Hypergroups Related to a Pair of Compact Hypergroups

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Abstract. The purpose of the present paper is to investigate a hypergroup associated with irreducible characters of a compact hypergroup H and a closed subhypergroup H_0 of H with $|H/H_0| < +\infty$. The convolution of this hypergroup is introduced by inducing irreducible characters of H_0 to H and by restricting irreducible characters of H to H_0. The method of proof relies on the notion of an induced character and an admissible hypergroup pair.

Key words: hypergroup; induced character; semi-direct product hypergroup; admissible hypergroup pair

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1 Introduction

The aim of the present paper is to contribute to the largely open problem of establishing a structure theory of hypergroups. Hypergroups are locally compact spaces on which the bounded measures convolve similar to the group case. The origin of the notion of hypergroup or generalized translation structure goes back to J. Delsarte and B.M. Levitan, the special class of double coset hypergroups appears already in the work of G. Frobenius.

There exists an axiomatic approach to hypergroups initiated by Charles F. Dunkl [2, 3], R.I. Jewett [13], and R. Spector [15], which lead to an extensive harmonic analysis of hypergroups. For the historical background of the theory we just refer to R.I. Jewett's fundamental paper [13] and the monograph [1] by W.R. Bloom and H. Heyer. In fact, hypergroups arose in the theory of second order differential equations and developed to be of significant applicability in probability theory where the hypergroup convolution of measure reflects a stochastic operation in the basic space of the hypergroup. Nowadays hypergroup structures are studied within various frameworks from non-commutative duality of groups to quantum groups and bimodules.

Since every investigation of the structures of hypergroups is oriented on the search of new, probably large examples, aspects of a partial solution to the structure problem are extension of hypergroups [4, 6], a cohomology theory for hypergroups [7] and imprimitivity of representations of hypergroups [8]. There are interesting results on hypergroup structures arising from

dual objects of a hypergroup including the group case [5]. Recent research on the structure of hypergroups relies on the application of induced characters [11, 12], hyperfields [9] and compact hypergroup pairs [10]. At this point we can outline our new results.

Let H be a strong compact hypergroup satisfying the second axiom of countability, and let H_0 be a subhypergroup of H with $|H/H_0| < +\infty$. By $\mathbb{Z}_q(2)$ we denote the q-deformation of $\mathbb{Z}(2)$, and the hats on H and H_0 signify their duals. In [11] the notion of an induced character of a finite-dimensional representation of H was introduced and studied in detail. The results obtained in that paper enable us in the present work to discuss character hypergroups of the type $\mathcal{K}\big(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)\big)$ which generalize those introduced in [10]. The admissible *group* pair of [10] will now be replaced by an admissible *hypergroup* pair, and the hypergroup structure of $\mathcal{K}\big(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)\big)$ will be characterized by the hypergroup pair (H, H_0) (Theorem 3.8). Applications to semi-direct product hypergroups follow (Theorem 4.7), and a list of new hypergroups appears in Section 5.

2 Preliminaries

In order to facilitate the reader's access to the problem discussed in this paper we recapitulate the notion of a hypergroup and of a few often applied facts. Details of the theory of hypergroups and standard examples can be found in the seminal paper [13] of R.I. Jewett and in the monograph [1] of W.R. Bloom and H. Heyer.

For a given locally compact (Hausdorff) space X we denote by $C^b(X)$ the space of bounded continuous functions on X, and by $C_c(X)$ and $C_0(X)$ its subspaces of functions with compact support or of functions vanishing at infinity respectively. For each compact subset K of X let $C_K(X)$ be the subset of functions $f \in C_c(X)$ with $\mathrm{supp}(f) \subset K$. By $M(X)$ we denote the set of Radon measures on X defined as linear functionals on $C_c(X)$ whose restriction to each $C_K(X)$ is continuous with respect to the topology of uniform convergence. $M^b(X)$ symbolizes the set of bounded measures on X. In fact, $M^b(X)$ is the dual of the Banach space $C_0(X)$, and it is furnished with the norm

$$\mu \mapsto \|\mu\| := \sup\{|\mu(f)| \colon \|f\| \leq 1\}.$$

Moreover, we shall refer to the subspaces $M_c(X)$ and $M^1(X)$ of measures with compact support or probability measures on X respectively.

Finally, $M_c^1(X) := M^1(X) \cap M_c(X)$. We denote the Dirac measure in $x \in X$ by ε_x.

A *hypergroup* is a locally compact (Hausdorff) space H together with a weakly continuous associative and bilinear convolution $*$ in the Banach space $M^b(H)$ satisfying the following axioms:

(HG1) For all $x, y \in H$, $\varepsilon_x * \varepsilon_y$ belongs to $M_c^1(H)$.

(HG2) There exist a neutral element $e \in H$ such that

$$\varepsilon_x * \varepsilon_e = \varepsilon_e * \varepsilon_x = \varepsilon_x$$

for all $x \in H$, and a continuous involution

$$x \longmapsto x^-$$

in H satisfying

$$e \in \mathrm{supp}(\varepsilon_x * \varepsilon_y) \quad \text{if and only if} \quad y = x^-$$

as well as

$$(\varepsilon_x * \varepsilon_y)^- = \varepsilon_{y^-} * \varepsilon_{x^-}$$

whenever $x, y \in H$.

(HG3) The mapping

$$(x, y) \longmapsto \operatorname{supp}(\varepsilon_x * \varepsilon_y)$$

from $H \times H$ into the space of compact subsets of H equipped with the Michael topology is continuous.

As a consequence of the weak continuity and bilinearity the convolution of arbitrary bounded measures on H is uniquely determined by the convolution of Dirac measures. In other words

$$\mu * \nu = \int_H \int_H \varepsilon_x * \varepsilon_y \mu(dx)\nu(dy),$$

where $\mu, \nu \in M^b(H)$.

A hypergroup H is called *commutative* if its convolution is commutative. Clearly, locally compact groups are hypergroups. Also double coset spaces $G//L$ arising from Gelfand pairs (G, L) are (commutative) hypergroups. Given a hypergroup H one can introduce subhypergroups, quotient hypergroups, direct and semi-direct product hypergroups (for the latter notion see [8, 18]), and hypergroup joins.

Every compact hypergroup H has the normalized *Haar measure* $\omega_H \in M(H)$ which is invariant with respect to the translation

$$f \mapsto f_x, \qquad x \in H,$$

where

$$f_x(y) := \varepsilon_x * \varepsilon_y(f) = \int_H f(z)(\varepsilon_x * \varepsilon_y)(dz)$$

for all $y \in H$.

Let $(H, *)$ and $(L, \circ)$ be two hypergroups with convolutions $*$ and $\circ$ as well with neutral elements ε_H and ε_L respectively. A continuous mapping $\varphi\colon H \to L$ is called a *hypergroup homomorphism* if $\varphi(\varepsilon_H) = \varepsilon_L$ and if φ is the unique linear weakly continuous extension from $M^b(H)$ to $M^b(L)$ satisfying the following conditions:

(1) $\varphi(\varepsilon_x) = \varepsilon_{\varphi(x)}$,

(2) $\varphi(\varepsilon_x^-) = \varphi(\varepsilon_x)^-$,

(3) $\varphi(\varepsilon_x * \varepsilon_y) = \varepsilon_{\varphi(x)} \circ \varepsilon_{\varphi(y)}$,

whenever $x, y \in H$.

If, in addition, φ is a homeomorphism from H onto L, it is called an *isomorphism* from H onto L, and in the case $L = H$ it is called an *automorphism* of H. The set $\operatorname{Aut}(H)$ of all automorphisms of H becomes a topological group furnished with the weak topology of $M^b(H)$.

An *action* of a locally compact group G on a hypergroup H is a continuous homomorphism from G into $\operatorname{Aut}(H)$.

Given an action α of G on H there is the notion of a semi-direct product hypergroup $K = H \rtimes_\alpha G$ which in general is a non-commutative hypergroup, efficiently applied all over in our work.

Let H be a hypergroup, and let $\mathcal{H}$ be a (separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$. By $\mathcal{B}(\mathcal{H})$ we denote the Banach $*$-algebra of bounded linear operator on $\mathcal{H}$. A $*$-homomorphism $\pi\colon M^b(H) \longrightarrow \mathcal{B}(\mathcal{H})$ is called a *representation* of H if

$$\pi(\varepsilon_e) = 1$$

and for all $u, v \in \mathcal{H}$ the mapping

$$\mu \mapsto \langle \pi(\mu)u, v \rangle$$

is continuous on $M^b(H)$.

In the sequel we shall deal with classes of representations and of irreducible representations of H under unitary equivalence.

Now let H be a compact hypergroup with a countable basis of its topology. $\hat{H}$ will denote the set of all equivalence classes of irreducible representations of H. H is said to be of *strong type* if $\hat{H}$ carries a hypergroup structure. If H is commutative, more structure is available. In this case $\hat{H}$ consists of characters of H which are defined as nonvanishing functions $\chi \in C^b(H)$ satisfying the equality

$$(\varepsilon_x * \varepsilon_y^-)(\chi) = \chi(x)\overline{\chi}(y)$$

valid for all $x, y \in H$. Once $\hat{H}$ is a hypergroup, the double dual $\hat{\hat{H}}$ can be formed, and the identification $\hat{\hat{H}} \cong H$ defines *Pontryagin hypergroups*.

Returning to an arbitrary compact hypergroup H and a closed subhypergroup H_0 of H, for a representation π_0 of H_0 with representing Hilbert space $\mathcal{H}(\pi_0)$ one introduces the *representation*

$$\pi := \operatorname{ind}_{H_0}^H \pi_0$$

induced by π_0 from H_0 to H as follows:

$$\mathcal{H}(\pi) := \left\{ \xi \in L^2(H, \mathcal{H}(\pi_0))\colon (\varepsilon_{h_0} * \varepsilon_x)(\xi) = \pi_0(h_0)\xi(x) \text{ for all } h_0 \in H_0 \right\}$$

and

$$(\pi(h)\xi)(x) := (\varepsilon_x * \varepsilon_h)(\xi)$$

for all $\xi \in \mathcal{H}(\pi)$, $x, h \in H$.

For further details on induced representations, see [8, 11].

3   Hypergroups related to admissible pairs

Let H be a strong compact hypergroup which satisfies the second axiom of countability, and $\hat{H}$ its dual. Then

$$\mathcal{K}(\hat{H}) = \left\{ \operatorname{ch}(\pi)\colon \pi \in \hat{H} \right\}$$

is a countable discrete commutative hypergroup, where

$$\operatorname{ch}(\pi)(h) = \frac{1}{\dim \pi} \operatorname{tr}(\pi(h)).$$

for all $\pi \in \hat{H}$, $h \in H$.

Now, let H_0 be a subhypergroup of H which is assumed to be also of strong type and such that $|H/H_0| < +\infty$. For $\tau \in \widehat{H_0}$ the induced representation $\mathrm{ind}_{H_0}^H \tau$ of τ from H_0 to H is finite-dimensional and decomposes as

$$\mathrm{ind}_{H_0}^H \tau \cong \pi_1 \oplus \cdots \oplus \pi_m,$$

where $\pi_1, \ldots, \pi_m \in \hat{H}$ ($m \geq 1$). The *induced character* of $\mathrm{ch}(\tau)$ is defined as

$$\mathrm{ind}_{H_0}^H \mathrm{ch}(\tau) := \frac{d(\pi_1)}{d(\pi)} \mathrm{ch}(\pi_1) + \cdots + \frac{d(\pi_m)}{d(\pi)} \mathrm{ch}(\pi_m),$$

where $d(\pi_j)$ for $j = 1, \ldots, m$ is the hyperdimension of π_j in the sense of Vrem [17] and

$$d(\pi) := d(\pi_1) + \cdots + d(\pi_m).$$

For $\tau_i, \tau_j \in \widehat{H_0}$, $\mathrm{ch}(\tau_i)\,\mathrm{ch}(\tau_j) \in M_c^1\big(\mathcal{K}(\widehat{H_0})\big)$ such that

$$\mathrm{ch}(\tau_i)\,\mathrm{ch}(\tau_j) = a_1\,\mathrm{ch}(\tau_1) + \cdots + a_\ell\,\mathrm{ch}(\tau_\ell),$$

with $a_k > 0$ ($k = 1, \ldots, \ell$) and $a_1 + \cdots + a_\ell = 1$. Concerning characters induced from H_0 to H and the following definition see [11].

Definition 3.1. For $\tau_i, \tau_j \in \widehat{H_0}$,

$$\mathrm{ind}_{H_0}^H(\mathrm{ch}(\tau_i)\,\mathrm{ch}(\tau_j)) := a_1\,\mathrm{ind}_{H_0}^H \mathrm{ch}(\tau_1) + \cdots + a_\ell\,\mathrm{ind}_{H_0}^H \mathrm{ch}(\tau_\ell).$$

Our main objective of study will be formulated in the subsequent

Definition 3.2. Let $\mathbb{Z}_q(2)$ be a hypergroup of order 2 with parameter $q \in (0, 1]$. The *twisted convolution* $* = *_q$ on the space

$$\mathcal{K}\big(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)\big) := \big\{(\mathrm{ch}(\pi), \circ), (\mathrm{ch}(\tau), \bullet) \colon \pi \in \hat{H}, \tau \in \widehat{H_0}\big\},$$

associated with $\mathbb{Z}_q(2)$ is given as follows:

(1) $(\mathrm{ch}(\pi_i), \circ) * (\mathrm{ch}(\pi_j), \circ) := (\mathrm{ch}(\pi_i)\,\mathrm{ch}(\pi_j), \circ),$

(2) $(\mathrm{ch}(\pi), \circ) * (\mathrm{ch}(\tau), \bullet) := \big(\big(\mathrm{res}_{H_0}^H \mathrm{ch}(\pi)\big)\mathrm{ch}(\tau), \bullet\big),$

(3) $(\mathrm{ch}(\tau), \bullet) * (\mathrm{ch}(\pi), \circ) := \big(\mathrm{ch}(\tau)\big(\mathrm{res}_{H_0}^H \mathrm{ch}(\pi)\big), \bullet\big),$

(4) $(\mathrm{ch}(\tau_i), \bullet) * (\mathrm{ch}(\tau_j), \bullet) := q\big(\mathrm{ind}_{H_0}^H(\mathrm{ch}(\tau_i)\,\mathrm{ch}(\tau_j)), \circ\big) + (1 - q)(\mathrm{ch}(\tau_i)\,\mathrm{ch}(\tau_j), \bullet).$

For details on deformation of hypergroups see [14].

Definition 3.3. Let (H, H_0) be a pair of consisting of a compact hypergroup H and a closed subhypergroup H_0 of H. We call (H, H_0) an *admissible hypergroup pair* if the following conditions are satisfied:

(1) for $\pi \in \hat{H}$ and $\tau \in \widehat{H_0}$

$$\mathrm{ind}_{H_0}^H \big(\big(\mathrm{res}_{H_0}^H \mathrm{ch}(\pi)\big)\mathrm{ch}(\tau)\big) = \mathrm{ch}(\pi)\,\mathrm{ind}_{H_0}^H \mathrm{ch}(\tau),$$

(2) for $\tau \in \widehat{H_0}$

$$\mathrm{res}_{H_0}^H \big(\mathrm{ind}_{H_0}^H \mathrm{ch}(\tau)\big) = \mathrm{ch}(\tau)\,\mathrm{res}_{H_0}^H \big(\mathrm{ind}_{H_0}^H \mathrm{ch}(\tau_0)\big),$$

where τ_0 is the trivial representation of H_0.

Remark 3.4. If a pair (G, G_0) consisting of a compact group and a closed subgroup G_0 is admissible in the sense of [10], then it is an admissible hypergroup pair.

Lemma 3.5. *If (H, H_0) is an admissible hypergroup pair, the following formulae hold:*

(1) *for $\pi \in \hat{H}$ and $\tau_i, \tau_j \in \widehat{H_0}$*

$$\operatorname{ind}_{H_0}^H \left(\left(\operatorname{res}_{H_0}^H \operatorname{ch}(\pi) \right) \operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j) \right) = \operatorname{ch}(\pi) \operatorname{ind}_{H_0}^H (\operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j)),$$

(2) *for $\tau_i, \tau_j \in \widehat{H_0}$*

$$\operatorname{res}_{H_0}^H \left(\operatorname{ind}_{H_0}^H \operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j) \right) = \operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j) \operatorname{res}_{H_0}^H \left(\operatorname{ind}_{H_0}^H \operatorname{ch}(\tau_0) \right).$$

Proof. It is easy to see the desired formulae by the definition of $\operatorname{ind}_{H_0}^H (\operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j))$.   ■

Proposition 3.6. *If a strong compact hypergroup H together with a strong subhypergroup H_0 of H with $|H/H_0| < +\infty$ forms an admissible hypergroup pair, then the following associativity relations hold. For $\pi_i, \pi_j, \pi_k, \pi \in \hat{H}$ and $\tau_i, \tau_j, \tau_k, \tau \in \widehat{H_0}$:*

$(A1)$ $((\operatorname{ch}(\pi_i), \circ) * (\operatorname{ch}(\pi_j), \circ)) * (\operatorname{ch}(\pi_k), \circ) = (\operatorname{ch}(\pi_i), \circ) * ((\operatorname{ch}(\pi_j), \circ) * (\operatorname{ch}(\pi_k), \circ)),$

$(A2)$ $((\operatorname{ch}(\tau), \bullet) * (\operatorname{ch}(\pi_i), \circ)) * (\operatorname{ch}(\pi_j), \circ) = (\operatorname{ch}(\tau), \bullet) * ((\operatorname{ch}(\pi_i), \circ) * (\operatorname{ch}(\pi_j), \circ)),$

$(A3)$ $((\operatorname{ch}(\tau_i), \bullet) * (\operatorname{ch}(\tau_j), \bullet)) * (\operatorname{ch}(\pi), \circ) = (\operatorname{ch}(\tau_i), \bullet) * ((\operatorname{ch}(\tau_j), \bullet) * (\operatorname{ch}(\pi), \circ)),$

$(A4)$ $((\operatorname{ch}(\tau_i), \bullet) * (\operatorname{ch}(\tau_j), \bullet)) * (\operatorname{ch}(\tau_k), \bullet) = (\operatorname{ch}(\tau_i), \bullet) * ((\operatorname{ch}(\tau_j), \bullet) * (\operatorname{ch}(\tau_k), \bullet)).$

Proof. $(A1)$ is clear by the assumption that H is strong, i.e., $\mathcal{K}(\hat{H})$ is a hypergroup.
   $(A2)$ For $\tau \in \widehat{H_0}$ and $\pi_i, \pi_j \in \hat{H}$,

$$\begin{aligned}((\operatorname{ch}(\tau), \bullet) * (\operatorname{ch}(\pi_i), \circ)) * (\operatorname{ch}(\pi_j), \circ) &= \left(\operatorname{ch}(\tau) \operatorname{res}_{H_0}^H \operatorname{ch}(\pi_i), \bullet\right) * (\operatorname{ch}(\pi_j), \circ) \\ &= \left(\operatorname{ch}(\tau) \left(\operatorname{res}_{H_0}^H \operatorname{ch}(\pi_i)\right) \left(\operatorname{res}_{H_0}^H \operatorname{ch}(\pi_j)\right), \bullet\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\operatorname{ch}(\tau), \bullet) * ((\operatorname{ch}(\pi_i), \circ) * (\operatorname{ch}(\pi_j), \circ)) &= (\operatorname{ch}(\tau), \bullet) * (\operatorname{ch}(\pi_i) \operatorname{ch}(\pi_j), \circ) \\ &= \left(\operatorname{ch}(\tau) \operatorname{res}_{H_0}^H (\operatorname{ch}(\pi_i) \operatorname{ch}(\pi_j)), \bullet\right) \\ &= \left(\operatorname{ch}(\tau) \left(\operatorname{res}_{H_0}^H \operatorname{ch}(\pi_i)\right) \left(\operatorname{res}_{H_0}^H \operatorname{ch}(\pi_j)\right), \bullet\right).\end{aligned}$$

   $(A3)$ For $\tau_i, \tau_j \in \widehat{H_0}$ and $\pi \in \hat{H}$,

$$\begin{aligned}&((\operatorname{ch}(\tau_i), \bullet) * (\operatorname{ch}(\tau_j), \bullet)) * (\operatorname{ch}(\pi), \circ) \\ &= \left(q\left(\operatorname{ind}_{H_0}^H (\operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j)), \circ\right) + (1-q)(\operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j), \bullet)\right) * (\operatorname{ch}(\pi), \circ) \\ &= q\left(\operatorname{ind}_{H_0}^H (\operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j)), \circ\right) * (\operatorname{ch}(\pi), \circ) + (1-q)(\operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j), \bullet) * (\operatorname{ch}(\pi), \circ) \\ &= q\left(\operatorname{ind}_{H_0}^H (\operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j)) \operatorname{ch}(\pi), \circ\right) + (1-q)\left(\operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j) \operatorname{res}_{H_0}^H \operatorname{ch}(\pi), \bullet\right) \\ &= q\left(\operatorname{ind}_{H_0}^H (\operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j) \operatorname{res}_{H_0}^H \operatorname{ch}(\pi)), \circ\right) + (1-q)\left(\operatorname{ch}(\tau_i) \operatorname{ch}(\tau_j) \operatorname{res}_{H_0}^H \operatorname{ch}(\pi), \bullet\right) \\ &= (\operatorname{ch}(\tau_i), \bullet) * \left(\operatorname{ch}(\tau_j) \operatorname{res}_{H_0}^H \operatorname{ch}(\pi), \bullet\right) \\ &= (\operatorname{ch}(\tau_i), \bullet) * ((\operatorname{ch}(\tau_j), \bullet) * (\operatorname{ch}(\pi), \circ)).\end{aligned}$$

   $(A4)$ For $\tau_i, \tau_j, \tau_k \in \widehat{H_0}$

$$((\operatorname{ch}(\tau_i), \bullet) * (\operatorname{ch}(\tau_j), \bullet)) * (\operatorname{ch}(\tau_k), \bullet)$$

$$= \left(q\left(\operatorname{ind}_{H_0}^H(\operatorname{ch}(\tau_i)\operatorname{ch}(\tau_j)),\circ\right) + (1-q)(\operatorname{ch}(\tau_i)\operatorname{ch}(\tau_j),\bullet)\right) * (\operatorname{ch}(\tau_k),\bullet)$$
$$= q\left(\operatorname{ind}_{H_0}^H(\operatorname{ch}(\tau_i)\operatorname{ch}(\tau_j)),\circ\right) * (\operatorname{ch}(\tau_k),\bullet) + (1-q)(\operatorname{ch}(\tau_i)\operatorname{ch}(\tau_j),\bullet) * (\operatorname{ch}(\tau_k),\bullet)$$
$$= q\left(\operatorname{res}_{H_0}^H\left(\operatorname{ind}_{H_0}^H(\operatorname{ch}(\tau_i)\operatorname{ch}(\tau_j))\right)\operatorname{ch}(\tau_k),\bullet\right)$$
$$\quad + (1-q)q\left(\operatorname{ind}_{H_0}^H(\operatorname{ch}(\tau_i)\operatorname{ch}(\tau_j)\operatorname{ch}(\tau_k)),\circ\right) + (1-q)^2(\operatorname{ch}(\tau_i)\operatorname{ch}(\tau_j)\operatorname{ch}(\tau_k),\bullet)$$
$$= q\left(\operatorname{res}_{H_0}^H\left(\operatorname{ind}_{H_0}^H\operatorname{ch}(\tau_0)\right)(\operatorname{ch}(\tau_i)\operatorname{ch}(\tau_j)\operatorname{ch}(\tau_k)),\bullet\right)$$
$$\quad + (1-q)q\left(\operatorname{ind}_{H_0}^H(\operatorname{ch}(\tau_i)\operatorname{ch}(\tau_j)\operatorname{ch}(\tau_k)),\circ\right) + (1-q)^2(\operatorname{ch}(\tau_i)\operatorname{ch}(\tau_j)\operatorname{ch}(\tau_k),\bullet).$$

This implies the associativity:

$$((\operatorname{ch}(\tau_i),\bullet) * (\operatorname{ch}(\tau_j),\bullet)) * (\operatorname{ch}(\tau_k),\bullet) = (\operatorname{ch}(\tau_i),\bullet) * ((\operatorname{ch}(\tau_j),\bullet) * (\operatorname{ch}(\tau_k),\bullet)). \qquad \blacksquare$$

Proposition 3.7. *Let H be a strong compact hypergroup and H_0 a subhypergroup of H which is also of strong type and such that $|H/H_0| < +\infty$. If $\mathcal{K}\left(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)\right)$ is a hypergroup, then (H, H_0) is an admissible hypergroup pair.*

Proof. (1) By the associativity $(A3)$ we have

$$(\operatorname{ch}(\tau),\bullet) * ((\operatorname{ch}(\tau_0),\bullet) * (\operatorname{ch}(\pi),\circ)) = ((\operatorname{ch}(\tau),\bullet) * (\operatorname{ch}(\tau_0),\bullet)) * (\operatorname{ch}(\pi),\circ),$$

and the formulae

$$(\operatorname{ch}(\tau),\bullet) * ((\operatorname{ch}(\tau_0),\bullet) * (\operatorname{ch}(\pi),\circ))$$
$$\quad = q\left(\operatorname{ind}_{H_0}^H\left(\operatorname{ch}(\tau)\operatorname{res}_{H_0}^H\operatorname{ch}(\pi)\right),\circ\right) + (1-q)\left(\operatorname{ch}(\tau)\operatorname{res}_{H_0}^H\operatorname{ch}(\pi),\bullet\right)$$

and

$$((\operatorname{ch}(\tau),\bullet) * (\operatorname{ch}(\tau_0),\bullet)) * (\operatorname{ch}(\pi),\circ)$$
$$\quad = q\left(\left(\operatorname{ind}_{H_0}^H\operatorname{ch}(\tau)\right)\operatorname{ch}(\pi),\circ\right) + (1-q)\left(\operatorname{ch}(\tau)\operatorname{res}_{H_0}^H\operatorname{ch}(\pi),\bullet\right).$$

Hence we obtain the admissibility condition (1)

$$\operatorname{ind}_{H_0}^H\left(\left(\operatorname{res}_{H_0}^H\operatorname{ch}(\pi)\right)\operatorname{ch}(\tau)\right) = \operatorname{ch}(\pi)\operatorname{ind}_{H_0}^H\operatorname{ch}(\tau).$$

(2) By the associativity $(A4)$

$$(\operatorname{ch}(\tau),\bullet) * ((\operatorname{ch}(\tau_0),\bullet) * (\operatorname{ch}(\tau_0),\bullet))$$
$$\quad = ((\operatorname{ch}(\tau),\bullet) * (\operatorname{ch}(\tau_0),\bullet)) * (\operatorname{ch}(\tau_0),\bullet), ((\operatorname{ch}(\tau),\bullet) * (\operatorname{ch}(\tau_0),\bullet)) * (\operatorname{ch}(\tau_0),\bullet)$$
$$\quad = q\left(\operatorname{res}_{H_0}^H\left(\operatorname{ind}_{H_0}^H\operatorname{ch}(\tau),\bullet\right) + (1-q)q\left(\operatorname{ind}_{H_0}^H\operatorname{ch}(\tau)\right),\circ\right) + (1-q)^2(\operatorname{ch}(\tau),\bullet),$$

and

$$(\operatorname{ch}(\tau),\bullet) * ((\operatorname{ch}(\tau_0),\bullet) * (\operatorname{ch}(\tau_0),\bullet))$$
$$\quad = q\left(\operatorname{ch}(\tau)\operatorname{res}_{H_0}^H\left(\operatorname{ind}_{H_0}^H\operatorname{ch}(\tau_0)\right),\bullet\right) + (1-q)q\left(\operatorname{ind}_{H_0}^H\operatorname{ch}(\tau),\circ\right) + (1-q)^2(\operatorname{ch}(\tau),\bullet).$$

Hence we obtain the admissibility condition (2)

$$\operatorname{res}_{H_0}^H\left(\operatorname{ind}_{H_0}^H\operatorname{ch}(\tau)\right) = \operatorname{ch}(\tau)\operatorname{res}_{H_0}^H\left(\operatorname{ind}_{H_0}^H\operatorname{ch}(\tau_0)\right). \qquad \blacksquare$$

Theorem 3.8. *Let H be a strong compact hypergroup and H_0 a subhypergroup of H which is also of strong type and such that $|H/H_0| < +\infty$. Then $\mathcal{K}\left(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)\right)$ is a hypergroup if and only if (H, H_0) is an admissible hypergroup pair.*

Proof. If (H, H_0) is an admissible hypergroup pair, then the associativity relations $(A1)$, $(A2)$, $(A3)$ and $(A4)$ are a consequence of Proposition 3.6. It is easy to check the remaining axioms of a hypergroup for $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$. The converse statement follows from Proposition 3.7. ∎

Remark 3.9.

(1) The above $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$ is a discrete commutative (at most countable) hypergroup such that the sequence

$$1 \longrightarrow \mathcal{K}(\hat{H}) \longrightarrow \mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)) \longrightarrow \mathbb{Z}_q(2) \longrightarrow 1$$

is exact.

(2) If $H_0 = H$, then $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$ is the direct product hypergroup $\mathcal{K}(\hat{H}) \times \mathbb{Z}_q(2)$.

(3) If H is a finite hypergroup and $H_0 = \{h_0\}$ where h_0 is unit of H, then $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$ is the hypergroup join $\mathcal{K}(\hat{H}) \vee \mathbb{Z}_q(2)$.

(4) If H is a compact commutative hypergroup of strong type and H_0 is a closed subhypergroup of H with $|H/H_0| < +\infty$. Then (H, H_0) is always an admissible hypergroup pair and $\mathcal{K}(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2))$ is a hypergroup. For more details see [9].

4 Semi-direct product hypergroups

We consider a non-commutative compact hypergroup, namely the semi-direct product hypergroup $K := H \rtimes_\alpha G$, where α is an action of a compact group G on a compact commutative hypergroup H of strong type. For a representation π of K we denote the restrictions of π to H and G by ρ and τ respectively. We shall write $\pi = \rho \odot \tau$ expressing

$$\pi(h, g) = \rho(h)\tau(g)$$

for all $h \in H$, $g \in G$. The action $\hat{\alpha}$ of G on $\hat{H}$ induced by α is given by

$$\hat{\alpha}_g(\chi)(h) := \chi(\alpha_g^{-1}(h)),$$

whenever $\chi \in \hat{H}$, $g \in G$, $h \in H$. Let

$$G(\chi) := \{g \in G \colon \hat{\alpha}_g(\chi) = \chi\}$$

be the stabilizer of $\chi \in \hat{H}$ under the action $\hat{\alpha}$ of G on $\hat{H}$.

Proposition 4.1. *Any irreducible representation π of $K = H \rtimes_\alpha G$ is given by*

$$\pi = \pi^{(\chi, \tau)} := \mathrm{ind}_{H \rtimes_\alpha G(\chi)}^{H \rtimes_\alpha G}(\chi \odot \tau),$$

where $\chi \in \hat{H}$, $\tau \in \widehat{G(\chi)}$.

Proof. This statement is obtained by an application of the Mackey machine as stated in Theorem 7.1 of [8]. ∎

We denote the orbit of $\chi \in \hat{H}$ by $O(\chi)$ under the action $\hat{\alpha}$ of G on $\hat{H}$, i.e.,

$$O(\chi) := \{\hat{\alpha}_g(\chi) \colon g \in G\}.$$

Then the representation ρ^χ associated with $O(\chi)$ is defined by

$$\rho^\chi := \int_{O(\chi)}^{\oplus} \sigma\mu(d\sigma),$$

where μ is the $\hat{\alpha}$-invariant probability measure supporting the orbit $O(\chi)$ of χ in $\hat{H}$. We denote the space $\{\rho^\chi : \chi \in \hat{H}\}$ by $\hat{H}^{\hat{\alpha}}$. By the assumption that H is strong we see that $\hat{H}^{\hat{\alpha}}$ is a hypergroup which is called the *orbital hypergroup* of $\hat{H}$ under the action $\hat{\alpha}$. Here we note that the irreducible representation $\pi^{(\chi,\tau)}$ of $K = H \rtimes_\alpha G$ is written as $\pi^{(\chi,\tau)} = \rho^\chi \odot u^\tau$, where the representation u^τ of G is given by

$$u^\tau = \mathrm{ind}_{G(\chi)}^G \tau.$$

Moreover we note that the hyperdimension $d(\pi)$ of $\pi = \pi^{(\chi,\tau)}$ is

$$d(\pi) := w(\chi)\dim\pi = w(\rho^\chi)\dim\tau,$$

where $w(\chi)$ and $w(\rho^\chi) = w(\chi)|O(\chi)|$ denote the weights of $\chi \in \hat{H}$ and $\rho^\chi \in \hat{H}^{\hat{\alpha}}$ respectively.

Proposition 4.2. *For a representation $\pi_0 = \chi \odot \tau$ of $K_0 = H \rtimes_\alpha G(\chi)$, where $\chi \in \hat{H}$, $\tau \in \widehat{G(\chi)}$, the character of the induced representation $\pi^{(\chi,\tau)} = \mathrm{ind}_{K_0}^K \pi_0$ of π_0 takes the form*

$$\mathrm{ch}\left(\mathrm{ind}_{K_0}^K \pi_0\right)(h,g) = \int_G \chi(\alpha_s(h))\,\mathrm{ch}(\tau)\left(sgs^{-1}\right)1_{G(\chi)}\left(sgs^{-1}\right)\omega_G(ds).$$

Moreover,

$$\left(\mathrm{ind}_{K_0}^K \mathrm{ch}(\pi_0)\right)(h,g) = \int_G \chi(\alpha_s(h))\,\mathrm{ch}(\tau)\left(sgs^{-1}\right)1_{G(\chi)}\left(sgs^{-1}\right)\omega_G(ds),$$

hence

$$\mathrm{ind}_{K_0}^K \mathrm{ch}(\pi_0) = \mathrm{ch}\left(\mathrm{ind}_{K_0}^K \pi_0\right) = \mathrm{ch}\left(\pi^{(\chi,\tau)}\right).$$

Proof. By an application of the character formulae as proved in Proposition 4.3 and Theorem 4.5 of [11], we obtain the desired formulae. ∎

Proposition 4.3 ([11, Theorem 4.6]). *Let H be a finite commutative hypergroup of strong type. Then the induced representation $\pi = \mathrm{ind}_G^{H \rtimes_\alpha G} \tau$ of an irreducible representation τ of G to $H \rtimes_\alpha G$ is finite-dimensional, and it is decomposed as*

$$\pi \cong \sum_{\rho^\chi \in \hat{H}^{\hat{\alpha}}}{}^{\oplus} \pi^{(\chi,\tau)}.$$

The character $\mathrm{ch}(\pi)$ of π is

$$\mathrm{ch}(\pi) = \sum_{\rho^\chi \in \hat{H}^{\hat{\alpha}}} \frac{w(\rho^\chi)}{w(\hat{H})} \mathrm{ch}\left(\pi^{(\chi,\tau)}\right).$$

Definition 4.4 ([5]). The action α of G on H is said to satisfy the *regularity condition* (or is called regular) provided

$$G(\chi_i) \cap G(\chi_j) \subset G(\chi_k)$$

for all $\chi_k \in \hat{H}$ such that $\chi_k \in \mathrm{supp}(\delta_{\chi_i} \hat{*} \delta_{\chi_j})$ whenever $\chi_i, \chi_j \in \hat{H}$, $k, i, j \in \{0, 1, \ldots, n\}$ and $\hat{*}$ symbolizes the convolution on $\hat{H}$.

Lemma 4.5 ([5, Lemma 3.1]). *If the action α satisfies the regularity condition, then the character set $\mathcal{K}\big(\widehat{H \rtimes_\alpha G}\big)$ of the semi-direct product hypergroup $H \rtimes_\alpha G$ is a commutative hypergroup.*

For $g \in G$, put

$$\hat{H}(g) := \big\{\sigma \in \hat{H} : g \in G(\sigma)\big\} = \big\{\sigma \in \hat{H} : \hat{\alpha}_g(\sigma) = \sigma\big\}.$$

Proposition 4.6. *Let H be a finite commutative hypergroup of strong type and G a compact group. Assume that the action α of G on H satisfies the regularity condition. Then the followings hold:*

(1) *$\hat{H}(g)$ is a subhypergroup of $\hat{H}$,*

(2) *for $g, t \in G$ the condition $tgt^{-1} \in G(\sigma)$ implies that $\hat{\alpha}_t^{-1}(\sigma) \in \hat{H}(g)$,*

(3) *for $\tau \in \hat{G}$*

$$\big(\mathrm{ind}_G^{H \rtimes_\alpha G} \mathrm{ch}(\tau)\big)(h, g) = \frac{w(\hat{H}(g))}{w(\hat{H})} \omega_{\hat{H}(g)}(h) \cdot \mathrm{ch}(\tau)(g),$$

where $\omega_{\hat{H}(g)}$ is the normalized Haar measure of $\hat{H}(g)$,

(4) *for $\tau \in \hat{G}$*

$$\mathrm{res}_G^{H \rtimes_\alpha G} \big(\mathrm{ind}_G^{H \rtimes_\alpha G} \mathrm{ch}(\tau)\big)(g) = \frac{w(\hat{H}(g))}{w(\hat{H})} \cdot \mathrm{ch}(\tau)(g).$$

Proof. (1) It is clear that $\sigma^- \in \hat{H}(g)$ for $\sigma \in \hat{H}(g)$. We show that $\mathrm{supp}(\sigma_i \hat{*} \sigma_j) \subset \hat{H}(g)$ for $\sigma_i, \sigma_j \in \hat{H}(g)$. The condition $\sigma_i, \sigma_j \in \hat{H}(g)$ implies that $g \in G(\sigma_i) \cap G(\sigma_j)$. Take $\sigma \in \mathrm{supp}(\sigma_i \hat{*} \sigma_j)$. Then by the regularity condition

$$G(\sigma_i) \cap G(\sigma_j) \subset G(\sigma)$$

we see that $g \in G(\sigma)$ which implies that $\sigma \in \hat{H}(g)$.

(2) By the condition $tgt^{-1} \in G(\sigma)$ we see that $\hat{\alpha}_{tgt^{-1}}(\sigma) = \sigma$. Then we obtain

$$\hat{\alpha}_t(\hat{\alpha}_g(\hat{\alpha}_{t^{-1}}(\sigma))) = \sigma,$$

so that

$$\hat{\alpha}_g\big(\hat{\alpha}_t^{-1}(\sigma)\big) = \hat{\alpha}_t^{-1}(\sigma),$$

which means that $\hat{\alpha}_t^{-1}(\sigma) \in \hat{H}(g)$.

(3) Applying Propositions 4.2 and 4.3 we obtain for $\tau \in \hat{G}$

$$\big(\mathrm{ind}_G^{H \rtimes_\alpha G} \mathrm{ch}(\tau)\big)(h, g) = \sum_{\rho^\chi \in \hat{H}^{\hat{\alpha}}} \frac{w(\rho^\chi)}{w(\hat{H})} \mathrm{ch}\big(\pi^{(\chi,\tau)}\big)(h, g)$$

$$= \sum_{\rho^\chi \in \hat{H}^{\hat{\alpha}}} \frac{w(\rho^\chi)}{w(\hat{H})} \int_G \chi(\alpha_s(h)) \mathrm{ch}(\tau)\big(sgs^{-1}\big) 1_{G(\chi)}\big(sgs^{-1}\big) d\omega_G(s)$$

$$= \sum_{\rho^\chi \in \hat{H}^{\hat{\alpha}}} \frac{w(\rho^\chi)}{w(\hat{H})} \int_G \chi(\alpha_s(h)) 1_{G(\chi)}\big(sgs^{-1}\big) d\omega_G(s) \cdot \mathrm{ch}(\tau)(g)$$

$$= \sum_{\rho^\chi \in \hat{H}^{\hat{\alpha}}} \frac{|O(\chi)| w(\chi)}{w(\hat{H})} \cdot \frac{1}{|O(\chi)|} \sum_{\sigma \in O(\chi)} \sigma(h) 1_{G(\sigma)}(g) \cdot \mathrm{ch}(\tau)(g)$$

$$= \sum_{\sigma \in \hat{H}} \frac{w(\sigma)}{w(\hat{H})} \sigma(h) 1_{G(\sigma)}(g) \cdot \mathrm{ch}(\tau)(g)$$

$$= \frac{w(\hat{H}(g))}{w(\hat{H})} \sum_{\sigma \in \hat{H}(g)} \frac{w(\sigma)}{w(\hat{H}(g))} \sigma(h) \cdot \mathrm{ch}(\tau)(g)$$

$$= \frac{w(\hat{H}(g))}{w(\hat{H})} \omega_{\hat{H}(g)}(h) \cdot \mathrm{ch}(\tau)(g).$$

(4) For $\tau \in \hat{G}$

$$\mathrm{res}_G^{H \rtimes_\alpha G} \big(\mathrm{ind}_G^{H \rtimes_\alpha G} \mathrm{ch}(\tau) \big)(g) = \big(\mathrm{ind}_G^{H \rtimes_\alpha G} \mathrm{ch}(\tau) \big)(h_0, g)$$

$$= \frac{w(\hat{H}(g))}{w(\hat{H})} \omega_{\hat{H}(g)}(h_0) \cdot \mathrm{ch}(\tau)(g) = \frac{w(\hat{H}(g))}{w(\hat{H})} \cdot \mathrm{ch}(\tau)(g). \qquad \blacksquare$$

Theorem 4.7. *Let H be a finite commutative hypergroup of strong type and G a compact group. Suppose that the action α of G on H satisfies the regularity condition. Then the pair $(H \rtimes_\alpha G, G)$ is an admissible hypergroup pair and $\mathcal{K}\big(\widehat{H \rtimes_\alpha G} \cup \hat{G}, \mathbb{Z}_q(2) \big)$ is a discrete commutative hypergroup.*

Proof. By the Mackey machine an irreducible representation π of $K = H \rtimes_\alpha G$ is given by

$$\pi = \pi^{(\chi, \tau_1)} = \mathrm{ind}_{H \rtimes_\alpha G(\chi)}^{H \rtimes_\alpha G} (\chi \odot \tau_1),$$

where $\chi \in \hat{H}, \tau_1 \in \widehat{G(\chi)}$ and

$$\pi^{(\chi, \tau_1)}(h, g) = \int_G \chi(\alpha_s(h)) \, \mathrm{ch}(\tau_1)\big(sgs^{-1}\big) 1_{G(\chi)}\big(sgs^{-1}\big) \omega_G(ds),$$

by Propositions 4.1 and 4.2.

(1) For $\tau \in \hat{G}$ and $\pi^{(\chi, \tau_1)} \in \hat{K} = \widehat{H \rtimes_\alpha G}$, applying Proposition 4.6,

$$\big(\mathrm{ind}_G^{H \rtimes_\alpha G} \mathrm{ch}(\tau) \cdot \mathrm{ch}\big(\pi^{(\chi, \tau_1)}\big) \big)(h, g) = \big(\mathrm{ind}_G^{H \rtimes_\alpha G} \mathrm{ch}(\tau) \big)(h, g) \, \mathrm{ch}\big(\pi^{(\chi, \tau_1)}\big)(h, g)$$

$$= \frac{w(\hat{H}(g))}{w(\hat{H})} \omega_{\hat{H}(g)}(h) \cdot \mathrm{ch}(\tau)(g) \int_G \chi(\alpha_s(h)) \, \mathrm{ch}(\tau_1)\big(sgs^{-1}\big) 1_{G(\chi)}\big(sgs^{-1}\big) \omega_G(ds)$$

$$= \frac{w(\hat{H}(g))}{w(\hat{H})} \cdot \mathrm{ch}(\tau)(g) \int_G \omega_{\hat{H}(g)}(h) \hat{\alpha}_s^{-1}(\chi)(h) \, \mathrm{ch}(\tau_1)\big(sgs^{-1}\big) 1_{G(\chi)}\big(sgs^{-1}\big) \omega_G(ds)$$

$$= \frac{w(\hat{H}(g))}{w(\hat{H})} \cdot \mathrm{ch}(\tau)(g) \int_G (\omega_{\hat{H}(g)} \hat{\alpha}_s^{-1}(\chi))(h) \, \mathrm{ch}(\tau_1)\big(sgs^{-1}\big) 1_{G(\chi)}\big(sgs^{-1}\big) \omega_G(ds)$$

$$= \frac{w(\hat{H}(g))}{w(\hat{H})} \cdot \mathrm{ch}(\tau)(g) \int_G \omega_{\hat{H}(g)}(h) \, \mathrm{ch}(\tau_1)\big(sgs^{-1}\big) 1_{G(\chi)}\big(sgs^{-1}\big) \omega_G(ds)$$

$$= \frac{w(\hat{H}(g))}{w(\hat{H})} \omega_{\hat{H}(g)}(h) \cdot \mathrm{ch}(\tau)(g) \int_G \mathrm{ch}(\tau_1)\big(sgs^{-1}\big) 1_{G(\chi)}\big(sgs^{-1}\big) \omega_G(ds)$$

$$= \big(\mathrm{ind}_G^{H \rtimes_\alpha G} \big(\mathrm{ch}(\tau) \big(\mathrm{res}_G^{H \rtimes_\alpha G} \mathrm{ch}(\pi) \big) \big) \big)(h, g).$$

Hence we obtain the admissibility condition (1).

(2) For $\tau \in \hat{G}$, applying Proposition 4.6,

$$\mathrm{res}_G^{H \rtimes_\alpha G} \big(\mathrm{ind}_G^{H \rtimes_\alpha G} \mathrm{ch}(\tau) \big)(g) = \frac{w(\hat{H}(g))}{w(\hat{H})} \cdot \mathrm{ch}(\tau)(g) = \frac{w(\hat{H}(g))}{w(\hat{H})} \cdot \mathrm{ch}(\tau_0)(g) \cdot \mathrm{ch}(\tau)(g)$$

$$= \mathrm{res}_G^{H \rtimes_\alpha G} \big(\mathrm{ind}_G^{H \rtimes_\alpha G} \tau_0 \big)(g) \cdot \mathrm{ch}(\tau)(g).$$

Hence we see the admissibility condition (2).

By Theorem 3.8 we see that $\mathcal{K}\big(\widehat{H \rtimes_\alpha G} \cup \hat{G}, \mathbb{Z}_q(2) \big)$ is a discrete commutative hypergroup. $\qquad \blacksquare$
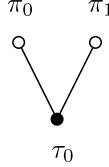
5   Examples of $\mathcal{K}\big(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)\big)$

In this section we illustrate some special case of

$$\mathcal{K}\big(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)\big) = \big\{(\mathrm{ch}(\pi_i), \circ), (\mathrm{ch}(\tau_j), \bullet) \colon \pi_i \in \hat{H}, \tau_j \in \widehat{H_0}\big\}.$$

We denote $(\mathrm{ch}(\pi_i), \circ)$ for $\pi_i \in \hat{H}$ by γ_i and $(\mathrm{ch}(\tau_j), \bullet)$ for $\tau_j \in \widehat{H_0}$ by ρ_j hereafter.

Example 5.1. Let $H = \mathbb{Z}_p(2) = \{h_0, h_1\}$ and $H_0 = \{h_0\}$ where h_0 is unit of H. In this case $\hat{H} = \{\pi_0, \pi_1\}$ and $\widehat{H_0} = \{\tau_0\}$. The Frobenius diagram (we refer to [10]) is

$$\begin{array}{cc} \pi_0 & \pi_1 \end{array}$$

$$\tau_0$$

The structure equations of $\mathcal{K}\big(\widehat{\mathbb{Z}_p(2)} \cup \widehat{\{h_0\}}, \mathbb{Z}_q(2)\big) = \{\gamma_0, \gamma_1, \rho_0\}$ are

$$\gamma_1\gamma_1 = p\gamma_0 + (1-p)\gamma_1, \qquad \rho_0\rho_0 = \frac{q}{2}\gamma_0 + \frac{q}{2}\gamma_1 + (1-q)\rho_0, \qquad \gamma_1\rho_0 = \rho_0.$$

We note that $\mathcal{K}\big(\widehat{\mathbb{Z}_p(2)} \cup \widehat{\{h_0\}}, \mathbb{Z}_q(2)\big) \cong \mathbb{Z}_p(2) \vee \mathbb{Z}_q(2)$ and (p, q)-deformations of the hypergroup associated with Dynkin diagram A_3 refer to Sunder–Wildberger [16].

Example 5.2. Let $H = \mathbb{Z}_p(3) = \{h_0, h_1, h_2\}$ and $H = \{h_0\}$ In this case $\hat{H} = \{\pi_0, \pi_1, \pi_2\}$ and $\widehat{H_0} = \{\tau_0\}$. The Frobenius diagram is

$$\begin{array}{ccc} \pi_0 & \pi_1 & \pi_2 \end{array}$$

$$\tau_0$$
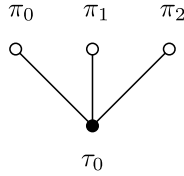
The structure equations of $\mathcal{K}\big(\widehat{\mathbb{Z}_p(3)} \cup \widehat{\{h_0\}}, \mathbb{Z}_q(2)\big) = \{\gamma_0, \gamma_1, \gamma_2, \rho_0\}$ are

$$\gamma_1\gamma_1 = \frac{1-p}{2}\gamma_1 + \frac{1+p}{2}\gamma_2, \qquad \gamma_2\gamma_2 = \frac{1+p}{2}\gamma_1 + \frac{1-p}{2}\gamma_2,$$

$$\gamma_1\gamma_2 = p\gamma_0 + \frac{1-p}{2}\gamma_1 + \frac{1-p}{2}\gamma_2, \qquad \rho_0\rho_0 = \frac{q}{3}\gamma_0 + \frac{q}{3}\gamma_1 + \frac{q}{3}\gamma_2 + (1-q)\rho_0,$$

$$\gamma_1\rho_0 = \gamma_2\rho_0 = \rho_0.$$

We note that $\mathcal{K}\big(\widehat{\mathbb{Z}_p(3)} \cup \widehat{\{h_0\}}, \mathbb{Z}_q(2)\big) \cong \mathbb{Z}_p(3) \vee \mathbb{Z}_q(2)$ and these hypergroups are (p, q)-deformations of the hypergroup associated with Dynkin diagram D_4 constructed by Sunder–Wildberger [16].

Example 5.3. Let $H = \mathbb{Z}_{(p,r)}(4) = \{h_0, h_1, h_2, h_3\}$ and $H_0 = \{h_0\}$. In this case $\hat{H} = \{\pi_0, \pi_1, \pi_2, \pi_3\}$ and $\widehat{H_0} = \{\tau_0\}$. The Frobenius diagram is
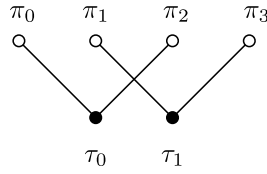
$$\begin{array}{cccc} \pi_0 & \pi_1 & \pi_2 & \pi_3 \end{array}$$

$$\tau_0$$
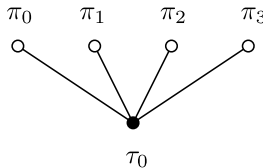
The structure equations of $\mathcal{K}\big(\widehat{\mathbb{Z}_{(p,r)}(4)} \cup \widehat{\{h_0\}}, \mathbb{Z}_q(2)\big) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \rho_0\}$ are

$$\gamma_1\gamma_1 = \gamma_3\gamma_3 = \frac{1-p}{2}\gamma_1 + p\gamma_2 + \frac{1-p}{2}\gamma_3, \qquad \gamma_2\gamma_2 = r\gamma_0 + (1-r)\gamma_2,$$

$$\gamma_1\gamma_2 = \frac{1-r}{2}\gamma_1 + \frac{1+r}{2}\gamma_3, \qquad \gamma_1\gamma_3 = \frac{2pr}{1+r}\gamma_0 + \frac{1-p}{2}\gamma_1 + \frac{p-pr}{1+r}\gamma_2 + \frac{1-p}{2}\gamma_3,$$

$$\gamma_2\gamma_3 = \frac{1+r}{2}\gamma_1 + \frac{1-r}{2}\gamma_3, \qquad \rho_0\rho_0 = \frac{q}{4}\gamma_0 + \frac{q}{4}\gamma_1 + \frac{q}{4}\gamma_2 + \frac{q}{4}\gamma_3 + (1-q)\rho_0,$$

$$\gamma_1\rho_0 = \gamma_2\rho_0 = \gamma_3\rho_0 = \rho_0.$$

We note that $\mathcal{K}\big(\widehat{\mathbb{Z}_{(p,r)}(4)} \cup \widehat{\{h_0\}}, \mathbb{Z}_q(2)\big) \cong \mathbb{Z}_{(p,r)}(4) \vee \mathbb{Z}_q(2)$.

Example 5.4. Let $H = \mathbb{Z}_{(p,r)}(4) = \{h_0, h_1, h_2, h_3\}$ and $H_0 = \mathbb{Z}_p(2) = \{h_0, h_2\}$. In this case $\hat{H} = \{\pi_0, \pi_1, \pi_2, \pi_3\}$ and $\widehat{H_0} = \{\tau_0, \tau_1\}$. The Frobenius diagram is

The structure equations of $\mathcal{K}\big(\widehat{\mathbb{Z}_{(p,r)}(4)} \cup \widehat{\mathbb{Z}_p(2)}, \mathbb{Z}_q(2)\big) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \rho_0, \rho_1\}$ are

$$\gamma_1\gamma_1 = \gamma_3\gamma_3 = \frac{1-p}{2}\gamma_1 + p\gamma_2 + \frac{1-p}{2}\gamma_3, \qquad \gamma_2\gamma_2 = r\gamma_0 + (1-r)\gamma_2,$$

$$\gamma_1\gamma_2 = \frac{1-r}{2}\gamma_1 + \frac{1+r}{2}\gamma_3, \qquad \gamma_1\gamma_3 = \frac{2pr}{1+r}\gamma_0 + \frac{1-p}{2}\gamma_1 + \frac{p-pr}{1+r}\gamma_2 + \frac{1-p}{2}\gamma_3,$$

$$\gamma_2\gamma_3 = \frac{1+r}{2}\gamma_1 + \frac{1-r}{2}\gamma_3, \qquad \rho_0\rho_0 = \rho_1\rho_1 = \frac{q}{2}\gamma_0 + \frac{q}{2}\gamma_2 + (1-q)\rho_0,$$

$$\rho_0\rho_1 = \frac{q}{2}\gamma_1 + \frac{q}{2}\gamma_3 + (1-q)\rho_1, \qquad \gamma_1\rho_0 = \gamma_3\rho_0 = \gamma_2\rho_1 = \rho_1,$$

$$\gamma_1\rho_1 = \gamma_3\rho_1 = \gamma_2\rho_0 = \rho_0.$$

Example 5.5. Let $H = S_p(3) = \mathbb{Z}_p(3) \rtimes_\alpha \mathbb{Z}_2$ and $H_0 = \mathbb{Z}_2$. In this case $\hat{H} = \{\pi_0, \pi_1, \pi_2\}$ $(\dim \pi_2 = 2)$ and $\widehat{H_0} = \{\tau_0, \tau_1\}$. The Frobenius diagram is
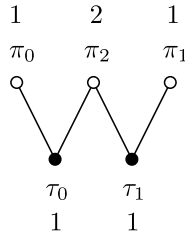
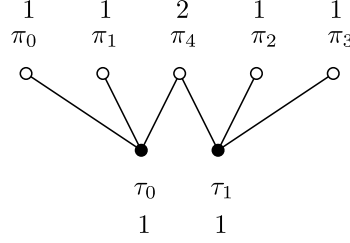The structure equations of $\mathcal{K}\big(\widehat{S_p(3)} \cup \widehat{\mathbb{Z}_2}, \mathbb{Z}_q(2)\big) = \{\gamma_0, \gamma_1, \gamma_2, \rho_0, \rho_1\}$ are

$$\gamma_1\gamma_1 = \gamma_0, \qquad \gamma_2\gamma_2 = \frac{p}{4}\gamma_0 + \frac{p}{4}\gamma_1 + \left(1 - \frac{p}{2}\right)\gamma_2, \qquad \gamma_1\gamma_2 = \gamma_2,$$

$$\rho_0\rho_0 = \rho_1\rho_1 = \frac{q}{3}\gamma_0 + \frac{2q}{3}\gamma_2 + (1-q)\rho_0, \qquad \rho_0\rho_1 = \frac{q}{3}\gamma_1 + \frac{2q}{3}\gamma_2 + (1-q)\rho_1,$$

$$\gamma_1\rho_0 = \rho_1, \qquad \gamma_1\rho_1 = \rho_0, \qquad \gamma_2\rho_0 = \gamma_2\rho_1 = \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1.$$

We note that these hypergroups are (p, q)-deformations of the hypergroup associated with Dynkin diagram A_5 refer to Sunder–Wildberger [16].
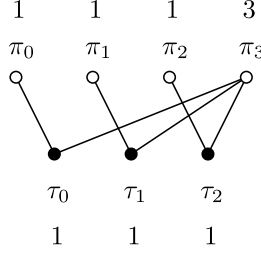
Example 5.6. Let $H = D_{(p,r)}(4) = \mathbb{Z}_{(p,r)}(4) \rtimes_\alpha \mathbb{Z}_2$ and $H_0 = \mathbb{Z}_2$. In this case $\hat{H} = \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$ (dim $\pi_4 = 2$) and $\widehat{H_0} = \{\tau_0, \tau_1\}$. The Frobenius diagram is

$$
\begin{array}{ccccc}
1 & 1 & 2 & 1 & 1 \\
\pi_0 & \pi_1 & \pi_4 & \pi_2 & \pi_3
\end{array}
$$

$$
\begin{array}{cc}
\tau_0 & \tau_1 \\
1 & 1
\end{array}
$$

The structure equations of $\mathcal{K}\big(\widehat{D_{(p,r)}(4)} \cup \widehat{\mathbb{Z}_2}, \mathbb{Z}_q(2)\big) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \rho_0, \rho_1\}$ are

$$\gamma_1\gamma_1 = \gamma_0, \qquad \gamma_2\gamma_2 = \gamma_3\gamma_3 = r\gamma_0 + (1-r)\gamma_2,$$

$$\gamma_4\gamma_4 = \frac{pr}{2(1+r)}\gamma_0 + \frac{pr}{2(1+r)}\gamma_1 + \frac{p}{2(1+r)}\gamma_2 + \frac{p}{2(1+r)}\gamma_3 + (1-p)\gamma_4,$$

$$\gamma_1\gamma_2 = \gamma_3, \qquad \gamma_1\gamma_3 = \gamma_2, \qquad \gamma_1\gamma_4 = \gamma_2\gamma_4 = \gamma_3\gamma_4 = \gamma_4, \qquad \gamma_2\gamma_3 = r\gamma_1 + (1-r)\gamma_3,$$

$$\rho_0\rho_0 = \rho_1\rho_1 = \frac{q}{4}\gamma_0 + \frac{q}{4}\gamma_1 + \frac{q}{2}\gamma_4 + (1-q)\rho_0, \qquad \rho_0\rho_1 = \frac{q}{4}\gamma_2 + \frac{q}{4}\gamma_3 + \frac{q}{2}\gamma_4 + (1-q)\rho_1,$$

$$\gamma_1\rho_0 = \gamma_2\rho_1 = \gamma_3\rho_1 = \rho_0, \qquad \gamma_1\rho_1 = \gamma_2\rho_0 = \gamma_3\rho_0 = \rho_1, \qquad \gamma_4\rho_0 = \gamma_4\rho_1 = \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1.$$
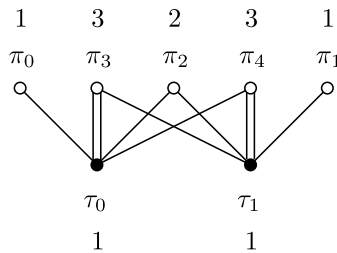
Example 5.7. Let $H = A_4 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\alpha \mathbb{Z}_3$ and $H_0 = \mathbb{Z}_3$. In this case $\hat{H} = \{\pi_0, \pi_1, \pi_2, \pi_3\}$ (dim $\pi_3 = 3$) and $\widehat{H_0} = \{\tau_0, \tau_1, \tau_2\}$. The Frobenius diagram is

$$
\begin{array}{cccc}
1 & 1 & 1 & 3 \\
\pi_0 & \pi_1 & \pi_2 & \pi_3
\end{array}
$$

$$
\begin{array}{ccc}
\tau_0 & \tau_1 & \tau_2 \\
1 & 1 & 1
\end{array}
$$

The structure equations of $\mathcal{K}\big(\widehat{A_4} \cup \widehat{\mathbb{Z}_3}, \mathbb{Z}_q(2)\big) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \rho_0, \rho_1, \rho_2\}$ are

$$\gamma_1\gamma_1 = \gamma_2, \qquad \gamma_2\gamma_2 = \gamma_1, \qquad \gamma_3\gamma_3 = \frac{1}{9}\gamma_0 + \frac{1}{9}\gamma_1 + \frac{1}{9}\gamma_2 + \frac{2}{3}\gamma_3, \qquad \gamma_1\gamma_2 = \gamma_0,$$

$$\gamma_1\gamma_3 = \gamma_3, \qquad \gamma_2\gamma_3 = \gamma_3, \qquad \rho_0\rho_0 = \rho_1\rho_2 = \frac{q}{4}\gamma_0 + \frac{3q}{4}\gamma_3 + (1-q)\rho_0,$$

$$\rho_1\rho_1 = \rho_0\rho_2 = \frac{q}{4}\gamma_2 + \frac{3q}{4}\gamma_3 + (1-q)\rho_2, \qquad \rho_2\rho_2 = \rho_0\rho_1 = \frac{q}{4}\gamma_1 + \frac{3q}{4}\gamma_3 + (1-q)\rho_1,$$

$$\gamma_1\rho_0 = \rho_1, \qquad \gamma_2\rho_0 = \gamma_1\rho_1 = \rho_2, \qquad \gamma_2\rho_1 = \rho_0,$$

$$\gamma_3\rho_0 = \gamma_3\rho_1 = \gamma_3\rho_2 = \frac{1}{3}\rho_0 + \frac{1}{3}\rho_1 + \frac{1}{3}\rho_2.$$

Example 5.8. Let $H = S_4 = A_4 \rtimes_\alpha \mathbb{Z}_2$ and $H_0 = \mathbb{Z}_2$. In this case $\hat{H} = \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$ and $\widehat{H_0} = \{\tau_0, \tau_1\}$. The Frobenius diagram is

$$
\begin{array}{ccccc}
1 & 3 & 2 & 3 & 1 \\
\pi_0 & \pi_3 & \pi_2 & \pi_4 & \pi_1
\end{array}
$$

$$
\begin{array}{cc}
\tau_0 & \tau_1 \\
1 & 1
\end{array}
$$

The structure equations of $\mathcal{K}\big(\widehat{S_4} \cup \widehat{\mathbb{Z}_2}, \mathbb{Z}_q(2)\big) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \rho_0, \rho_1\}$ are

$$\gamma_1\gamma_1 = \gamma_0, \qquad \gamma_2\gamma_2 = \frac{1}{4}\gamma_0 + \frac{1}{4}\gamma_1 + \frac{1}{2}\gamma_2,$$

$$\gamma_3\gamma_3 = \gamma_4\gamma_4 = \frac{1}{9}\gamma_0 + \frac{2}{9}\gamma_2 + \frac{1}{3}\gamma_3 + \frac{1}{3}\gamma_4,$$

$$\gamma_1\gamma_2 = \gamma_2, \qquad \gamma_1\gamma_3 = \gamma_4, \qquad \gamma_1\gamma_4 = \gamma_3, \qquad \gamma_2\gamma_3 = \gamma_2\gamma_4 = \frac{1}{2}\gamma_3 + \frac{1}{2}\gamma_4,$$

$$\gamma_3\gamma_4 = \frac{1}{9}\gamma_1 + \frac{2}{9}\gamma_2 + \frac{1}{3}\gamma_3 + \frac{1}{3}\gamma_4,$$

$$\rho_0\rho_0 = \rho_1\rho_1 = \frac{q}{12}\gamma_0 + \frac{q}{6}\gamma_2 + \frac{q}{2}\gamma_3 + \frac{q}{4}\gamma_4 + (1-q)\rho_0,$$

$$\rho_0\rho_1 = \rho_1\rho_0 = \frac{q}{12}\gamma_1 + \frac{q}{6}\gamma_2 + \frac{q}{4}\gamma_3 + \frac{q}{2}\gamma_4 + (1-q)\rho_1,$$

$$\gamma_0\rho_0 = \gamma_1\rho_1 = \rho_0, \qquad \gamma_0\rho_1 = \gamma_1\rho_0 = \rho_1, \qquad \gamma_2\rho_0 = \gamma_2\rho_1 = \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1,$$

$$\gamma_3\rho_0 = \gamma_4\rho_1 = \frac{2}{3}\rho_0 + \frac{1}{3}\rho_1, \qquad \gamma_4\rho_0 = \gamma_3\rho_1 = \frac{1}{3}\rho_0 + \frac{2}{3}\rho_1.$$

Example 5.9. Let $H = S_4$ and $H_0 = S_3$. In this case $\hat{H} = \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4\}$ and $\widehat{H_0} = \{\tau_0, \tau_1, \tau_2\}$. The Frobenius diagram is
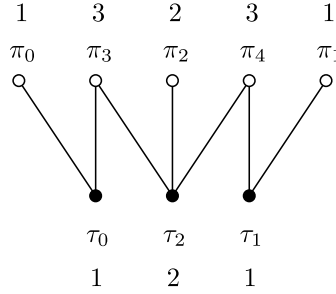
The structure equations of $\mathcal{K}\big(\widehat{S_4} \cup \widehat{S_3}, \mathbb{Z}_q(2)\big) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \rho_0, \rho_1, \rho_2\}$ are

$$\gamma_1\gamma_1 = \gamma_0, \qquad \gamma_2\gamma_2 = \frac{1}{4}\gamma_0 + \frac{1}{4}\gamma_1 + \frac{1}{2}\gamma_2, \qquad \gamma_3\gamma_3 = \gamma_4\gamma_4 = \frac{1}{9}\gamma_0 + \frac{2}{9}\gamma_2 + \frac{1}{3}\gamma_3 + \frac{1}{3}\gamma_4,$$

$$\gamma_1\gamma_2 = \gamma_2, \qquad \gamma_1\gamma_3 = \gamma_4, \qquad \gamma_1\gamma_4 = \gamma_3, \qquad \gamma_2\gamma_3 = \gamma_2\gamma_4 = \frac{1}{2}\gamma_3 + \frac{1}{2}\gamma_4,$$

$$\gamma_3\gamma_4 = \frac{1}{9}\gamma_1 + \frac{2}{9}\gamma_2 + \frac{1}{3}\gamma_3 + \frac{1}{3}\gamma_4, \qquad \rho_0\rho_0 = \rho_1\rho_1 = \frac{q}{4}\gamma_0 + \frac{3q}{4}\gamma_3 + (1-q)\rho_0,$$

$$\rho_2\rho_2 = \frac{q}{16}\gamma_0 + \frac{q}{16}\gamma_1 + \frac{q}{8}\gamma_2 + \frac{3q}{8}\gamma_3 + \frac{3q}{8}\gamma_4 + \frac{1-q}{4}\rho_0 + \frac{1-q}{4}\rho_1 + \frac{1-q}{2}\rho_2,$$

$$\rho_1\rho_2 = \frac{q}{4}\gamma_2 + \frac{3q}{8}\gamma_3 + \frac{3q}{8}\gamma_4 + (1-q)\rho_2, \qquad \gamma_0\rho_0 = \gamma_1\rho_1 = \rho_0, \qquad \gamma_0\rho_1 = \gamma_1\rho_0 = \rho_1,$$

$$\gamma_0\rho_2 = \gamma_1\rho_2 = \gamma_2\rho_0 = \gamma_2\rho_1 = \rho_2, \qquad \gamma_3\rho_0 = \gamma_4\rho_1 = \frac{1}{3}\rho_0 + \frac{2}{3}\rho_2,$$

$$\gamma_4\rho_0 = \gamma_3\rho_1 = \frac{1}{3}\rho_1 + \frac{2}{3}\rho_2, \qquad \gamma_2\rho_2 = \frac{1}{4}\rho_0 + \frac{1}{4}\rho_1 + \frac{1}{2}\rho_2,$$

$$\gamma_3\rho_2 = \gamma_4\rho_0 = \frac{1}{6}\rho_0 + \frac{1}{6}\rho_1 + \frac{2}{3}\rho_2.$$

Example 5.10. Let $H = \mathbb{Z}_p(3) \rtimes_\alpha (\mathbb{Z}_2 \times \mathbb{T}) \cong S_p(3) \times \mathbb{T}$ and $H_0 = \mathbb{Z}_2 \times \mathbb{T}$. In this case $\hat{H} = \widehat{S_p(3) \times \mathbb{T}} = \widehat{S_p(3)} \times \hat{\mathbb{T}} = \widehat{S_p(3)} \times \mathbb{Z}$ and $\widehat{H_0} = \widehat{\mathbb{Z}_2 \times \mathbb{T}} = \widehat{\mathbb{Z}_2} \times \hat{\mathbb{T}} = \mathbb{Z}_2 \times \mathbb{Z}$. The Frobenius diagram is
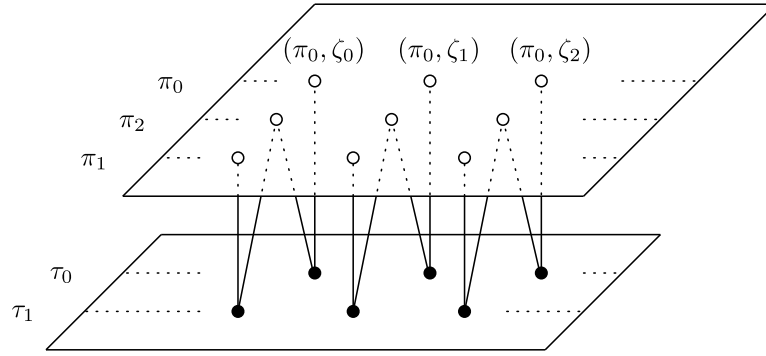
We denote $(\mathrm{ch}((\pi_i, \zeta_j)), \circ)$ for $(\pi_i, \zeta_j) \in \hat{H}$ by (γ_i, ξ_j) and $(\mathrm{ch}((\tau_k, \zeta_\ell)), \bullet)$ for $(\tau_k, \zeta_\ell) \in \widehat{H_0}$ by (ρ_k, ξ_ℓ). The structure equations of

$$\mathcal{K}\big(\hat{H} \cup \widehat{H_0}, \mathbb{Z}_q(2)\big) = \{(\gamma_i, \xi_j), (\rho_k, \xi_\ell) \colon i = 0, 1, 2, \ k = 0, 1, \ j, \ell \in \mathbb{Z}\}$$

are

$$(\gamma_1, \xi_j)(\gamma_1, \xi_\ell) = (\gamma_0, \xi_{j+\ell}), \qquad (\gamma_1, \xi_j)(\gamma_2, \xi_\ell) = (\gamma_2, \xi_{j+\ell}),$$

$$(\gamma_2, \xi_j)(\gamma_2, \xi_\ell) = \frac{p}{4}(\gamma_0, \xi_{j+\ell}) + \frac{p}{4}(\gamma_1, \xi_{j+\ell}) + \left(1 - \frac{p}{2}\right)(\gamma_2, \xi_{j+\ell}),$$

$$(\rho_0, \xi_j)(\rho_0, \xi_\ell) = (\rho_1, \xi_j)(\rho_1, \xi_\ell) = \frac{q}{3}(\gamma_0, \xi_{j+\ell}) + \frac{2q}{3}(\gamma_2, \xi_{j+\ell}) + (1 - q)(\rho_0, \xi_{j+\ell}),$$

$$(\rho_0, \xi_j)(\rho_1, \xi_\ell) = (\rho_1, \xi_\ell)(\rho_0, \xi_j) = \frac{q}{3}(\gamma_1, \xi_{j+\ell}) + \frac{2q}{3}(\gamma_2, \xi_{j+\ell}) + (1 - q)(\rho_1, \xi_{j+\ell}),$$

$$(\gamma_0, \xi_j)(\rho_0, \xi_\ell) = (\gamma_1, \xi_j)(\rho_1, \xi_\ell) = (\rho_0, \xi_{j+\ell}),$$

$$(\gamma_0, \xi_j)(\rho_1, \xi_\ell) = (\gamma_1, \xi_j)(\rho_0, \xi_\ell) = (\rho_1, \xi_{j+\ell}),$$

$$(\gamma_2, \xi_j)(\rho_0, \xi_\ell) = (\gamma_2, \xi_j)(\rho_1, \xi_\ell) = \frac{1}{2}(\rho_0, \xi_{j+\ell}) + \frac{1}{2}(\rho_1, \xi_{j+\ell}).$$

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