

Continuous Choreographies as Limiting Solutions of N -body Type Problems with Weak Interaction

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Abstract. We consider the limit $N \rightarrow +\infty$ of N -body type problems with weak interaction, equal masses and $-\sigma$ -homogeneous potential, $0 < \sigma < 1$. We obtain the integro-differential equation that the motions must satisfy, with limit choreographic solutions corresponding to travelling waves of this equation. Such equation is the Euler–Lagrange equation of a corresponding limiting action functional. Our main result is that the circle is the absolute minimizer of the action functional among zero mean (travelling wave) loops of class H^1 .

Key words: N -body problem; continuous coreography; Lagrangian action

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1 Introduction

It is well known that the planar regular N -gon relative equilibrium is a solution of the equations of motion for the Newtonian N -body problem with $N \geq 3$ and equal mass bodies [1, 3, 6]. We consider N -body type problems with weak interaction, equal masses in \mathbb{R}^d , $d \geq 2$, and $-\sigma$ -homogeneous potential, $0 < \sigma < 1$, and study the resulting equation of motion when $N \rightarrow +\infty$, which we will refer to as *continuous system*, as well as a particular type of solutions which we call *continuous choreographies* using a variational approach. We can roughly say that a continuous choreography is a limiting configuration of classical choreographies when the number of particles grows without limit and indeed the circle turns out to be the continuous choreography associated to regular N -gons on the plane. G. Buck [2] considered curves which are locally approximated by solutions of the N -body problem. There is an essential difficulty in trying to study a continuous curve of particles interacting under the Newtonian law, since the combined force exerted by the neighbours blows up, except at points where the curvature vanishes. It seems that overcome this difficulty is impossible. Therefore we confined ourselves to consider weak interaction.

The paper is structured as follows: In the introduction we recall the variational approach of the N -body problem, as well as the particular type of solutions that we consider in this paper, the choreographic solutions. In Section 2 we obtain the limit equation of the N -body problem when $N \rightarrow +\infty$ and look at the particular case of travelling wave type continuous distributions of masses. Then we consider the limiting action functional and its critical points. In Section 3 we first show that the circle is a continuous choreography for $0 < \sigma < 1$, and then prove our main result that it is in fact the absolute minimizer of the action functional among zero mean loops of class H^1 with period 1. This is the continuous distributions of masses counterpart of Theorem 1 in [1] according to which, under the choreographic restriction, the Lagrangian action attains its absolute minimum at the planar regular N -gon relative equilibrium.

1.1 Choreographies for the N -body problem

The N -body problem with homogeneous potential of degree $\sigma \in (-1, 0)$, is the study of the dynamics of N positive mass points m_1, \dots, m_N moving in \mathbb{R}^d , $d \geq 2$, interacting according to equations

$$m_i \ddot{q}_i = - \sum_{\substack{j=1, \dots, N \\ j \neq i}} \sigma m_i m_j \frac{q_i - q_j}{\|q_i - q_j\|^{2+\sigma}}, \quad i = 1, \dots, N, \quad (1.1)$$

where $q_i(t) \in \mathbb{R}^d$ denotes the position of m_i at time t . The case $\sigma = 1$ is the *Newtonian N -body problem*.

Equivalently, we have the second-order equation

$$M \ddot{q} = \nabla U(q), \quad q = (q_1, \dots, q_N), \quad (1.2)$$

where $U: \mathbb{R}^{d \times N} \rightarrow [0, \infty]$ given by

$$U(q) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{\|q_i - q_j\|^\sigma},$$

is the *potential function* (the negative of the *potential energy*), ∇ is the *gradient* in $\mathbb{R}^{d \times N}$, $M = \text{diag}[m_1 I_d, \dots, m_N I_d]$.

Since the *center of mass* describes an uniform motion, we can fix it at the origin, and take

$$\mathcal{X} = \left\{ q \in \mathbb{R}^{d \times N} \mid \sum_{i=1}^N m_i q_i = 0 \right\}$$

as configuration space for (1.2).

According to the variational approach, solving (1.1) is equivalent to seeking the critical points of the Lagrangian action

$$A^\sigma: H^1([0, 1], \mathcal{X}) \rightarrow [0, +\infty], \quad A^\sigma(q) = \int_0^1 L^\sigma(q(t), \dot{q}(t)) dt, \quad (1.3)$$

where $L^\sigma: T\mathcal{X} \rightarrow [0, +\infty]$ is defined on the *tangent bundle*

$$T\mathcal{X} = \{(q, v) \mid q \in \mathcal{X}, v \in \mathbb{R}^{d \times N}\}$$

as

$$L^\sigma(q, v) := K(v) + U(q) = \frac{1}{2} \sum_{i=1}^N m_i \|v_i\|^2 + \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{\|q_i - q_j\|^\sigma}.$$

Equations (1.1) are the *Euler–Lagrange equations* associated to the functional (1.3).

There is an important class of solutions of (1.1) known as *choreographies*. A *choreography* of (1.1) is a periodic solution $q(t)$ whose orbit is the union of closed curves, each of these is the trajectory of at least two bodies. If the solution consists of only one closed curve, then we call this solution a *simple choreography*.

Definition 1.1 (see [4, 5, 8]). We say that a solution of the N -body problem (1.1) is a simple choreography if it is T -periodic and all bodies move on the same curve, interchanging their mutual positions after a time fixed, $\tau = T/N$, that is, there exists a function $x: \mathbb{R} \rightarrow \mathbb{R}^d$ such that:

$$q_i(t) = x(t + (i - 1)\tau), \quad i = 1, \dots, N, \quad t \in \mathbb{R}.$$

An example of a simple choreography is the relative equilibrium associated to Lagrange's equilateral triangle and this fact was extended in [6] to the case of N equal masses. For details on choreographies we refer the reader to [8] and the references therein.

2 Continuous choreographies as the limit $N \rightarrow \infty$ of N -body type problems

2.1 The equation of motion

Consider the N -body problem (1.1) with equal masses ($m_1 = \dots = m_N = 1/N$) on \mathbb{R}^d and $0 < \sigma < 1$

$$\ddot{q}_i(t) = - \sum_{\substack{j=1, \dots, N \\ j \neq i}} \frac{\sigma}{N} \frac{q_i(t) - q_j(t)}{\|q_i(t) - q_j(t)\|^{2+\sigma}}, \quad i = 1, \dots, N. \quad (2.1)$$

Consider $q: [0, 1] \times [a, b] \rightarrow \mathbb{R}^d$, periodic on the first variable and twice differentiable on the second variable. Let $\Delta s = 1/N$ and suppose that the position of the i -th mass at time t is given by

$$q_i(t) = q((i-1)\Delta s, t),$$

then equation (2.1) becomes

$$\ddot{q}_i(t) = - \sum_{\substack{j=1, \dots, N \\ j \neq i}} \sigma \frac{q((i-1)\Delta s, t) - q((j-1)\Delta s, t)}{\|q((i-1)\Delta s, t) - q((j-1)\Delta s, t)\|^{2+\sigma}} \Delta s, \quad i = 1, \dots, N. \quad (2.2)$$

So when $N \rightarrow +\infty$, $\Delta s \rightarrow 0$, the limit of the right-hand side of equation (2.2) can be written as a Cauchy principal value

$$\frac{\partial^2 q}{\partial t^2}(s, t) = - \lim_{\delta \rightarrow 0} \int_{s+\delta}^{1+s-\delta} \sigma \frac{q(s, t) - q(r, t)}{\|q(s, t) - q(r, t)\|^{2+\sigma}} dr. \quad (2.3)$$

A natural way to make the positions $q_i(t)$ satisfy the choreography condition is to take $q(s, t)$ of travelling wave type, that is

$$q(s, t) = y(s - vt), \quad (2.4)$$

where y is a 1-periodic function in $C^2(\mathbb{R}, \mathbb{R}^d)$.

Using expression (2.4) in equation (2.3), we obtain that y must satisfy

$$v^2 \ddot{y}(s) = - \lim_{\delta \rightarrow 0} \int_{s+\delta}^{1+s-\delta} \sigma \frac{y(s) - y(r)}{\|y(s) - y(r)\|^{2+\sigma}} dr, \quad s \in \mathbb{R}. \quad (2.5)$$

We denote by $C_1^2(\mathbb{R}, \mathbb{R}^d)$ the set of 1-periodic functions in $C^2(\mathbb{R}, \mathbb{R}^d)$ and by $H_1^1(\mathbb{R}, \mathbb{R}^d)$ the set of 1-periodic functions whose restriction to $[0, 1]$ is in $H^1([0, 1], \mathbb{R}^d)$.

Claim 2.1. *Assume $y \in C_1^2(\mathbb{R}, \mathbb{R}^d)$ defines a regular simple closed curve. Then the right-hand side of (2.5) is well defined.*

Proof. There is a continuous function $g: \mathbb{R} \times [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}^d$ such that

$$y(s+t) = y(s) + \dot{y}(s)t + g(s, t)t^2.$$

Since y is regular, for s fixed, the function

$$F(t) = \frac{(y(s+t) - y(s))|t|^{2+\sigma}}{t\|y(s+t) - y(s)\|^{2+\sigma}} = \frac{\dot{y}(s) + tg(s, t)}{\|\dot{y}(s) + tg(s, t)\|^{2+\sigma}}$$

can be considered as a continuous function on $[-\varepsilon, \varepsilon]$ with $F(0) = \dot{y}(s)\|\dot{y}(s)\|^{-2-\sigma}$ and differentiable at 0. Thus

$$G(t) = \left(\frac{y(s+t) - y(s)}{\|y(s+t) - y(s)\|^{2+\sigma}} - \frac{\dot{y}(s)t}{\|\dot{y}(s)t\|^{2+\sigma}} \right) |t|^\sigma = \frac{F(t) - F(0)}{t}$$

is bounded on $[-\varepsilon, \varepsilon] - \{0\}$ and then

$$\lim_{\delta \rightarrow 0} \int_{-\varepsilon}^{-\delta} \frac{G(t)dt}{|t|^\sigma} + \int_{\delta}^{\varepsilon} \frac{G(t)dt}{|t|^\sigma}$$

exists. Since $t \mapsto t|t|^{-2-\sigma}$ is an odd function we have

$$\int_{-\varepsilon}^{-\delta} \frac{tdt}{|t|^{2+\sigma}} + \int_{\delta}^{\varepsilon} \frac{tdt}{|t|^{2+\sigma}} = 0,$$

and then

$$\lim_{\delta \rightarrow 0} \left(\int_{s-\varepsilon}^{s-\delta} + \int_{s+\delta}^{s+\varepsilon} \right) \frac{(y(r) - y(s))dr}{\|y(s) - y(r)\|^{2+\sigma}} = \lim_{\delta \rightarrow 0} \int_{-\varepsilon}^{-\delta} \frac{G(t)dt}{|t|^\sigma} + \int_{\delta}^{\varepsilon} \frac{G(t)dt}{|t|^\sigma}. \quad \blacksquare$$

2.2 Variational approach to continuous choreographies (2.5)

Consider the action functional

$$\mathcal{A}^\sigma: \Lambda \rightarrow [0, +\infty]$$

given by

$$\mathcal{A}^\sigma(y) = \int_0^1 \frac{v^2}{2} \|\dot{y}(s)\|^2 ds + \frac{1}{2} \int_0^1 \int_0^1 \frac{dr ds}{\|y(s) - y(r)\|^\sigma}, \quad (2.6)$$

where

$$\Lambda := \left\{ y \in H_1^1(\mathbb{R}, \mathbb{R}^d) \mid \int_0^1 y(s) ds = 0 \right\}.$$

For $y \in \Lambda$, we have that $\|y\|_{L^2[0,1]} \leq \|\dot{y}\|_{L^2[0,1]}$ and therefore $\|y\|_{H^1[0,1]}$ is equivalent to $\|y\|_{L^2[0,1]}$.

Proposition 2.2. *Assume $y \in C_1^2(\mathbb{R}, \mathbb{R}^d)$ defines a regular simple closed curve. Then y is an extremal of the functional (2.6) if and only if it satisfies (2.5).*

Proof. For $y \in C_1^2(\mathbb{R}, \mathbb{R}^d)$ and any $z \in C_1^2(\mathbb{R}, \mathbb{R}^d)$ we have

$$\begin{aligned} \frac{d\mathcal{A}^\sigma(y + \varepsilon z)}{d\varepsilon} \Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon} \left\{ \int_0^1 \frac{v^2}{2} \|\dot{y}(s) + \varepsilon \dot{z}(s)\|^2 ds \right\}_{\varepsilon=0} \\ &\quad + \frac{1}{2} \frac{d}{d\varepsilon} \left\{ \int_0^1 \int_0^1 \frac{dr ds}{\|y(s) - y(r) + \varepsilon(z(s) - z(r))\|^\sigma} \right\}_{\varepsilon=0}. \end{aligned} \quad (2.7)$$

Differentiating and then integrating by parts the first term on the right-hand side of (2.7) we obtain

$$\frac{d}{d\varepsilon} \left\{ \int_0^1 \frac{v^2}{2} \|\dot{y}(s) + \varepsilon \dot{z}(s)\|^2 ds \right\}_{\varepsilon=0} = \int_0^1 -v^2 \ddot{y}(s) \cdot z(s) ds.$$

We now consider the second term on the right-hand side of (2.7). For $\delta > 0$ small

$$\begin{aligned} & \int_0^1 \int_0^1 \frac{drds}{\|y(s) - y(r) + \varepsilon(z(s) - z(r))\|^\sigma} \\ &= \int_0^1 \int_{s-\delta}^{1+s-\delta} \frac{drds}{\|y(s) - y(r) + \varepsilon(z(s) - z(r))\|^\sigma} \\ &= \int_0^1 \left(\int_{s+\delta}^{1+s-\delta} + \int_{s-\delta}^{s+\delta} \right) \frac{drds}{\|y(s) - y(r) + \varepsilon(z(s) - z(r))\|^\sigma}. \end{aligned}$$

We have

$$\begin{aligned} & \frac{1}{2} \frac{d}{d\varepsilon} \left\{ \int_0^1 \int_{s+\delta}^{1+s-\delta} \frac{drds}{\|y(r) - y(s) + \varepsilon(z(r) - z(s))\|^\sigma} \right\}_{\varepsilon=0} \\ &= -\frac{1}{2} \int_0^1 \int_{s+\delta}^{1+s-\delta} \frac{\sigma(y(s) - y(r)) \cdot (z(s) - z(r))}{\|y(s) - y(r)\|^{2+\sigma}} drds \\ &= -\frac{1}{2} \int_0^{1-\delta} \int_{s+\delta}^{1-\max(0, \delta-s)} \frac{\sigma(y(s) - y(r)) \cdot (z(s) - z(r))}{\|y(s) - y(r)\|^{2+\sigma}} drds \\ &\quad - \frac{1}{2} \int_0^{1-\delta} \int_{r+\delta}^{1-\max(0, \delta-r)} \frac{\sigma(y(s) - y(r)) \cdot (z(s) - z(r))}{\|y(s) - y(r)\|^{2+\sigma}} dsdr \\ &= -\int_0^1 \int_{s+\delta}^{1+s-\delta} \frac{\sigma(y(s) - y(r)) \cdot z(s)}{\|y(s) - y(r)\|^{2+\sigma}} drds. \end{aligned}$$

There are continuous functions $g, h: \mathbb{R} \times [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}^d$ such that

$$y(t+s) = y(s) + \dot{y}(s)t + g(s, t)t^2, \quad z(t+s) = z(s) + \dot{z}(s)t + h(s, t)t^2.$$

Since y is regular, $|\dot{y}(s)| \geq \beta > 0$, for $\delta, \varepsilon > 0$ small we have that

$$|t| \leq \delta, \quad |u| \leq \varepsilon \Rightarrow |\dot{y}(s) + u\dot{z}(s) + (g(s, t) + uh(s, t))t| \geq \frac{\beta}{2}.$$

The fundamental theorem of calculus gives

$$\begin{aligned} & \left| \int_{s-\delta}^{s+\delta} \frac{dr}{\|y(s) - y(r) + \varepsilon(z(s) - z(r))\|^\sigma} - \frac{dr}{\|y(s) - y(r)\|^\sigma} \right| \\ &= \left| \int_{s-\delta}^{s+\delta} \int_0^\varepsilon \sigma \frac{(y(s) - y(r) + u(z(s) - z(r))) \cdot (z(s) - z(r))}{\|y(s) - y(r) + u(z(s) - z(r))\|^{2+\sigma}} du dr \right| \\ &= \left| \int_{-\delta}^\delta \int_0^\varepsilon \sigma \frac{(\dot{y}(s) + u\dot{z}(s) + (g(s, t) + uh(s, t))t) \cdot (\dot{z}(s) + h(s, t)t)}{\|\dot{y}(s) + u\dot{z}(s) + (g(s, t) + uh(s, t))t\|^{2+\sigma} |t|^\sigma} du dt \right| \\ &\leq C \int_{-\delta}^\delta \int_0^\varepsilon |t|^{-\sigma} du dt = \frac{2C}{1-\sigma} \varepsilon \delta^{1-\sigma} \end{aligned}$$

with C constant. Thus, for $\delta > 0$ small we have

$$\begin{aligned} \frac{d\mathcal{A}^\sigma(y + \varepsilon z)}{d\varepsilon} \Big|_{\varepsilon=0} &= \int_0^1 -v^2 \ddot{y}(s) \cdot z(s) ds + O(\delta^{1-\sigma}) \\ &\quad - \int_0^1 \int_{s+\delta}^{1+s-\delta} \frac{\sigma(y(s) - y(r)) \cdot z(s)}{\|y(s) - y(r)\|^{2+\sigma}} drds, \end{aligned}$$

and then

$$\frac{d\mathcal{A}^\sigma(y + \varepsilon z)}{d\varepsilon} \Big|_{\varepsilon=0} = -\int_0^1 \left[v^2 \ddot{y}(s) + \lim_{\delta \rightarrow 0} \int_{s+\delta}^{1+s-\delta} \frac{\sigma(y(s) - y(r))}{\|y(s) - y(r)\|^{2+\sigma}} dr \right] \cdot z(s) ds.$$

Therefore, y satisfies the condition

$$\left. \frac{d\mathcal{A}^\sigma(y + \varepsilon z)}{d\varepsilon} \right|_{\varepsilon=0} = 0$$

for any $z \in C_1^2(\mathbb{R}, \mathbb{R}^d)$, if and only if it satisfies (2.5). \blacksquare

3 Circular choreography as minimizer of the action

In this section we consider circular planar curves as solutions of equation (2.5) and in fact as absolute minimizers of the action functional. This is motivated by papers [6, 7], which prove that the planar regular N -gon relative equilibrium is a solution of the N -body problem, and mainly by Theorem 1 in [1]. We naturally follow the ideas from that paper.

Proposition 3.1. *The function given by $x(s) := e^{2\pi is}$ is a solution of (2.5) on the plane $\mathbb{R}^2 \cong \mathbb{C}$ if and only if*

$$v^2 = \frac{\sigma}{4\pi^2} \lim_{\delta \rightarrow 0} \int_{\delta}^{1-\delta} \frac{1 - e^{2\pi it}}{|1 - e^{2\pi it}|^{2+\sigma}} dt = \frac{\sigma}{8\pi^2} \int_0^1 \frac{dt}{(2 \sin(\pi t))^\sigma}. \quad (3.1)$$

Proof. We have

$$\begin{aligned} \int_{s+\delta}^{1+s-\delta} \frac{e^{2\pi is} - e^{2\pi ir}}{|e^{2\pi is} - e^{2\pi ir}|^{2+\sigma}} dr &= e^{2\pi is} \int_{s+\delta}^{1+s-\delta} \frac{1 - e^{2\pi i(r-s)}}{|1 - e^{2\pi i(r-s)}|^{2+\sigma}} dr \\ &= e^{2\pi is} \int_{\delta}^{1-\delta} \frac{1 - e^{2\pi it}}{|1 - e^{2\pi it}|^{2+\sigma}} dt, \end{aligned}$$

and

$$\ddot{x}(s) = -4\pi^2 e^{2\pi is}.$$

Thus, (2.5) is equivalent to (3.1) \blacksquare

Proposition 3.2. *For $\beta > 0$ let $\mu \in C((0, 1), \mathbb{R}^+) \cap L^{\beta/\beta+1}(0, 1)$. For any $\xi: (0, 1) \rightarrow \mathbb{R}^+$ we have*

$$\left(\int_0^1 \mu^{\beta/\beta+1} \right)^{\beta+1} \leq \left(\int_0^1 \mu \xi \right)^\beta \int_0^1 \xi^{-\beta},$$

and equality holds if and only if $\mu \xi^{\beta+1}$ is constant.

Proof. By Hölder's inequality

$$\int_0^1 \mu^{\beta/\beta+1} = \int_0^1 (\mu \xi)^{\beta/\beta+1} \xi^{-\beta/\beta+1} \leq \left(\int_0^1 \mu \xi \right)^{\beta/\beta+1} \left(\int_0^1 \xi^{-\beta} \right)^{1/\beta+1},$$

and equality holds if and only if $\mu \xi^{\beta+1}$ is constant. \blacksquare

Henceforth we let v be given by (3.1). Define

$$\begin{aligned} \xi_y(t) &= \int_0^1 \|y(s+t) - y(s)\|^2 ds, & \hat{\xi}(t) &= |1 - e^{2\pi it}|^2, \\ c &= \int_0^1 \hat{\xi}^{-\sigma/2} dt = \int_0^1 \frac{dt}{(2 \sin(\pi t))^\sigma} = \frac{8\pi^2 v^2}{\sigma}, & \mu &= \frac{\hat{\xi}^{-\sigma/2-1}}{c}. \end{aligned}$$

If y is twice differentiable, then the integral

$$\Delta^\mu y(t) = \int_0^1 \mu(s)(2y(t) - y(t+s) - y(t-s))ds = \int_0^1 \frac{2y(t) - y(t+s) - y(t-s)}{c(2\sin(\pi s))^{2+\sigma}} ds$$

converges since the numerator is $\mathcal{O}((s-b)^2)$ as $s \rightarrow b$ for $b = 0, 1$.

Note that Δ^μ is linear and for $f_k(t) = ae^{2k\pi it}$ we have

$$\Delta^\mu f_k(t) = ae^{2k\pi it} \int_0^1 \frac{2 - e^{2k\pi is} - e^{-2k\pi is}}{c(2\sin(\pi s))^{2+\sigma}} ds = d_k f_k(t),$$

where the integral

$$d_k = \int_0^1 \frac{4\sin^2(k\pi s)}{c(2\sin(\pi s))^{2+\sigma}} ds$$

converges since the integrand is $\mathcal{O}(|s-b|^{-\sigma})$ as $s \rightarrow b$ for $b = 0, 1$.

By Proposition 2 in [1], for $k \geq 2$, $u \in (0, \pi)$ we have $\sin^2(ku) < k^2 \sin^2 u$ and so $d_k < k^2 d_1$.

Remark 3.3. We write $a \in \mathbb{C}^d$ as $a = \operatorname{Re} a + i \operatorname{Im} a$ and $\bar{a} = \operatorname{Re} a - i \operatorname{Im} a$.

Proposition 3.4. For every $y \in \Lambda$

$$\int_0^1 \|\dot{y}\|^2 \geq 4\pi^2 \int_0^1 \mu \xi_y, \tag{3.2}$$

and the equality holds if and only if $y(t) = ae^{2\pi it} + \bar{a}e^{-2\pi it}$.

Proof.

$$\begin{aligned} \int_0^1 \Delta^\mu y \cdot y &= \int_0^1 \mu(s) \int_0^1 (2\|y(t)\|^2 - y(t) \cdot y(t+s) - y(t) \cdot y(t-s)) dt ds \\ &= \int_0^1 \mu(s) \int_0^1 (2\|y(t)\|^2 - 2y(t) \cdot y(t+s)) dt ds \\ &= \int_0^1 \mu(s) \int_0^1 \|y(t) - y(t+s)\|^2 dt ds = \int_0^1 \mu \xi_y. \end{aligned}$$

Observe that the inequality (3.2) holds true for y constant. We can then assume that ξ_y never vanishes. Our aim is to show that the functional

$$J(y) := \frac{\int_0^1 \|\dot{y}\|^2}{\int_0^1 \Delta^\mu y \cdot y}$$

defined on Λ attains its infimum and that its minimal value is $4\pi^2$. Since the functional J is homogeneous of degree zero, minimizing J is equivalent to minimizing the coercive functional $\tilde{J}(y) = \int_0^1 \|\dot{y}(t)\|^2 dt$ on the constraint

$$M = \left\{ y \in \Lambda \mid \int_0^1 \Delta^\mu y \cdot y dt = 1 \right\},$$

which is closed with respect to the weak H_1^1 -topology. Let $(y_n)_n$ be a minimizing sequence, then $\int_0^1 \|\dot{y}_n\|^2 dt = \|\dot{y}_n\|_{L^2}^2$ is bounded. By lower semi-continuity of the norm we can select a subsequence (y_{n_k}) , weakly convergent to $y \in M$ and obtain

$$\tilde{J}(y) \leq \liminf_{n_k} \tilde{J}(y_{n_k}).$$

So we can state that the minimum of \tilde{J} exist. The minimal value of \tilde{J} corresponds to the first eigenvalue λ_{\min} for the problem

$$-\ddot{y} = \lambda \Delta^\mu y, \quad y \in H_1^1(\mathbb{R}, \mathbb{R}^d), \quad \int_0^1 y = 0. \quad (3.3)$$

To study problem (3.3) write a solution as

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{2k\pi it}, \quad a_0 = 0, \quad a_{-k} = \bar{a}_k$$

to get

$$\ddot{y}(t) = - \sum_{k=-\infty}^{\infty} 4\pi k^2 a_k e^{2k\pi it}, \quad \Delta^\mu y(t) = \sum_{k=-\infty}^{\infty} a_k d_k e^{2k\pi it},$$

so that the eigenvalues of the problem (3.3) are $\lambda_k = 4\pi^2 k^2 / d_k$, $k \in \mathbb{N}$. Thus $\lambda_{\min} = \lambda_1 = 4\pi^2$ and the minimum is attained if and only if $y(t) = ae^{2\pi it} + \bar{a}e^{-2\pi it}$. ■

Theorem 3.5. *The absolute minimum of \mathcal{A}^σ on Λ , with $0 < \sigma < 1$ and v given by (3.1), is attained at and only at a unit circle*

$$y(t) = ae^{2\pi it} + \bar{a}e^{-2\pi it} = E_1 \cos(2\pi t) + E_2 \sin(2\pi t), \quad (3.4)$$

where $E_1 = 2 \operatorname{Re}(a)$, $E_2 = 2 \operatorname{Im}(a)$ are orthogonal unit vectors in \mathbb{R}^d .

Proof. By Jensen's inequality

$$\xi_y(t)^{-\sigma/2} = \left(\int_0^1 \|y(s+t) - y(s)\|^2 ds \right)^{-\sigma/2} \leq \int_0^1 \frac{ds}{\|y(s+t) - y(s)\|^\sigma},$$

and equality holds if and only if $\xi_y(t) = \|y(t) - y(0)\| = \|y(s+t) - y(s)\|$ for any s . Thus

$$\mathcal{A}^\sigma(y) \geq \int_0^1 \frac{v^2}{2} \|\dot{y}\|^2 + \frac{1}{2} \int_0^1 \xi_y^{-\sigma/2}. \quad (3.5)$$

Define

$$\tilde{\mathcal{A}}^\sigma(y) := \int_0^1 \frac{v^2}{2} \|\dot{y}\|^2 + \frac{4\pi^2 v^2}{\sigma} \left(\int_0^1 \mu \xi_y \right)^{-\sigma/2}.$$

By Proposition 3.2, the minimum of the functional

$$\Phi(\xi) = \left(\int_0^1 \mu \xi \right)^{\sigma/2} \int_0^1 \xi^{-\sigma/2}$$

is attained at ξ if and only if ξ is proportional to $\hat{\xi}$ and its value is $c = 8\pi^2 v^2 / \sigma$, which together with (3.5) gives

$$\mathcal{A}^\sigma(y) \geq \tilde{\mathcal{A}}^\sigma(y), \quad (3.6)$$

and equality in (3.6) holds at a circle

$$y(t) = ae^{2\pi it} + \bar{a}e^{-2\pi it} = E_1 \cos(2\pi t) + E_2 \sin(2\pi t),$$

where $E_1 = 2 \operatorname{Re}(a)$, $E_2 = 2 \operatorname{Im}(a)$ are orthogonal vectors in \mathbb{R}^d of the same length.

Define

$$\bar{\mathcal{A}}^\sigma(y) := 2\pi^2 v^2 \left(\int_0^1 \mu \xi_y + \frac{2}{\sigma} \left(\int_0^1 \mu \xi_y \right)^{-\sigma/2} \right).$$

By Proposition 3.4 we have

$$\tilde{\mathcal{A}}^\sigma(y) \geq \bar{\mathcal{A}}^\sigma(y), \tag{3.7}$$

and the equality in (3.7) holds if only if $y(t) = ae^{2\pi it} + \bar{a}e^{-2\pi it}$.

The function $g: \mathbb{R}^+ \rightarrow \mathbb{R}$, $g(u) = u + \frac{2}{\sigma}u^{-\sigma/2}$ has a unique minimum at $u = 1$, therefore $\bar{\mathcal{A}}^\sigma$ attains its absolute minimum $2\pi^2 v^2(1 + 2/\sigma)$ at functions $y \in \Lambda$ with $\int_0^1 \mu \xi_y = 1$, among others at a unit circle (3.4), and at a unit circle all \mathcal{A}^σ , $\tilde{\mathcal{A}}^\sigma$, $\bar{\mathcal{A}}^\sigma$ coincide. Conversely, if \mathcal{A}^σ attains its minimum at $y \in \Lambda$ then $\int_0^1 \mu \xi_y = 1$ and (3.6), (3.7) must be equalities. Thus $\xi_y(t) = \|y(t) - y(0)\| = \hat{\xi}(t)$ and $y(t) = ae^{2\pi it} + \bar{a}e^{-2\pi it}$. It is not difficult to see that $E_1 = 2 \operatorname{Re}(a)$, $E_2 = 2 \operatorname{Im}(a)$ are orthogonal unit vectors in \mathbb{R}^d . ■

4 Open problems

It is possible that imposing additional symmetries beyond the choreography condition, could give some other critical points of the action besides the absolute minimizers. One also could impose some topological non-trivial condition to the curve defining the continuous choreography. Connections with other topics such as vortex lines in fluid dynamics might also be of interest.

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