sequence are constructed. An interplay by these PPC fractions in finding a relation between this chain sequence, its complementary chain sequence and their respective Carathéodry functions is obtained in this section. In Section 4, another illustration of characterizing the Szegő polynomials using Gaussian hypergeometric functions is provided. For particular values, using complementary chain sequences, the corresponding Verblunsky coefficients of these Szegő polynomials are also shown to be perturbed Verblunsky coefficients obtained earlier.

2 Complementary chain sequences

As is obvious from the definition of chain sequences, the minimal and maximal parameter sequences are uniquely defined for any given chain sequence. Also, the chain sequence for which the minimal and maximal parameter sequences coincide, that is, $M_0 = 0$, has its own importance as illustrated in the previous section. Such a chain sequence is said to determine its parameters uniquely and is referred to as a single parameter positive chain sequence (SPPCS) [2]. By Wall's criteria for maximal parameter sequence [26, p. 82], this is equivalent to

$$\sum_{n=1}^{\infty} \frac{m_1}{1 - m_1} \cdot \frac{m_2}{1 - m_2} \cdot \frac{m_3}{1 - m_3} \cdots \frac{m_n}{1 - m_n} = \infty. \quad (2.1)$$

Thus, introducing a perturbation in the minimal parameters m_n will lead to a uniquely defined change in the chain sequence.

Definition 2.1. Suppose $\{d_n\}_{n=1}^{\infty}$ is a chain sequence with $\{m_n\}_{n=0}^{\infty}$ as its minimal parameter sequence. Let $\{k_n\}_{n=0}^{\infty}$ be another sequence given by $k_0 = 0$ and $k_n = 1 - m_n$ for $n \geq 1$. Then the chain sequence $\{a_n\}_{n=1}^{\infty}$ having $\{k_n\}_{n=0}^{\infty}$ as its minimal parameter sequence is called the complementary chain sequence of $\{d_n\}$.

Such chain sequences enjoy interesting relations like [26, equation (75.3)]

$$1 + \frac{\sqrt{1+z}}{1 + \frac{d_1 z}{1 + \frac{d_2 z}{1 + \frac{d_3 z}{\ddots}}}} \cdot \frac{\sqrt{1+z}}{1 + \frac{a_1 z}{1 + \frac{a_2 z}{1 + \frac{a_3 z}{\ddots}}}} = 1.$$

They also satisfy

$$d_1 - a_1 = 1 - 2k_1 = 2m_1 - 1$$

and

$$d_n - a_n = \Delta m_{n-1} = -\nabla k_n, \quad n \geq 2.$$

where Δ and ∇ are the forward and backward difference operators respectively. Further of particular interest is the ratio of these two chain sequences given by

$$\frac{d_1}{a_1} = \frac{m_1}{1 - m_1}, \quad \frac{d_n}{a_n} = \frac{k_{n-1}}{1 - k_{n-1}} \frac{m_n}{1 - m_n}, \quad n \geq 2.$$

This implies

$$\frac{m_n}{1 - m_n} = \frac{d_n}{a_n} \frac{m_{n-1}}{1 - m_{n-1}} = \dots = \frac{d_n d_{n-1} \cdots d_1}{a_n a_{n-1} \cdots a_1}, \quad n \geq 1. \quad (2.2)$$

Substituting (2.2) in (2.1), we have the following lemma.

Lemma 2.2. *Let $\{d_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ be two complementary chain sequences of each other. Then $\{d_n\}_{n=1}^{\infty}$ will be a SPPCS if and only if*

$$\sum_{n=1}^{\infty} \prod_{j=1}^n \frac{d_1 d_2 \cdots d_j}{a_1 a_2 \cdots a_j} = \infty.$$

Remark 2.3. The above lemma is useful while considering a chain sequence and its complementary chain sequence without using the information on the corresponding minimal parameters.

Lemma 2.4. *Let $\{d_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ be two complementary chain sequences of each other. If $\{d_n\}_{n=1}^{\infty}$ is not a SPPCS, then $\{a_n\}_{n=1}^{\infty}$ is a SPPCS.*

Proof. If $\{d_n\}_{n=1}^{\infty}$ is not a SPPCS then its minimal parameter sequence $\{m_n\}_{n=0}^{\infty}$ is such that

$$\sum_{n=1}^{\infty} \prod_{j=1}^n \frac{m_j}{1 - m_j} < \infty.$$

Hence, $\lim_{n \rightarrow \infty} \prod_{j=1}^n m_j / (1 - m_j) = 0$, and we have

$$\sum_{n=1}^{\infty} \prod_{j=1}^n \frac{k_j}{1 - k_j} = \sum_{n=1}^{\infty} \prod_{j=1}^n \frac{1 - m_j}{m_j} = \infty.$$

Thus, concluding the proof of the lemma. ■

Lemma 2.5. *Let $\{d_n\}_{n=1}^{\infty}$ be a chain sequence and $\{a_n\}_{n=1}^{\infty}$ be its complementary chain sequence with minimal parameter sequences $\{m_n\}_{n=0}^{\infty}$ and $\{k_n\}_{n=0}^{\infty}$ respectively.*

- If $0 < m_n < 1/2$, $n \geq 1$, then a_n is a SPPCS.
- If $1/2 < m_n < 1$, $n \geq 1$, then d_n is a SPPCS.

Proof. Observe that if $0 < m_n < 1/2$, $k_n / (1 - k_n) > 1$ for all $n \geq 1$. Similarly, $1/2 < m_n < 1$ implies $m_n / (1 - m_n) > 1$ for all $n \geq 1$. The results now follow from (2.1). ■

It is known that [26, p. 79] if $d_n \geq 1/4$, $n \geq 1$, every parameter sequence $\{g_n\}$, in particular the minimal parameter sequence $\{m_n\}$ of $\{d_n\}$ is non-decreasing. For the special case when $d_n = 1/4$, $n \geq 1$, $m_n \rightarrow 1/2$ as $n \rightarrow \infty$. This implies $0 < m_n < 1/2$, $n \geq 1$. By Lemma 2.5, $\{a_n\}$ is a SPPCS. In other words, the chain sequence complementary to the constant chain sequence $\{1/4\}$ determines its parameters g_n uniquely, which are further given by

$$g_0 = 0, \quad g_n = \frac{n+2}{2(n+1)}, \quad n \geq 1.$$

Moreover, if $d_n \geq 1/4$, there exist some $n \in \mathbb{N}$ such that $a_n < 1/4 \leq d_n$. Indeed,

$$d_n = (1 - m_{n-1})m_n \geq m_{n-1}(1 - m_n) = a_n, \quad n \geq 2,$$

with the sign of the difference of d_1 and a_1 depending on whether $m_1 \in (0, 1/2)$ or $(1/2, 1)$. If $a_n \in (1/4, 1)$ for $n \geq 1$, k_n has to be non-decreasing. This is a contradiction as $k_n = 1 - m_n$ for $n \geq 1$.

The effect of complementary chain sequences in studying perturbation of Verblunsky coefficients given by (1.6) has interesting consequences. In this context, we give the following result.

Theorem 2.6. *Let $\{c_n\}_{n=1}^{\infty}$ and $\{d_{n+1}\}_{n=1}^{\infty}$ be, respectively, the real sequence and positive chain sequence as given in (1.4). Let $\{m_n^{(t)}\}_{n=0}^{\infty}$ be the minimal parameter sequence of the augmented positive chain sequence $\{d_n\}_{n=1}^{\infty}$, where $d_1 = (1-t)M_1$ and $\{M_{n+1}\}_{n=0}^{\infty}$ is the maximal parameter sequence of $\{d_{n+1}\}_{n=1}^{\infty}$. Let $\{k_n^{(t)}\}_{n=0}^{\infty}$ be the minimal parameter sequence of the positive chain sequence $\{a_n\}_{n=1}^{\infty}$ obtained as complementary to $\{d_n\}_{n=1}^{\infty}$. Set $\tau_n = \frac{1-ic_n}{1+ic_n}\tau_{n-1}$,*

$$\alpha_{n-1}^{(t)} = \bar{\tau}_n \left[\frac{1 - 2m_n^{(t)} - ic_n}{1 + ic_n} \right] \quad \text{and} \quad \beta_{n-1}^{(t)} = \bar{\tau}_n \left[\frac{1 - 2k_n^{(t)} - ic_n}{1 + ic_n} \right],$$

for $n \geq 1$, with $\tau_0 = 1$. Let $\mu^{(t)}(z)$ and $\nu^{(t)}(z)$ be, respectively, the probability measures having $\alpha_{n-1}^{(t)}$ and $\beta_{n-1}^{(t)}$ as the corresponding Verblunsky coefficients. Then the following can be stated:

1. For $0 < t < 1$, the measure $\mu^{(t)}(z)$ has a pure point of size t at $z = 1$, while $\nu^{(t)}(z)$ does not.
2. $\beta_{n-1}^{(t)} = -\bar{\tau}_n \bar{\tau}_{n-1} \bar{\alpha}_{n-1}^{(t)}$, $n \geq 1$.
3. For $n \geq 1$, if $c_n = (-1)^n c$, $c \in \mathbb{R}$, $\beta_{n-1}^{(t)} = -\frac{1-ic}{1+ic} \alpha_{n-1}^{(t)}$, $n \geq 1$.
4. If $c_n = 0$, $n \geq 1$ then the Verblunsky coefficients, which are real, are such that $\beta_{n-1}^{(t)} = -\alpha_{n-1}^{(t)}$, $n \geq 1$.

Proof. First we observe that $\alpha_{n-1}^{(t)}$ are the generalized Verblunsky coefficients of the measure $\mu^{(t)}(z)$ as given by (1.5). Consequently, for $0 < t < 1$ the probability measure $\mu^{(t)}(z)$ has a pure point of size t at $z = 1$. Since $d_1 = (1-t)M_1$, choosing $M_0 = t > 0$, the sequence $\{t, M_1, M_2, M_3, \dots\}$ is the maximal parameter sequence of $\{d_n\}_{n=1}^{\infty}$. Since $t > 0$, $\{d_n\}_{n=1}^{\infty}$ is a non SPPCS and hence, by Lemma 2.4 the sequence $\{a_n\}_{n=1}^{\infty}$ is a SPPCS so that $\{k_n^{(t)}\}_{n=0}^{\infty}$ is also its maximal parameter sequence. Thus, by results established in [6], the measure $\nu^{(t)}(z)$ does not have a pure point at $z = 1$. This proves the first part of the theorem.

Now to prove the second part, we first have

$$\beta_{n-1}^{(t)} = \bar{\tau}_n \left[\frac{1 - 2k_n^{(t)} - ic_n}{1 + ic_n} \right] = \bar{\tau}_n \left[\frac{-1 + 2m_n^{(t)} - ic_n}{1 + ic_n} \right].$$

By conjugation of the expression for $\alpha_{n-1}^{(t)}$, we have

$$-\bar{\alpha}_{n-1}^{(t)} = \tau_n \left[\frac{-1 + 2m_n^{(t)} - ic_n}{1 - ic_n} \right],$$

which leads to the second part of the theorem.

Clearly with $c_n = (-1)^n c$, $n \geq 1$ we have $\bar{\tau}_{2n} = 1$ and $\bar{\tau}_{2n+1} = \frac{1-ic}{1+ic}$. Thus, the third part of the theorem is established.

The last part follows by taking $\bar{\tau}_n \bar{\tau}_{n-1} = 1$, $n \geq 1$. This is only possible if $c_n = 0$, $n \geq 1$. ■

The perturbation of the Verblunsky coefficients in case of OPUC and of the recurrence coefficients in case of the real line play an important role in the spectral theory of orthogonal polynomials. The reader is referred to [9] and [18] for some details. For a recent work in this direction, we refer to [4].

The last two parts of Theorem 2.6 are important cases of Aleksandrov transformation and, in the case of last part gives rise to second kind polynomials for the measure $\mu^{(t)}$ [22]. In this particular case, the recurrence relation (1.4) assumes a very simple form, similar to that considered in [7].

In the next section, starting with particular minimal parameter sequences and assuming $c_n = 0$, $n \geq 1$, we construct the para-orthogonal polynomials and the related Szegő polynomials to illustrate our results.

3 An illustration involving Carathéodory functions

In a series of papers [13, 14, 15], Jones et al. during their investigation of the connection between Szegő polynomials and continued fractions introduced the following

$$\delta_0 - \frac{2\delta_0}{1} + \frac{1}{\bar{\delta}_1 z} + \frac{(1 - |\delta_1|^2)z}{\delta_1} + \frac{1}{\bar{\delta}_2 z} + \frac{(1 - |\delta_2|^2)z}{\delta_2} + \dots \quad (3.1)$$

These are called Hermitian Perron–Carathéodory fractions or HPC-fractions and are also used to solve the trigonometric moment problem. They are completely determined by $\delta_n \in \mathbb{C}$, where $\delta_0 \neq 0$ and $|\delta_n| \neq 1$ for $n \geq 1$. Under the stronger conditions $\delta_0 > 0$ and $|\delta_n| < 1$, for $n \geq 1$, (3.1) is called a positive PC fraction (PPC-fractions). Let $\mathcal{P}_n(z)$ and $\mathcal{Q}_n(z)$ be respectively the numerator and denominator of the n^{th} approximant of a PPC-fraction where $\mathcal{Q}_n(z)$ is a polynomial of degree n and $\mathcal{P}_n(z)$ of degree at most n . Then [15, Theorems 3.1 and 3.2] $\Phi_n(z)$ are precisely the odd ordered denominators $\mathcal{Q}_{2n+1}(z)$ and $\Phi_n^*(z)$ the even ordered denominators $\mathcal{Q}_{2n}(z)$. The δ'_n s are then given by $\delta_n = \Phi_n(0)$ and are called the Schur parameters or the reflection coefficients. This gives the following equivalent set of recurrence relations for the Szegő polynomials:

$$\begin{aligned} \Phi_n^*(z) &= \bar{\delta}_n z \Phi_{n-1} z + \Phi_{n-1}^*(z), \\ \Phi_n(z) &= \delta_n \Phi_n^*(z) + (1 - |\delta_n|^2) z \Phi_{n-1}(z), \quad n \geq 1. \end{aligned}$$

Further, if (3.1) is a positive PC-fraction, there exists a pair of formal power series

$$\mathcal{L}_0 = \mu_0 + 2 \sum_{k=1}^{\infty} \mu_k z^k, \quad \mathcal{L}_\infty = -\mu_0 - 2 \sum_{k=1}^{\infty} \mu_{-k} z^{-k},$$

where μ_k are the moments as defined earlier and such that

$$\mathcal{L}_0 - \Lambda_0 \left(\frac{\mathcal{P}_{2n}}{\mathcal{Q}_{2n}} \right) = \mathcal{O}(z^{n+1}), \quad \mathcal{L}_\infty - \Lambda_\infty \left(\frac{\mathcal{P}_{2n+1}}{\mathcal{Q}_{2n+1}} \right) = \mathcal{O} \left(\frac{1}{z^{n+1}} \right).$$

Here, $\Lambda_0(\mathcal{R}(z))$ and $\Lambda_\infty(\mathcal{R}(z))$ are the Laurent series expansion of the rational function $\mathcal{R}(z)$ about 0 and ∞ respectively. For details regarding correspondence of continued fractions to power series, see [16, 17].

For $|\zeta| < 1$, the polynomials

$$\Psi_n(z) = \int_{\partial\mathbb{D}} \frac{z + \zeta}{z - \zeta} (\Phi_n(z) - \Phi_n(\zeta)) d\mu(\zeta), \quad n \geq 1,$$

are known in literature as the associated Szegő polynomials or polynomials of the second kind [10]. They arise as the odd ordered numerators of (3.1). The function $-\Psi_n^*(z)$ is called the polynomial associated with $\Phi_n^*(z)$ and are the even ordered numerators in (3.1). It is also known that for $|z| < 1$, there exists a function $\mathcal{C}(z) = \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)$ with $\operatorname{Re} \mathcal{C}(z) > 0$ such that

$$\mathcal{C}(z) - \frac{\Psi_n^*(z)}{\Phi_n^*(z)} = \mathcal{O}(z^{n+1}).$$

$\mathcal{C}(z)$ is called the Carathéodory function associated with the PPC-fraction (3.1) or with the Szegő polynomials $\Phi_n(z)$ obtained from this PPC-fraction. The ratio $\Psi_n(z)/\Phi_n(z)$ also converges to a function $\hat{\mathcal{C}}(z)$ called the Carathéodory reciprocal of $\mathcal{C}(z)$ [14] and is defined by

$$\mathcal{C}(z) = -\overline{\hat{\mathcal{C}}(1/\bar{z})}.$$

The convergence is uniform on compact subsets of $|z| < 1$ and $|z| > 1$ respectively. Also, \mathcal{L}_0 is the Taylor series expansion of $\mathcal{C}(z)$ about 0 and \mathcal{L}_∞ is that of $\hat{\mathcal{C}}(z)$ about ∞ .

Consider the sequence $\{\delta_n\}_{n=1}^\infty$, which satisfies $\delta_0 > 0$, $|\delta_n| < 1$ and

$$\delta_{n+1} - \delta_n = \delta_n \delta_{n+1}, \quad n \geq 1. \quad (3.2)$$

Our aim in this section is to use a chain sequence to construct the Szegő polynomials $\Phi_n^{(t)}(z)$, having $\delta_n \in \mathbb{R}$ and satisfying (3.2) as the Verblunsky coefficients. We will also use the complementary chain sequence to get another sequence of Szegő polynomials $\tilde{\Phi}_n^{(t)}(z)$ which has $-\delta_n$ as the Verblunsky coefficients. The associated Carathéodory function in each case is also given and it is shown that there exists a relation between them.

We start with the sequence $\{m_n^{(t)}\}_{n=0}^\infty$, where $m_0^{(t)} = 0$ and $m_n^{(t)} = (1 - \delta_n)/2$, $n \geq 1$. These minimal parameters are obtained by first substituting $c_k = 0$, $k \geq 1$ in the Verblunsky coefficients (1.6) and then equating them to δ_n . The corresponding chain sequence is

$$d_1 = \frac{1 - \delta_1}{2} \quad \text{and} \quad d_n = \frac{1}{4}(1 + \delta_{n-1})(1 - \delta_n) = \frac{1}{4}(1 - 2\delta_{n-1}\delta_n), \quad n \geq 2.$$

The following are two algebraic relations of δ_n which will be needed later and can be proved by simple induction using (3.2).

$$\delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_4 + \cdots + \delta_n\delta_{n+1} = \delta_{n+1} - \delta_1, \quad n \in \mathbb{N}.$$

and

$$\delta_n = \frac{\delta_{n+1}}{1 + \delta_{n+1}} = \cdots = \frac{\delta_{n+k}}{1 + k\delta_{n+k}}, \quad k \in \mathbb{N}. \quad (3.3)$$

Proposition 3.1. *The monic polynomial*

$$R_n(z) = 1 + \sum_{k=1}^n [1 + 2k(n-k)\delta_1\delta_n] z^k \quad (3.4)$$

satisfies the recurrence relation

$$R_{n+1}(z) = (z+1)R_n(z) - (1 - 2\delta_n\delta_{n+1})zR_{n-1}(z), \quad n \geq 1,$$

with the initial conditions, $R_0(z) = 1$ and $R_1(z) = z + 1$.

Proof. First, note that $R_1(z)$ given by (3.4) satisfies the initial condition. Suppose $R_n(z)$ has this form and satisfies the recurrence relation for $n = 1, 2, \dots, j$. We shall now show

$$R_{j+1}(z) + (1 - 2\delta_j\delta_{j+1})zR_{j-1}(z) = (z + 1)R_j(z). \quad (3.5)$$

Using (3.3), the coefficient of z^k in the left-hand side of (3.5) is

$$\begin{aligned} & 1 + 2k(j - k + 1)\delta_1\delta_{j+1} + (1 - 2\delta_j\delta_{j+1})[1 + 2(k - 1)(j - k)\delta_1\delta_{j-1}] \\ &= 1 + 2\frac{k(j - k + 1)}{j}(\delta_{j+1} - \delta_1) + 1 - 2(\delta_{j+1} - \delta_j) + 2\frac{(k - 1)(j - k)}{j - 2}(\delta_{j-1} - \delta_1) \\ &\quad - \frac{2 \cdot 2(k - 1)(j - k)}{j - 2}(\delta_{j-1} - \delta_1)(\delta_{j+1} - \delta_j). \end{aligned} \quad (3.6)$$

It is easy to verify that the coefficients of δ_{j+1} and δ_{j-1} vanish in (3.6). The coefficient of δ_1 is

$$\begin{aligned} & -\frac{2k(j - k + 1)}{j} - \frac{2(k - 1)(j - k)}{j - 2} - \frac{2 \cdot 2(k - 1)(j - k)}{j(j - 2)} + \frac{2 \cdot 2(k - 1)(j - k)}{(j - 1)(j - 2)} \\ &= -\frac{2k(j - k)}{j - 1} - \frac{2(k - 1)(j - k + 1)}{j - 1}. \end{aligned} \quad (3.7)$$

Similarly, the coefficient of δ_j is

$$2 + \frac{2 \cdot 2(k - 1)(j - k)}{j - 1} = \frac{2k(j - k)}{j - 1} + \frac{2(k - 1)(j - k + 1)}{j - 1}. \quad (3.8)$$

Using (3.7) and (3.8) in (3.6), the coefficient of z^k in the left-hand side of (3.5) is given by

$$[1 + 2(k - 1)(j - k + 1)\delta_1\delta_j] + [1 + 2k(j - k)\delta_1\delta_j],$$

which is nothing but the coefficient of z^k in the right-hand side of (3.5). Hence, by induction the proof is complete. \blacksquare

We now obtain the Szegő polynomials $\Phi_n^{(t)}(z)$ from the para-orthogonal polynomials $R_n(z)$ given by (3.4). Using (1.7) and (3.4), it can be seen that the coefficient of z^k , $1 \leq k \leq n - 1$, in $\Phi_n^{(t)}(z)$ is $-\delta_n(1 - 2k\delta_1)$. Hence, the Szegő polynomials are given by

$$\Phi_n^{(t)}(z) = z^n - \delta_n[(1 - 2(n - 1)\delta_1)z^{n-1} + \dots + (1 - 2\delta_1)z + 1], \quad n \geq 1, \quad (3.9)$$

with $\alpha_{n-1}^{(t)} = -\delta_n$.

We now give the Carathéodory function associated with the parameters δ_n 's given by (3.2). Consider

$$\mathcal{C}(z) = 1 - \frac{2(1 - \sigma)z}{1 + (1 - 2\sigma)z} = \frac{1 - z}{1 + (1 - 2\sigma)z}, \quad |z| < 1,$$

where $0 < \sigma < 1$. That $\mathcal{C}(z)$ corresponds to a PPC-fraction with the parameter γ_n , where

$$\gamma_n = \frac{1}{n + \frac{\sigma}{1 - \sigma}}, \quad n \geq 1. \quad (3.10)$$

can be shown by applying the algorithm [14] which is similar to the Schur algorithm. With the initial values $\mathcal{C}_0(z) = (1 - z)/(1 + (1 - 2\sigma)z)$, $\gamma_0 = \mathcal{C}_0(0) = 1$, define

$$\mathcal{C}_1(z) = \frac{\gamma_0 - \mathcal{C}_0(z)}{\gamma_0 + \mathcal{C}_0(z)}, \quad \gamma_1 = \mathcal{C}'_1(0).$$

Then

$$\mathcal{C}_1(z) = \frac{z}{1 + \frac{\sigma}{1-\sigma} - \left(1 - \frac{1-2\sigma}{1-\sigma}\right)z}, \quad \text{and} \quad \gamma_1 = \frac{1}{1 + \frac{\sigma}{1-\sigma}}.$$

Assume for $k \geq 1$ the following

$$\mathcal{C}_k(z) = \frac{z}{k + \frac{\sigma}{1-\sigma} - \left(k - \frac{1-2\sigma}{1-\sigma}\right)z}, \quad \gamma_k = \mathcal{C}'_k(0).$$

This is true for $k = 1$. Now define

$$\mathcal{C}_{k+1}(z) = \frac{\gamma_k z - \mathcal{C}_k(z)}{\gamma_k \mathcal{C}_k(z) - z}, \quad n \geq 1. \quad (3.11)$$

It can be shown that

$$\gamma_k = \frac{1 - \sigma}{k - (k-1)\sigma} = \frac{1}{k + \frac{\sigma}{1-\sigma}},$$

which is also true for $k = 1$. Simplifying (3.11), we obtain

$$\mathcal{C}_{k+1} = \frac{z}{\left(k + 1 + \frac{\sigma}{1-\sigma}\right) - \left(k + 1 - \frac{1-2\sigma}{1-\sigma}\right)z},$$

from which $\gamma_{k+1} = \frac{1}{k+1 + \frac{\sigma}{1-\sigma}}$. Hence by induction, (3.10) and because of the uniqueness of the Carathéodory function that corresponds to a given PPC-fraction, the assertion follows. Moreover, observe that $\delta_n = -\gamma_n$ satisfies (3.2) and so $\Phi_n^{(t)}(0) = \frac{1}{n + \frac{\sigma}{1-\sigma}}$.

From the power series expansion of $\mathcal{C}(z)$, we also obtain the moments as

$$\mu_0 = 1, \quad \mu_k = (-1)^k (1 - \alpha)(1 - 2\alpha)^{k-1}, \quad k \geq 1.$$

Using the fact that the Verblunsky coefficients are all real, from (1.2), we have

$$\chi_n^{-2} = \prod_{k=1}^n (1 - \delta_k^2).$$

Further

$$\delta_n = \frac{1}{n + \frac{\sigma}{1-\sigma}} = \frac{1 - \sigma}{n(1 - \sigma) + \sigma}, \quad n \geq 1,$$

and we obtain

$$\begin{aligned} 1 - \delta_n^2 &= \frac{[n(1 - \sigma) + \sigma - 1 + \sigma][n(1 - \sigma) + \sigma + 1 - \sigma]}{[n(1 - \sigma) + \sigma]^2} \\ &= \frac{[(n - 1) - (n - 2)\sigma][(n + 1) - n\sigma]}{[n - (n - 1)\sigma]^2}, \end{aligned}$$

which yields the fact that

$$\chi_n^{-2} = \frac{\sigma[(n + 1) - n\sigma]}{[n - (n - 1)\sigma]}.$$

Rewriting the right-hand expression as $\sigma(1 + \frac{1-\sigma}{n(1-\sigma)+\sigma})$ gives

$$\chi_n^{-2} = \|\Phi_n^{(t)}(z)\|^2 = \sigma(1 + \delta_n),$$

which tends to $\sigma > 0$ as $n \rightarrow \infty$.

Consider now the parameter sequence $\{k_n^{(t)}\}_{n=0}^\infty$, defined by $k_0^{(t)} = 0$ and $k_n^{(t)} = 1 - m_n^{(t)} = (1 + \delta_n)/2$, $n \geq 1$. From (3.2), it is easy to check that $1 + \delta_{n+1} = 1/(1 - \delta_n)$, $n \geq 1$. In this case, the constant sequence $\{1/4\}$ becomes the complementary chain sequence so that equation (1.4) assumes the form

$$\tilde{R}_{n+1}(z) = [1 + z]\tilde{R}_n(z) - z\tilde{R}_{n-1}(z), \quad n \geq 1.$$

The polynomials satisfying the above recurrence relation are the palindromic polynomials $z^n + \lambda(z^{n-1} + \dots + z) + 1$. For $\lambda = 1$, the para-orthogonal polynomials are the partial sums of the geometric series given by

$$\tilde{R}_n(z) = 1 + z + z^2 + \dots + z^n = \frac{1 - z^{n+1}}{1 - z}, \quad n \geq 1.$$

Then (1.7) yields the Szegő polynomial

$$\tilde{\Phi}_n^{(t)}(z) = z^n + \delta_n z^{n-1} + \dots + \delta_n z + \delta_n, \quad n \geq 1, \quad (3.12)$$

with $\alpha_{n-1}^{(t)} = -\delta_n$. The polynomials $\tilde{\Phi}_n^{(t)}(z)$ have been considered in [20] where it is proved that

$$\tilde{\Phi}_n^{(t)}(0) = \delta_n = -\frac{1}{n + \frac{\sigma}{1-\sigma}}, \quad n \geq 1. \quad (3.13)$$

Further, the corresponding Carathéodory function is $\tilde{\mathcal{C}}(z) = \frac{1+(1-2\sigma)z}{1-z}$, $|z| < 1$, where $0 < \sigma < 1$. This is a special case when all the moments are equal to $\tilde{\mu} = (1 - \sigma)$. We summarize the above facts as a theorem.

Theorem 3.2. *Consider the real sequence $\{\delta_n\}_{n=0}^\infty$ satisfying $\delta_n - \delta_{n-1} = \delta_{n-1}\delta_n$, $n \geq 1$ under the restrictions $\delta_0 > 0$ and $|\delta_n| < 1$, $n \geq 1$. If $\mathcal{C}(z)$ is a Carathéodory function whose PPC-fraction can be obtained from the minimal parameter sequence $\{m_n\}$, where $2m_n = 1 - \delta_n$, $n \geq 1$, then $1 - m_n$ gives the PPC-fraction corresponding to the Carathéodory function $1/\mathcal{C}(z)$.*

Note that an equivalent statement using Schur parameters is given in [21]. Further, let $\mu^{(t)}(z)$ be the probability measure associated with the positive chain sequence $\{d_n\}_{n=1}^\infty$. Since its complementary chain sequence $\{1/4\}$ is not a SPPCS, by Lemma (2.4) $\{d_n\}_{n=1}^\infty$ is a SPPCS and hence $\mu^{(t)}(z)$ has zero jump ($t = 0$) at $z = 1$. If $\nu^{(t)}(z)$ is the measure associated with $\{1/4\}$, $\nu^{(t)}(z)$ has a jump $t = 1/2$ at $z = 1$. Finally as shown in [20], $\nu^{(1/2)}(\theta)$ is of the form,

$$d\nu^{(1/2)}(\theta) = d\nu_s^{(1/2)}(\theta) + (1 - \tilde{\mu})d(\theta),$$

where $d\nu_s^{(1/2)}(\theta)$ is a point measure with mass $\tilde{\mu}$ at $z=1$ and mass zero elsewhere.

We end this illustration with two observations which we state as remarks.

Remark 3.3. Suppose the minimal parameters are given in terms of some variable ε . It follows that the coefficients of the polynomial $R_n(z)$ satisfying (1.4) with $c_n = 0$ for $n \geq 1$ will be given in terms of ε . Since, it is clear that $R_n(z)$ is palindromic for the chain sequence $\{d_n\} = \{1/4\}$, $R_n(z)$ can always be expressed as the sum of two polynomials, one of them being a palindromic and the other one being such that it vanishes whenever ε is chosen so that $d_n = 1/4$.

Remark 3.4. As $n \rightarrow \infty$, both the minimal parameter sequences approach $1/2$. From the expressions (3.9) and (3.12) it is clear that for fixed z , $\Phi_n^{(t)}(z)$ and $\tilde{\Phi}_n^{(t)}(z)$ approach z^n as n becomes large. The polynomials z^n are called the Szegő–Chebyshev polynomials and correspond to the standard Lebesgue measure on the unit circle.

4 An illustration using Gaussian hypergeometric functions

The Gaussian hypergeometric function, with the complex parameters a , b and c is defined by the power series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad |z| < 1,$$

where $c \neq 0, -1, -2, \dots$ and $(a)_n$ is the Pochhammer symbol. With specialized values of the parameters a , b and c , many elementary functions can be represented by the Gaussian hypergeometric functions or their ratios. If $\operatorname{Re}(c - a - b) > 0$, the series converges for $|z| = 1$ to the value given by

$$F(a, b; c; 1) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

In case the series is terminating, we have the Chu–Vandermonde identity [1]

$$F(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}. \quad (4.1)$$

Two hypergeometric functions $F(a_1, b_1; c_1; z)$ and $F(a_2, b_2; c_2; z)$ are said to be contiguous if the difference between the corresponding parameters is at most unity. A linear combination of two contiguous hypergeometric functions is again a hypergeometric function. Such relations are called contiguous relations and have been used to explore many hidden properties of the hypergeometric functions, for example by Gauss who found continued fraction expansions for ratios of hypergeometric functions [19] and hence for the special functions that these ratios represent. In some special cases, the contiguous relations can also be related to the recurrence relations for orthogonal polynomials. Consider one such relation [1]

$$(c-a)F(a-1, b; c; z) = (c-2a-(b-a)z)F(a, b; c; z) + a(1-z)F(a+1, b; c; z),$$

which as shown in [24], can be transformed to the three term recurrence relation

$$\varrho_{n+1}(z) = \left(z + \frac{c-b+n}{b+n} \right) \varrho_n(z) - \frac{n(c+n-1)}{(b+n-1)(b+n)} \varrho_{n-1}(z), \quad n \geq 1, \quad (4.2)$$

satisfied by the monic polynomial

$$\varrho_n(z) = \frac{(c)_n}{(b)_n} F(-n, b; c; 1-z). \quad (4.3)$$

It was also shown that for the specific values $b = \lambda \in \mathbb{R}$ and $c = 2\lambda - 1$, the polynomials (4.3) are Szegő polynomials. We note that with $b = \lambda + 1$, $\varrho_n(z)$ given by (4.3) are called the circular Jacobi polynomials [12, Example 8.2.5]. For other specialized values of b and c in (4.2), $\varrho_n(z)$ also becomes the para-orthogonal polynomial.

Let $\lambda > -1/2 \in \mathbb{R}$. Taking $b = \lambda + 1$ and $c = 2\lambda + 2$, (4.2) reduces to

$$\varrho_{n+1}(z) = (z+1)\varrho_n(z) - \frac{n(2\lambda+n+1)}{(\lambda+n)(\lambda+n+1)} z \varrho_{n-1}(z), \quad n \geq 1,$$

satisfied by

$$\varrho_n(z) = R_n(z) = \frac{(2\lambda+2)_n}{(\lambda+1)_n} F(-n, \lambda+1; 2\lambda+2; 1-z), \quad n \geq 1.$$

Consider now the sequence $\{d_{n+1}\}_{n=1}^{\infty}$, where

$$d_{n+1} = \frac{1}{4} \frac{n(2\lambda + n + 1)}{(\lambda + n)(\lambda + n + 1)}, \quad n \geq 1.$$

As established in [2, Example 3], for $\lambda > -1$, the sequence $\{d_{n+1}\}_{n=1}^{\infty}$ is a positive chain sequence and $\{\mathbf{m}_n\}_{n=0}^{\infty}$, where

$$\mathbf{m}_n = \frac{n}{2(\lambda + n + 1)}, \quad n \geq 0,$$

is its minimal parameter sequence. When $-1/2 \geq \lambda > -1$, $\{\mathbf{m}_n\}_{n=0}^{\infty}$ is also the maximal parameter sequence of $\{d_{n+1}\}_{n=1}^{\infty}$, which makes it a SPPCS. However, when $\lambda > -1/2$ then $\{d_{n+1}\}_{n=1}^{\infty}$ is not a SPPCS and its maximal parameter sequence $\{M_{n+1}\}_{n=0}^{\infty}$ is such that

$$M_{n+1} = \frac{2\lambda + n + 1}{2(\lambda + n + 1)}, \quad n \geq 0.$$

The coefficients d_{n+1} , $n \geq 1$ are the same coefficients occurring in the recurrence formula for ultraspherical (or Gegenbauer) polynomials.

Further, for $\lambda > -1/2$ and $0 \leq t < 1$, if $\{m_n^{(t)}\}_{n=0}^{\infty}$ is the minimal parameter sequence of the positive chain sequence $\{d_n\}_{n=1}^{\infty}$, obtained by adding $d_1 = (1-t)M_1$ to $\{d_{n+1}\}_{n=1}^{\infty}$, then from (1.7)

$$\Phi_n^{(t)}(z) = R_n(z) - 2(1 - m_n^{(t)})R_{n-1}(z), \quad n \geq 1$$

and are the monic OPUC with respect to the measure $\mu^{(t)}(z)$, where $\mu^{(t)}(z)$ is as defined by (1.5). To find $\mu^{(t)}(z)$, we first find the measure $\mu^{(0)}(z)$ arising when $\{d_n\}_{n=1}^{\infty}$ becomes a SPPCS ($t = 0$). As shown in [24], the monic OPUC are given by

$$\Phi_n^{(0)}(z) = R_n(z) - 2(1 - M_n)R_{n-1}(z) = \frac{(2\lambda + 1)_n}{(\lambda + 1)_n} F(-n, \lambda + 1; 2\lambda + 1; 1 - z), \quad n \geq 1.$$

Using the identity (4.1), the Verblunsky coefficients are given by

$$\alpha_{n-1}^{(0)} = -\Phi_n^{(0)}(0) = -\frac{(\lambda)_n}{(\lambda + 1)_n}, \quad n \geq 1. \quad (4.4)$$

The Verblunsky coefficients $\alpha_{n-1}^{(0)}$ are associated with the non-trivial probability measure given by [24]

$$d\mu^{(0)}(e^{i\theta}) = \tau^{(\lambda)} \sin^{2\lambda}(\theta/2) d\theta,$$

where

$$\tau^{(\lambda)} = \frac{|\Gamma(1 + \lambda)|^2}{\Gamma(2\lambda + 1)} 4^\lambda.$$

Hence

$$\int_{\partial\mathbb{D}} f(\zeta) d\mu^{(t)}(\zeta) = (1-t)\tau^{(\lambda)} \int_0^{2\pi} f(e^{i\theta}) \sin^{2\lambda}(\theta/2) d\theta + tf(1).$$

Further characterization of Szegő polynomials is provided below as it is not possible to find closed form expressions for the coefficients of the para-orthogonal polynomials and Szegő polynomials. Since $\{R_n(z)\}$, depends on the parameter $b (= \lambda + 1)$, in what follows, we denote $R_n(z)$ by $R_n^{(b)}(z)$. We also denote c_n and d_n by $c_n^{(b)}$ and $d_n^{(b)}$ respectively. Now, note that if

$$Q_n^{(b)}(z) = \frac{1}{2(1-t)M_1} \int_{\mathbb{T}} \frac{R_n^{(b)}(z) - R_n^{(b)}(\zeta)}{z - \zeta} (1 - \zeta) d\mu^{(t)}(\zeta), \quad n \geq 0,$$

then $\{Q_n^{(b)}(z)\}_{n=0}^\infty$ satisfies

$$Q_{n+1}^{(b)}(z) = [(1 + ic_{n+1}^{(b)})z + (1 - ic_{n+1}^{(b)})]Q_n^{(b)}(z) - 4d_{n+1}^{(b)}zQ_{n-1}^{(b)}(z), \quad n \geq 1,$$

with $Q_0^{(b)}(z) = 0$ and $Q_1^{(b)}(z) = 1$. That is, the three term recurrence for $\{Q_n^{(b)}(z)\}_{n=0}^\infty$ is the same as for $\{R_n^{(b)}(z)\}_{n=0}^\infty$, with the difference being only on the initial conditions. The polynomials $\{Q_n^{(b)}(z)\}$ are generally called the numerator polynomials associated with $\{R_n^{(b)}(z)\}$. Further, observe that the three term recurrence for $\{Q_n^{(b)}(z)\}_{n=0}^\infty$ can also be given in the shifted form

$$Q_{n+2}^{(b)}(z) = [(1 + ic_{n+2}^{(b)})z + (1 - ic_{n+2}^{(b)})]Q_{n+1}^{(b)}(z) - 4d_{n+2}^{(b)}zQ_n^{(b)}(z), \quad n \geq 1, \quad (4.5)$$

with $Q_1^{(b)}(z) = 1$ and $Q_2^{(b)}(z) = (1 + ic_2^{(b)})z + (1 - ic_2^{(b)})$.

Consider now the parameter sequence given by $k_n^{(t)} = 1 - m_n^{(0)} = n/[2(\lambda + n)]$ for $n \geq 1$. For sake of clarity, we would like to note that t need not be necessarily 0. It depends on whether the resulting chain sequence for $\{k_n^{(t)}\}$, given by

$$a_1^{(b)} = \frac{1}{2\lambda + 2} \quad \text{and} \quad a_{n+1}^{(b)} = \frac{1}{4} \frac{(n+1)(2\lambda + n)}{(\lambda + n)(\lambda + n + 1)}, \quad n \geq 1, \quad (4.6)$$

is a SPPCS or not.

Let $\nu^{(t)}(z)$ be the measure associated with the Verblunsky coefficients $\{\beta_{n-1}^{(t)}\}_{n=1}^\infty$ given by

$$\beta_{n-1}^{(t)} = \bar{\tau}_n \left[\frac{1 - 2k_n^{(t)} - ic_n^{(b)}}{1 + ic_n^{(b)}} \right], \quad n \geq 1.$$

Following Theorem 2.6, the corresponding OPUC are

$$\tilde{\Phi}_n^{(t)}(z) = \frac{\tilde{R}_n^{(b)}(z) - 2(1 - k_n^{(t)})\tilde{R}_{n-1}^{(b)}(z)}{\prod_{k=1}^n (1 + ic_k^{(b)}), \quad n \geq 1,$$

where the polynomials $\tilde{R}_n^{(b)}$ are given by

$$\tilde{R}_{n+1}^{(b)}(z) = [(1 + ic_{n+1}^{(b)})z + (1 - ic_{n+1}^{(b)})]\tilde{R}_n^{(b)}(z) - 4a_{n+1}^{(b)}z\tilde{R}_{n-1}^{(b)}(z), \quad n \geq 1, \quad (4.7)$$

with $\tilde{R}_0^{(b)}(z) = 1$ and $\tilde{R}_1^{(b)}(z) = (1 + ic_1^{(b)})z + (1 - ic_1^{(b)})$. Observing that $c_n^{(b)} = c_{n+1}^{(b-1)}$, $a_{n+1}^{(b)} = d_{n+2}^{(b-1)}$, $n \geq 1$, we have from (4.5) and (4.7)

$$\tilde{R}_n^{(b)}(z) = Q_{n+1}^{(b-1)}(z), \quad n \geq 0,$$

and thus

$$\tilde{\Phi}_n^{(t)}(z) = \frac{Q_{n+1}^{(b-1)}(z) - 2(1 - k_n^{(t)})Q_n^{(b-1)}(z)}{\prod_{k=1}^n (1 + ic_{k+1}^{(b-1)}), \quad n \geq 1.$$

That is, if $R_n^{(b)}(z)$ generates the OPUC $\Phi_n^{(t)}(z)$, $Q_n^{(b-1)}(z)$, which are the numerator polynomials for $R_n^{(b-1)}(z)$ generates the OPUC $\tilde{\Phi}_n^{(t)}(z)$ associated with the complementary chain sequences. We note that, in the present case too, $c_n^{(b)} (= c_n) = 0$, $n \geq 1$ and so by Theorem 2.6 $\beta_{n-1}^{(t)} = -\alpha_{n-1}^{(0)}$ for $n \geq 1$. Hence $d\nu^{(t)}(z)$ are the Aleksandrov measures associated with $d\mu^{(0)}(z)$ [22].

Further, we note that such Szegő polynomials result from perturbations of the Verblunsky coefficients obtained in Section 3. Indeed, for $\sigma = \lambda/(1 + \lambda)$, $\{\lambda\delta_n\}$ corresponds to the Verblunsky coefficients given by (4.4), whereas by Verblunsky theorem, $\{\lambda\gamma_n\}$ corresponds to those given by the complementary chain sequence $\{a_{n+1}^{(b)}\}$ given by (4.6). Here $\{\delta_n\}$ and $\{\gamma_n\}$ are the ones chosen respectively by (3.10) and (3.13).

Further, when $\{a_{n+1}^{(b)}\}_{n=1}^{\infty}$ is the constant chain sequence $\{1/4\}$, $\tilde{R}_n^{(b)}(z)$ are the palindromic polynomials given by

$$\tilde{R}_n^{(b)}(z) = z^n + \nu^{(\lambda)}(z^{n-1} + \dots + z) + 1, \quad n \geq 1,$$

where $\nu^{(\lambda)}$ is a constant depending on λ . Here we study the cases $\lambda = 0$ and $\lambda = 1$ for which the complementary chain sequence $a_{n+1}^{(b)} = 1/4$.

Case 1, $\lambda = 0$. Let

$$\tilde{R}_n^{(b)}(z) = z^n + \nu^{(0)}(z^{n-1} + \dots + z) + 1, \quad n \geq 1.$$

The complementary chain sequence is $\{1/2, 1/4, 1/4, \dots\}$ which is known to be a SPPCS. Hence $\{k_n^{(t)}\}_{n=0}^{\infty}$ where $k_0^{(t)} = 0$, $k_n^{(t)} = 1/2$, $n \geq 1$ is also the maximal parameter sequence implying that $t = 0$ and so

$$\tilde{\Phi}_n^{(0)}(z) = z^n + (\nu^{(0)} - 1)z^{n-1}.$$

For $\nu^{(0)} = 1$, $\tilde{\Phi}_n^{(0)}(z) = z^n$ and from Remark 3.4, $\lambda = 0$ can be viewed as the limiting case for the Verblunsky coefficients obtained in Section 3. Note that the Verblunsky coefficients are 0, as can be verified from (4.4).

Case 2, $\lambda = 1$. Let

$$\tilde{R}_n^{(b)}(z) = z^n + \nu^{(1)}(z^{n-1} + \dots + z) + 1, \quad n \geq 1.$$

The complementary chain sequence is $\{1/4, 1/4, 1/4, \dots\}$ and $k_0^{(t)} = 0$, $k_n^{(t)} = n/2(n+1)$, $n \geq 1$. In this case, $t = 1/2$ and

$$\tilde{\Phi}_n^{(1/2)}(z) = z^n + \left(\nu^{(1)} - \frac{n+2}{n+1}\right)z^{n-1} - \frac{\nu^{(1)}}{n+1}(z^{n-2} + \dots + z) - \frac{1}{n+1}, \quad n \geq 1,$$

so that the Verblunsky coefficients are given by $1/(n+1)$. Again it can be verified from (4.4) that the Verblunsky coefficients corresponding to $\lambda = 1$ are $(1)_n/(2)_n = 1/(n+1)$. Finally, for $\nu^{(1)} = 0$, $\tilde{R}_n^{(b)} = z^n + 1$, which has been considered as Example 1 in [2].

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