

Singular Degenerations of Lie Supergroups of Type $D(2, 1; a)$

Kenji IOHARA [†] and Fabio GAVARINI [‡]

[†] *Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 Boulevard du 11 Novembre 1918, F 69622 Villeurbanne Cedex, France*

E-mail: iohara@math.univ-lyon1.fr

URL: <http://math.univ-lyon1.fr/~iohara/>

[‡] *Dipartimento di Matematica, Università di Roma “Tor Vergata”,*

Via della ricerca scientifica 1, I-00133 Roma, Italy

E-mail: gavarini@mat.uniroma2.it

Received October 31, 2017, in final form December 11, 2018; Published online December 31, 2018

<https://doi.org/10.3842/SIGMA.2018.137>

Abstract. The complex Lie superalgebras \mathfrak{g} of type $D(2, 1; a)$ – also denoted by $\mathfrak{osp}(4, 2; a)$ – are usually considered for “non-singular” values of the parameter a , for which they are simple. In this paper we introduce five suitable integral forms of \mathfrak{g} , that are well-defined at singular values too, giving rise to “singular specializations” that are no longer simple: this extends the family of *simple* objects of type $D(2, 1; a)$ in five different ways. The resulting five families coincide for general values of a , but are different at “singular” ones: here they provide non-simple Lie superalgebras, whose structure we describe explicitly. We also perform the parallel construction for complex Lie supergroups and describe their singular specializations (or “degenerations”) at singular values of a . Although one may work with a single complex parameter a , in order to stress the overall \mathfrak{S}_3 -symmetry of the whole situation, we shall work (following Kaplansky) with a two-dimensional parameter $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ ranging in the complex affine plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$.

Key words: Lie superalgebras; Lie supergroups; singular degenerations; contractions

2010 Mathematics Subject Classification: 14A22; 17B20; 13D10

Contents

1	Introduction	2
2	Preliminaries	3
2.1	Basic superobjects	4
2.2	Lie superalgebras	4
2.3	Lie supergroups	4
2.4	Super Harish-Chandra pairs and Lie supergroups	6
3	Lie superalgebras of type $D(2, 1; \sigma)$	7
3.1	Construction of \mathfrak{g}_σ (after Kaplansky)	8
3.2	Kac’ realization	9
3.3	Bases of \mathfrak{g}_σ	12
4	Integral forms & degenerations for Lie superalgebras of type $D(2, 1; \sigma)$	14
4.1	First family: the Lie superalgebras $\mathfrak{g}(\sigma)$	15
4.2	Second family: the Lie superalgebras $\mathfrak{g}'(\sigma)$	16
4.3	Third family: the Lie superalgebras $\mathfrak{g}''(\sigma)$	18

4.4	Degenerations from contractions: the $\widehat{\mathfrak{g}}(\sigma)$'s and the $\widehat{\mathfrak{g}}'(\sigma)$'s	20
5	Lie supergroups of type $D(2, 1; \sigma)$: presentations and degenerations	23
5.1	First family: the Lie supergroups \mathbf{G}_σ	23
5.2	Second family: the Lie supergroups \mathbf{G}'_σ	25
5.3	Third family: the Lie supergroups \mathbf{G}''_σ	27
5.4	Lie supergroups from contractions: the family of the $\widehat{\mathbf{G}}_\sigma$'s	30
5.5	Lie supergroups from contractions: the family of the $\widehat{\mathbf{G}}'_\sigma$'s	32
	References	35

1 Introduction

In the classification of simple finite-dimensional complex Lie superalgebras – due to Kac (cf. [12]) – a special one-parameter family occurs, whose elements \mathfrak{g}_a depend on a parameter $a \in \mathbb{C} \setminus \{0, -1\}$. These are “generically non-isomorphic”, and all isomorphisms between them are encoded in a free action of the symmetric group \mathfrak{S}_3 on the family $\{\mathfrak{g}_a\}_{a \in \mathbb{C} \setminus \{0, -1\}}$. It was pointed out in [12] that the Cartan matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & a \\ 0 & -1 & 2 \end{pmatrix}$ used to define this Lie superalgebra had already appeared in [18], as a Cartan matrix of a one-parameter family of 16-dimensional simple Lie algebras over a field k of characteristic 2 with $a \in k \setminus \{0, 1\}$.

For any $a \in \{1, -2, -1/2\}$ one has $\mathfrak{g}_1 \cong \mathfrak{osp}(4, 2)$, which is of type $D(2, 1)$: thus Kac called each \mathfrak{g}_a to be “of type $D(2, 1; a)$ ” – while $D(m, n)$ is the type of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m, 2n)$. For the same reason, some authors, for example [3] – cf. also [2] – use instead notation $\mathfrak{osp}(4, 2; a)$. By general theory, one can complete each of the (simple) Lie superalgebras \mathfrak{g}_a and form a so-called super Harish-Chandra pair: and then one associates to the latter a corresponding complex Lie supergroup, say \mathbf{G}_a , whose tangent Lie superalgebra is \mathfrak{g}_a – as prescribed in Kac’ classification of simple algebraic supergroups, cf. [11]. All these \mathbf{G}_a ’s form a family $\{\mathbf{G}_a\}_{a \in \mathbb{C} \setminus \{0, -1\}}$, which bears a free \mathfrak{S}_3 -action that induces the \mathfrak{S}_3 -action on $\{\mathfrak{g}_a\}_{a \in \mathbb{C} \setminus \{0, -1\}}$. The starting point of the present paper is the following question: can we “take the limit” (in some sense) of \mathfrak{g}_a for a approaching to the “singular values” $a = 0$ and $a = -1$? And if yes, what is the structure of the resulting “limit” Lie superalgebra? Similarly, we raise the same questions for the family of the supergroups \mathbf{G}_a .

In this article, we show that there are several ways to answer, in the positive, these questions. In fact, we present *five* possible ways to complete the family of simple Lie superalgebras $D(2, 1; a)$ with additional Lie superalgebras for the “singular values” $a \in \{0, -1\}$. Each one of these new, extra objects can be thought of as a “limit” of the older ones; however, the existence of different options show that such “limits” have no intrinsic meaning, but strongly depend on some choice – roughly, on “how you approach the singular point”. For each of these choices, the corresponding new objects that are “limits” of the (original) simple Lie superalgebras $D(2, 1; a)$ happen to be non-simple, and we describe explicitly their structure, which is different for the different choices. Therefore, we extend the old family $\{\mathfrak{g}_a = \mathfrak{osp}(4, 2; a)\}_{a \in \mathbb{C} \setminus \{0, -1\}}$ of simple Lie superalgebras to five larger families, indexed by the points of $\mathbb{P}^1(\mathbb{C}) \cup \{*\}$, whose elements at “non-singular values” $a \in \{0, -1, \infty, *\}$ are *non-simple* – which is why we call them “degenerations” – and (when comparing one family with a different one) non-isomorphic.

By the way, our analysis is by no means exhaustive: one can still provide further ways to complete the family of the simple \mathfrak{g}_a ’s (for non-singular values of a) by adding some extra objects at singular values of a , right in the same spirit but with different outcomes. Our goal here is only to explain the *existence* and *non-uniqueness* of such constructions. A few words about our construction. First, instead of working with Lie superalgebras \mathfrak{g}_a indexed by a single parameter $a \in \mathbb{C} \setminus \{0, -1\}$ – later extended to $a \in \mathbb{C}$ – we rather deal with a multiparameter

$\sigma \in V := \{(\sigma_1, \sigma_2, \sigma_3) \in \mathbb{C}^3 \mid \sum_i \sigma_i = 0\}$. The starting point is a construction – due to Kaplansky, cf. [13]; see also [15] – that for each $\sigma \in V$ provides a Lie superalgebra \mathfrak{g}_σ : this yields a full family of Lie superalgebras $\{\mathfrak{g}_\sigma\}_{\sigma \in V}$, forming a bundle over V , naturally endowed with an action of the group $\mathcal{G} := \mathbb{C}^\times \times \mathfrak{S}_3$ via Lie superalgebra isomorphisms. For each σ in the “general locus” $V^\times := V \setminus (\bigcup_{i=1}^3 \{\sigma_i = 0\})$ we have $\mathfrak{g}_\sigma \cong \mathfrak{g}_a$ for some $a \in \mathbb{C} \setminus \{0, -1\}$ so the original family $\{\mathfrak{g}_a = \mathfrak{osp}(4, 2; a)\}_{a \in \mathbb{C} \setminus \{0, -1\}}$ of simple Lie superalgebras is taken into account; in addition, the \mathfrak{g}_σ ’s are well-defined also at singular values $\sigma \in V \cap (\bigcup_{i=1}^3 \{\sigma_i = 0\})$, but there they are non-simple instead.

Thus Kaplansky’s family of Lie superalgebras provides a first solution to our problem. In addition, we re-visit this construction and devise five recipes to construct similar families, as follows. For $\sigma \in V^\times$, we fix in \mathfrak{g}_σ a particular \mathbb{C} -basis, call it B , in such a way that the structure constants are polynomials in σ . When we replace $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ with a formal parameter $\mathbf{x} = (x_1, x_2, x_3)$, the previous multiplication table defines a Lie superalgebra structure on the free $\mathbb{C}[\mathbf{x}]$ -module with basis B , denoted by $\mathfrak{g}_B(\mathbf{x})$. Then for each $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$ the quotient $\mathfrak{g}_B(\sigma) := \mathfrak{g}_B(\mathbf{x}) / (x_i - \sigma_i)_{i=1,2,3} \mathfrak{g}_B(\mathbf{x})$ is a well-defined complex Lie superalgebra, such that $\mathfrak{g}_B(\sigma) \cong \mathfrak{g}_\sigma$ for $\sigma \in V^\times$; thus we get a whole family $\{\mathfrak{g}_B(\sigma)\}_{\sigma \in V}$ as requested, that actually depends on the choice of the basis B . We present five explicit examples that give rise to five different outcomes – one being Kaplansky’s family. Indeed, at each point of the “singular locus” $V \cap (\bigcup_{i=1}^3 \{\sigma_i = 0\})$ these families present different (non-isomorphic) non-simple Lie superalgebras, that we describe in detail. As a second contribution, we perform a parallel construction at the level of Lie supergroups: namely, for each $\sigma \in V$ we “complete” each Lie superalgebra $\mathfrak{g}_B(\sigma)$ to form a super Harish-Chandra pair, and then take the corresponding (complex holomorphic) Lie supergroup. This yields a family $\{\mathbf{G}_B(\sigma)\}_{\sigma \in V}$ of Lie supergroups, with \mathbf{G}_σ isomorphic to \mathbf{G}_a for a suitable $a \in \mathbb{C} \setminus \{0, -1\}$ for non-singular values of σ , while $\mathbf{G}_B(\sigma)$ is not simple for singular values instead; moreover, the group $\mathcal{G} := \mathbb{C}^\times \times \mathfrak{S}_3$ freely acts on this family via Lie supergroup isomorphisms. In other words, we complete the “old” family of the simple Lie supergroups \mathbf{G}_a ’s (isomorphic to suitable \mathbf{G}_σ ’s) by suitably adding new, non-simple Lie supergroups at singular values of σ . The construction depends on B , and with our five, previously fixed choices we find five different families: for each of them, we describe explicitly the non-simple supergroups \mathbf{G}_σ at singular values of σ – which are referred to as “degenerations” of the (previously known, simple) \mathbf{G}_a ’s.

This analysis might be reformulated in the language of *deformation theory* of supermanifold – e.g., as treated in [17]. However, this goes beyond the scope of the present article. This article is organized as follows. In Section 2, we briefly recall the basic algebraic background necessary for this work, in particular, some language about *supermathematics*. In Section 3, we introduce our Lie superalgebras $\mathfrak{g}_\sigma = \mathfrak{osp}(4, 2; \sigma)$. Several *integral forms* of the Lie superalgebra \mathfrak{g}_σ are introduced in Section 4. In particular, as an application, the structure of their *singular degenerations* is studied in detail (Theorems 4.1, 4.2, 4.3, 4.4 and 4.5). Section 5 is the last highlight of this paper: we introduce and analyze the Lie supergroups whose Lie superalgebras are studied in Section 4, and we describe the (non-simple) structure of their degenerations – i.e., the member of the families at singular values of σ (Theorems 5.1, 5.2, 5.3, 5.4 and 5.5).

As the main objects treated in this article have many special features, most of the above descriptions are given in a down-to-earth manner, so that even the readers who are not familiar with the subject could follow easily our exposition.

2 Preliminaries

In this section, we recall the notions and language of Lie superalgebras and Lie supergroups. Our purpose is to fix the terminology, but everything indeed is standard matter.

2.1 Basic superobjects

All throughout the paper, we work over the field \mathbb{C} of complex numbers (nevertheless, immediate generalizations are possible), unless otherwise stated. By \mathbb{C} -*supermodule*, or \mathbb{C} -*super vector space*, any \mathbb{C} -module V endowed with a \mathbb{Z}_2 -grading $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ is the group with two elements. Then $V_{\bar{0}}$ and its elements are called *even*, while $V_{\bar{1}}$ and its elements *odd*. By $|x| (\in \mathbb{Z}_2)$ we denote the *parity* of any non-zero homogeneous element, defined by the condition $x \in V_{|x|}$.

We call \mathbb{C} -*superalgebra* any associative, unital \mathbb{C} -algebra A which is \mathbb{Z}_2 -graded: so A has a \mathbb{Z}_2 -grading $A = A_{\bar{0}} \oplus A_{\bar{1}}$, and $A_{\mathbf{a}}A_{\mathbf{b}} \subseteq A_{\mathbf{a}+\mathbf{b}}$. Any such A is said to be *commutative* if $xy = (-1)^{|x||y|}yx$ for all homogeneous $x, y \in A$; so, in particular, $z^2 = 0$ for all $z \in A_{\bar{1}}$. All \mathbb{C} -superalgebras form a category, whose morphisms are those of unital \mathbb{C} -algebras preserving the \mathbb{Z}_2 -grading; inside it, commutative \mathbb{C} -superalgebras form a subcategory, that we denote by **(salg)**. We denote by **(alg)** the category of (associative, unital) commutative \mathbb{C} -algebras, and by **(mod)** that of \mathbb{C} -modules. Note also that there is an obvious functor $(\)_{\bar{0}}: \mathbf{(salg)} \rightarrow \mathbf{(alg)}$ given on objects by $A \mapsto A_{\bar{0}}$. We call *Weil superalgebra* any finite-dimensional commutative \mathbb{C} -superalgebra A such that $A = \mathbb{C} \oplus \mathfrak{N}(A)$ where \mathbb{C} is even and $\mathfrak{N}(A) = \mathfrak{N}(A)_{\bar{0}} \oplus \mathfrak{N}(A)_{\bar{1}}$ is a \mathbb{Z}_2 -graded nilpotent ideal (the *nilradical* of A). Every Weil superalgebra A is endowed with a canonical epimorphisms $p_A: A \twoheadrightarrow \mathbb{C}$ and an embedding $u_A: \mathbb{C} \hookrightarrow A$, such that $p_A \circ u_A = \text{id}$. Weil superalgebras over \mathbb{C} form a full subcategory of **(salg)**, denoted by **(Wsalg)**. Finally, let **(Walg)** := **(Wsalg)** \cap **(alg)** be the category of *Weil algebras (over \mathbb{C})*, i.e., the full subcategory of all totally even objects in **(Wsalg)** – namely, those whose odd part is trivial. Then the functor $(\)_{\bar{0}}: \mathbf{(salg)} \rightarrow \mathbf{(alg)}$ obviously restricts to a similar functor $(\)_{\bar{0}}: \mathbf{(Wsalg)} \rightarrow \mathbf{(Walg)}$ given again by $A \mapsto A_{\bar{0}}$.

2.2 Lie superalgebras

By definition, a *Lie superalgebra* is a \mathbb{C} -supermodule $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a (*Lie super*)*bracket* $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, $(x, y) \mapsto [x, y]$, which is \mathbb{C} -bilinear, preserving the \mathbb{Z}_2 -grading and satisfies the following (for all homogenous $x, y, z \in \mathfrak{g}$):

- (a) $[x, y] + (-1)^{|x||y|}[y, x] = 0$ (*anti-symmetry*);
- (b) $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$ (*Jacobi identity*).

In this situation, we write $Y^{(2)} := 2^{-1}[Y, Y] (\in \mathfrak{g}_{\bar{0}})$ for all $Y \in \mathfrak{g}_{\bar{1}}$. All Lie \mathbb{C} -superalgebras form a category, denoted by **(sLie)**, whose morphisms are \mathbb{C} -linear, preserving the \mathbb{Z}_2 -grading and the bracket. Note that if \mathfrak{g} is a Lie \mathbb{C} -superalgebra, then its even part $\mathfrak{g}_{\bar{0}}$ is automatically a Lie \mathbb{C} -algebra.

Lie superalgebras can also be described in functorial language. Indeed, let **(Lie)** be the category of Lie \mathbb{C} -algebras. Then every Lie \mathbb{C} -superalgebra $\mathfrak{g} \in \mathbf{(sLie)}$ defines a functor

$$\mathcal{L}_{\mathfrak{g}}: \mathbf{(Wsalg)} \longrightarrow \mathbf{(Lie)}, \quad A \mapsto \mathcal{L}_{\mathfrak{g}}(A) := (A \otimes \mathfrak{g})_{\bar{0}} = (A_{\bar{0}} \otimes \mathfrak{g}_{\bar{0}}) \oplus (A_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}}).$$

Indeed, $A \otimes \mathfrak{g}$ is a Lie superalgebra (in a suitable, more general sense, over A) on its own, with Lie bracket $[a \otimes X, a' \otimes X'] := (-1)^{|X||a'|}aa' \otimes [X, X']$; now $\mathcal{L}_{\mathfrak{g}}(A)$ is the even part of $A \otimes \mathfrak{g}$, hence it is a Lie algebra on its own.

2.3 Lie supergroups

We shall now recall, in steps, the notion of complex holomorphic “Lie supergroups”, as a special kind of “supermanifold”. The following is a very concise summary of a long, detailed theory: further details are, for instance, in [1, 4, 17].

2.3.1. Supermanifolds. By *superspace* we mean a pair $S = (|S|, \mathcal{O}_S)$ of a topological space $|S|$ and a sheaf of commutative superalgebras \mathcal{O}_S on it such that the stalk $\mathcal{O}_{S,x}$ of \mathcal{O}_S at each point $x \in |S|$ is a local superalgebra. A *morphism* $\phi: S \rightarrow T$ between superspaces S and T is a pair $(|\phi|, \phi^*)$ where $|\phi|: |S| \rightarrow |T|$ is a continuous map of topological spaces and the induced morphism $\phi^*: \mathcal{O}_T \rightarrow |\phi|_*(\mathcal{O}_S)$ of sheaves on $|T|$ is such that $\phi_x^*(\mathfrak{m}_{|\phi|(x)}) \subseteq \mathfrak{m}_x$, where $\mathfrak{m}_{|\phi|(x)}$ and \mathfrak{m}_x denote the maximal ideals in the stalks $\mathcal{O}_{T,|\phi|(x)}$ and $\mathcal{O}_{S,x}$, respectively.

As basic model, the *linear supervariety* $\mathcal{C}_{\mathbb{C}}^{p|q}$ (in Leites' terminology) is, by definition, the topological space \mathbb{C}^p endowed with the following sheaf of commutative superalgebras: $\mathcal{O}_{\mathcal{C}_{\mathbb{C}}^{p|q}}(U) := \mathcal{H}_{\mathbb{C}^p}(U) \otimes_{\mathbb{C}} \Lambda_{\mathbb{C}}(\xi_1, \dots, \xi_q)$ for any open set $U \subseteq \mathbb{C}^p$, where $\mathcal{H}_{\mathbb{C}^p}$ is the sheaf of holomorphic functions on \mathbb{C}^p and $\Lambda_{\mathbb{C}}(\xi_1, \dots, \xi_q)$ is the complex Grassmann algebra on q variables ξ_1, \dots, ξ_q of odd parity. A (*complex holomorphic*) *supermanifold* of (super)dimension $p|q$ is a superspace $M = (|M|, \mathcal{O}_M)$ such that $|M|$ is Hausdorff and second-countable and M is locally isomorphic to $\mathcal{C}_{\mathbb{C}}^{p|q}$, i.e., for each $x \in |M|$ there is an open set $V_x \subseteq |M|$ with $x \in V_x$ and $U \subseteq \mathbb{C}^p$ such that $\mathcal{O}_M|_{V_x} \cong \mathcal{O}_{\mathcal{C}_{\mathbb{C}}^{p|q}}|_U$ (in particular, it is locally isomorphic to $\mathcal{C}_{\mathbb{C}}^{p|q}$). A *morphism* between holomorphic supermanifolds is just a morphism (between them) as superspaces.

We denote the category of (complex holomorphic) supermanifolds by **(hsmfd)**.

Let now M be a holomorphic supermanifold and U an open subset in $|M|$. Let $\mathcal{I}_M(U)$ be the (nilpotent) ideal of $\mathcal{O}_M(U)$ generated by the odd part of the latter: then $\mathcal{O}_M/\mathcal{I}_M$ defines a sheaf of purely even superalgebras over $|M|$, locally isomorphic to $\mathcal{H}_{\mathbb{C}^p}$. Then $M_{\text{rd}} := (|M|, \mathcal{O}_M/\mathcal{I}_M)$ is a *classical* holomorphic manifold, called the *underlying holomorphic (sub)manifold* of M ; the standard projection $s \mapsto \tilde{s} := s + \mathcal{I}_M(U)$ (for all $s \in \mathcal{O}_M(U)$) at the sheaf level yields an embedding $M_{\text{rd}} \rightarrow M$, so M_{rd} can be seen as an embedded sub(super)manifold of M . The whole construction is clearly functorial in M .

Finally, each “classical” manifold can be seen as a “supermanifold”, just regarding its structure sheaf as one of superalgebras that are actually *totally even*, i.e., with trivial odd part. Conversely, any supermanifold enjoying the latter property is actually a manifold, nothing more. In other words, every manifold identify with a supermanifold M that actually coincides with its underlying (sub)manifold M_{rd} .

2.3.2. Lie supergroups and the functorial approach. A *group object* in the category **(hsmfd)** is called (*complex holomorphic*) *Lie supergroup*. These objects, together with the obvious morphisms, form a subcategory among supermanifolds, denoted **(Lsgrp)**.

Lie supergroups – as well as supermanifolds – can also be conveniently studied via a functorial approach that we now briefly recall (cf. [1] or [9] for details). Let M be a supermanifold. For every $x \in |M|$ and every $A \in \mathbf{(Wsalg)}$ we set $M_{A,x} = \text{Hom}_{(\mathbf{salg})}(\mathcal{O}_{M,x}, A)$ and $M_A = \bigsqcup_{x \in |M|} M_{A,x}$; then we define $\mathcal{W}_M: \mathbf{(Wsalg)} \rightarrow \mathbf{(set)}$ to be the functor given by $A \mapsto M_A$ and $\rho \mapsto \rho^{(M)}$ with $\rho^{(M)}: M_A \rightarrow M_B$, $x_A \mapsto \rho \circ x_A$. Overall, this provides a functor $\mathcal{B}: \mathbf{(hsmfd)} \rightarrow [(\mathbf{Wsalg}), \mathbf{(set)}]$ given on objects by $M \mapsto \mathcal{W}_M$; we can now refine still more.

Given a finite dimensional commutative algebra $A_{\bar{0}}$ over \mathbb{C} , a (complex holomorphic) $A_{\bar{0}}$ -*manifold* is any manifold that is locally modelled on some open subset of some finite dimensional $A_{\bar{0}}$ -module, so that the differential of every change of charts is an $A_{\bar{0}}$ -module isomorphism. An $A_{\bar{0}}$ -*morphism* between two $A_{\bar{0}}$ -manifolds is any smooth morphism whose differential is everywhere $A_{\bar{0}}$ -linear (we then say that “it is $A_{\bar{0}}$ -smooth”). Gathering all $A_{\bar{0}}$ -manifolds (for all possible A), and suitably defining morphisms among them, one defines the category **($A_{\bar{0}}$ -hmfd)** of all “ $A_{\bar{0}}$ -manifolds”.

A key point now is that *each* \mathcal{W}_M *turns out to be a functor from* **(Wsalg)** *into* **($A_{\bar{0}}$ -hmfd)**. Furthermore, let $[[\mathbf{(Wsalg)}, (\mathcal{A}_{\bar{0}}\text{-hmfd)}]]$ be the subcategory of $[(\mathbf{Wsalg}), (\mathcal{A}_{\bar{0}}\text{-hmfd})]$ with the same objects but whose morphisms are all natural transformations $\phi: \mathcal{G} \rightarrow \mathcal{H}$ such that for every $A \in \mathbf{(Wsalg)}$ the induced $\phi_A: \mathcal{G}(A) \rightarrow \mathcal{H}(A)$ is $A_{\bar{0}}$ -smooth. Then *the second key*

point is that if $\phi: M \longrightarrow N$ is a morphism of supermanifolds, then ϕ_A is a morphism in $[[(\mathbf{Wsalg}), (\mathcal{A}_{\bar{0}}\text{-hmf})]]$. The final outcome is that we have a functor $\mathcal{S}: (\mathbf{hmf}) \longrightarrow [[(\mathbf{Wsalg}), (\mathcal{A}_{\bar{0}}\text{-hmf})]]$, given on objects by $M \mapsto \mathcal{W}_M$; the key result is that *this embedding is full and faithful*, so that for any two supermanifolds M and N one has $M \cong N$ if and only if $\mathcal{S}(M) \cong \mathcal{S}(N)$, i.e., $\mathcal{W}_M \cong \mathcal{W}_N$.

Still relevant to us, is that the embedding \mathcal{S} preserves products, hence also group objects. Therefore, *a supermanifold M is a Lie supergroup if and only if $\mathcal{S}(M) := \mathcal{W}_M$ takes values in the subcategory (among $\mathcal{A}_{\bar{0}}$ -manifolds) of group objects* – thus each $\mathcal{W}_M(A)$ is a group.

Finally, in the functorial approach the “classical” manifolds (i.e., totally even supermanifolds) can be recovered as follows: in the previous construction one simply has to replace the words “Weil superalgebras” with “Weil algebras” everywhere. It then follows, in particular, that the functor of points \mathcal{W}_M of any holomorphic, manifold \mathcal{M} is actually a functor from (\mathbf{Walg}) to $(\mathcal{A}_{\bar{0}}\text{-hmf})$; one can still see it as (the functor of points of) a *supermanifold* – that is totally even, though – by composing it with the natural functor $(\)_{\bar{0}}: (\mathbf{Wsalg}) \longrightarrow (\mathbf{Walg})$. On the other hand, given any supermanifold M , say holomorphic, the functor of points of its underlying submanifold M_{rd} is given by $\mathcal{W}_{M_{\text{rd}}}(\mathcal{A}) = \mathcal{W}_M(\mathcal{A})$ for each $\mathcal{A} \in (\mathbf{Walg})$, or in short $\mathcal{W}_{M_{\text{rd}}} = \mathcal{W}_M|_{(\mathbf{Walg})}$.

Finally, it is worth stressing that the functorial point of view on supermanifolds was originally developed – by Leites, Berezin, Deligne, Molotkov, Voronov and many others – in a slightly different way. Namely, they considered functors defined, rather than on Weil superalgebras, on *Grassmann (super)algebras*. Actually, the two approaches are equivalent: see [1] for a detailed, critical analysis of the matter.

There are some advantages in restricting the focus onto Grassmann algebras. For instance, they are the superalgebras of global sections onto the superdomains of dimension $0|q$ – i.e., “super-points”. Therefore, if M is a supermanifold considered as a super-ringed space, its description via a functor defined on Grassmann algebras (only) can be really seen as the true restriction of the functor of points of M , considered as a super-ringed space.

On the other hand, the use of Weil superalgebras has the advantage that one can use it to perform differential calculus on functors \mathcal{W}_M , much in the spirit of Weil’s approach to differential calculus in algebraic geometry. Note also that some peculiar properties for Grassmann algebras are still available for every Weil superalgebra A : e.g., the existence of the maps $p_A: A \longrightarrow \mathbb{C}$ and $u_A: \mathbb{C} \longrightarrow A$, key tools in the theory (for instance, for any Lie supergroup G this implies the existence of a semidirect product splitting of the group $G(A)$ of A -points of G). See [1] for further details.

2.4 Super Harish-Chandra pairs and Lie supergroups

A different way to deal with Lie supergroups (or algebraic supergroups) is via the notion of “super Harish-Chandra pair”, that gathers together the infinitesimal counterpart – that of Lie superalgebra – and the classical (i.e., “non-super”) counterpart – that of Lie group – of the notion of Lie supergroup. We recall it shortly, referring to [9] (and [8]) for further details.

2.4.1. Super Harish-Chandra pairs. We call *super Harish-Chandra pair* – or just “*sHCp*” in short – any pair (G, \mathfrak{g}) such that G is a (complex holomorphic) Lie group, \mathfrak{g} a complex Lie superalgebra such that $\mathfrak{g}_{\bar{0}} = \text{Lie}(G)$, and there is a (holomorphic) G -action on \mathfrak{g} by Lie superalgebra automorphisms, denoted by $\text{Ad}: G \longrightarrow \text{Aut}(\mathfrak{g})$, such that its restriction to $\mathfrak{g}_{\bar{0}}$ is the adjoint action of G on $\text{Lie}(G) = \mathfrak{g}_{\bar{0}}$ and the differential of this action is the restriction to $\text{Lie}(G) \times \mathfrak{g} = \mathfrak{g}_{\bar{0}} \times \mathfrak{g}$ of the adjoint action of \mathfrak{g} on itself. Then a *morphism* $(\Omega, \omega): (G', \mathfrak{g}') \longrightarrow (G'', \mathfrak{g}'')$ between sHCp’s is given by a morphism of Lie groups $\Omega: G' \longrightarrow G''$ and a morphism of Lie superalgebras $\omega: \mathfrak{g}' \longrightarrow \mathfrak{g}''$ such that $\omega|_{\mathfrak{g}'_{\bar{0}}} = d\Omega$ and $\omega \circ \text{Ad}_g = \text{Ad}_{\Omega_+(g)} \circ \omega$ for all $g \in G$.

We denote the category of all super Harish-Chandra pairs by (\mathbf{sHCp}) .

2.4.2. From Lie supergroups to sHCp's. For any $A \in (\mathbf{Wsalg})$, let $A[\varepsilon] := A[x]/(x^2)$, with $\varepsilon := x \bmod (x^2)$ being *even*. Then $A[\varepsilon] = A \oplus A\varepsilon \in (\mathbf{Wsalg})$, and there exists a natural morphism $p_A: A[\varepsilon] \rightarrow A$ given by $(a + a'\varepsilon) \mapsto a$. For a Lie supergroup \mathbf{G} , thought of as a functor $\mathbf{G}: (\mathbf{Wsalg}) \rightarrow (\mathbf{groups})$ – i.e., identifying $\mathbf{G} \cong \mathcal{W}_{\mathbf{G}}$ – let $\mathbf{G}(p_A): \mathbf{G}(A[\varepsilon]) \rightarrow \mathbf{G}(A)$ be the morphism associated with $p_A: A[\varepsilon] \rightarrow A$. Then there exists a unique functor $\text{Lie}(\mathbf{G}): (\mathbf{Wsalg}) \rightarrow (\mathbf{groups})$ given on objects by $\text{Lie}(\mathbf{G})(A) := \text{Ker}(\mathbf{G}(p_A))$. The key fact now is that $\text{Lie}(\mathbf{G})$ is actually valued in the category (\mathbf{Lie}) of Lie algebras, i.e., it is a functor $\text{Lie}(\mathbf{G}): (\mathbf{Wsalg}) \rightarrow (\mathbf{Lie})$. Furthermore, there exists a Lie superalgebra \mathfrak{g} – identified with the tangent superspace to \mathbf{G} at the unit point – such that $\text{Lie}(\mathbf{G}) = \mathcal{L}_{\mathfrak{g}}$ (cf. Section 2.2). Moreover, for any $A \in (\mathbf{Wsalg})$ one has $\text{Lie}(\mathbf{G})(A) = \text{Lie}(\mathbf{G}(A))$, the latter being the Lie algebra of the Lie group $\mathbf{G}(A)$.

Finally, the construction $\mathbf{G} \mapsto \text{Lie}(\mathbf{G})$ for Lie supergroups is actually natural, i.e., provides a functor $\text{Lie}: (\mathbf{Lsgrp}) \rightarrow (\mathbf{sLie})$ from Lie supergroups to Lie superalgebras.

On the other hand, each Lie supergroup \mathbf{G} is a group object in the category of (holomorphic) supermanifolds: therefore, its underlying submanifold \mathbf{G}_{rd} is in turn a group object in the category of (holomorphic) manifolds, i.e., it is a Lie group. Indeed, the naturality of the construction $\mathbf{G} \mapsto \mathbf{G}_{\text{rd}}$ provides a functor from Lie supergroups to (complex) Lie groups.

On top of this analysis, if \mathbf{G} is any Lie supergroup, then $(\mathbf{G}_{\text{rd}}, \text{Lie}(\mathbf{G}))$ is a super Harish-Chandra pair; more precisely, we have a functor $\Phi: (\mathbf{Lsgrp}) \rightarrow (\mathbf{sHCp})$ given on objects by $\mathbf{G} \mapsto (\mathbf{G}_{\text{rd}}, \text{Lie}(\mathbf{G}))$ and on morphisms by $\phi \mapsto (\phi_{\text{rd}}, \text{Lie}(\phi))$.

2.4.3. From sHCp's to Lie supergroups. The functor $\Phi: (\mathbf{Lsgrp}) \rightarrow (\mathbf{sHCp})$ has a quasi-inverse $\Psi: (\mathbf{sHCp}) \rightarrow (\mathbf{Lsgrp})$ that we can describe explicitly (see [8, 9]).

Indeed, let $\mathcal{P} := (G, \mathfrak{g})$ be a super Harish-Chandra pair, and let $B := \{Y_i\}_{i \in I}$ be a \mathbb{C} -basis of $\mathfrak{g}_{\bar{1}}$. For any $A \in (\mathbf{Wsalg})$, we define $\mathbf{G}_{\mathcal{P}}(A)$ as being the group with generators the elements of the set $\Gamma_A^B := G(A) \cup \{(1 + \eta_i Y_i)\}_{(i, \eta_i) \in I \times A_{\bar{1}}}$ and relations

$$\begin{aligned} 1_G &= 1, & g' \cdot g'' &= g' \cdot_G g'', \\ (1 + \eta_i Y_i) \cdot g &= g \cdot (1 + c_{j_1} \eta_i Y_{j_1}) \cdots (1 + c_{j_k} \eta_i Y_{j_k}) \\ &\quad \text{with } \text{Ad}(g^{-1})(Y_i) &= c_{j_1} Y_{j_1} + \cdots + c_{j_k} Y_{j_k}, \\ (1 + \eta'_i Y_i) \cdot (1 + \eta''_i Y_i) &= (1_G + \eta'_i \eta''_i Y_i^{(2)})_G \cdot (1 + (\eta'_i + \eta''_i) Y_i), \\ (1 + \eta_i Y_i) \cdot (1 + \eta_j Y_j) &= (1_G + \eta_j \eta_i [Y_i, Y_j])_G \cdot (1 + \eta_j Y_j) \cdot (1 + \eta_i Y_i) \end{aligned}$$

for $g, g', g'' \in G(A)$, $\eta_i, \eta'_i, \eta''_i, \eta_j \in A_{\bar{1}}$, $i, j \in I$. This defines the functor $\mathbf{G}_{\mathcal{P}}$ on objects, and one then defines it on morphisms as follows: for any $\varphi: A' \rightarrow A''$ in (\mathbf{Wsalg}) we let $\mathbf{G}_{\mathcal{P}}(\varphi): \mathbf{G}_{\mathcal{P}}(A') \rightarrow \mathbf{G}_{\mathcal{P}}(A'')$ be the group morphism uniquely defined on generators by $\mathbf{G}_{\mathcal{P}}(\varphi)(g') := G(\varphi)(g')$, $\mathbf{G}_{\mathcal{P}}(\varphi)(1 + \eta' Y_i) := (1 + \varphi(\eta') Y_i)$.

One proves (see [8, 9]) that every such $\mathbf{G}_{\mathcal{P}}$ is in fact a Lie supergroup – thought of as a special functor, i.e., identified with its associated Weil–Berezin functor. In addition, *the construction $\mathcal{P} \mapsto \mathbf{G}_{\mathcal{P}}$ is natural in \mathcal{P} , i.e., it yields a functor $\Psi: (\mathbf{sHCp}) \rightarrow (\mathbf{Lsgrp})$; moreover, the latter is a quasi-inverse to $\Phi: (\mathbf{Lsgrp}) \rightarrow (\mathbf{sHCp})$.*

3 Lie superalgebras of type $D(2, 1; \sigma)$

In this section, we present the complex Lie superalgebras of type $D(2, 1; a)$. On the one hand, one can construct them directly, through Kaplansky's representation (cf. [14]; a widely accessible account of it is also in Scheunert's book [15, Chapter I, Section 1.5]), which depend on parameters. On the other hand, for non-singular values of the parameters one can realize them via Kac' method, choosing a suitable Cartan matrix, which still depends on parameters.

3.1 Construction of \mathfrak{g}_σ (after Kaplansky)

To fix notation, we set $V := \{(\sigma_1, \sigma_2, \sigma_3) \in \mathbb{C}^3 \mid \sum_i \sigma_i = 0\}$ – a plane in \mathbb{C}^3 – and $V^\times := V \cap (\mathbb{C}^\times)^3$, where $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$; also, $\mathbf{0} := (0, 0, 0) \in V \setminus V^\times$. The Lie superalgebras \mathfrak{g}_σ we deal with will depend on a parameter $\sigma := (\sigma_1, \sigma_2, \sigma_3) \in V$.

3.1.1. Kaplansky’s realization. We recall hereafter the construction of Lie superalgebras introduced by Kaplansky (cf. [14]) who denoted them by $\Gamma(A, B, C)$, with $A, B, C \in \mathbb{C}$. With a suitable normalization, and different terminology, they form the family nowadays called “of type $D(2, 1; a)$ ”, with $a \in \mathbb{C}$ and $a \notin \{0, -1\}$ to ensure simplicity; we shall stick to Kaplansky’s point of view, but using the parameter $\sigma \in \mathbb{C}^3$ (and later in V) and adopting the convention of denoting by $\mathfrak{osp}(4, 2; \sigma)$ the Lie superalgebra of type $D(2, 1; \sigma)$. A detailed account of Kaplansky’s realization of these Lie algebras can be found in Scheunert’s book (cf. [15, Chapter I, Section 1, Example 5]). Recall that having a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_\bar{0} \oplus \mathfrak{g}_\bar{1}$ amounts to having: (a) a Lie algebra $\mathfrak{g}_\bar{0}$; (b) a $\mathfrak{g}_\bar{0}$ -module $\mathfrak{g}_\bar{1}$; (c) a $\mathfrak{g}_\bar{0}$ -valued symmetric product on $\mathfrak{g}_\bar{1}$, such that the $\mathfrak{g}_\bar{0}$ -action is by derivations of the (symmetric) product.

Indeed, in the family of Lie superalgebras $\mathfrak{g}_\sigma := \mathfrak{osp}(4, 2; \sigma)$ parts (a) and (b) above will stand the same for any $\sigma \in V$, while the dependence (of the Lie superalgebra structure) on σ will actually occur only for part (c).

Step (a): Let $\mathfrak{sl}_i(2) := \mathbb{C}e_i \oplus \mathbb{C}h_i \oplus \mathbb{C}f_i$ ($i = 1, 2, 3$) be three isomorphic copy of $\mathfrak{sl}(2)$, in its standard realization. Then we consider their direct sum $\mathfrak{sl}_1(2) \oplus \mathfrak{sl}_2(2) \oplus \mathfrak{sl}_3(2)$ with its natural structure of Lie algebra: this will be the even part $(\mathfrak{g}_\sigma)_\bar{0}$ of our \mathfrak{g}_σ .

Step (b): Let $\square := \mathbb{C}|+\rangle \oplus \mathbb{C}|-\rangle$ be the (natural) tautological 2-dimensional $\mathfrak{sl}(2)$ -module, and \square_i (for all $i = 1, 2, 3$) its i -th copy for $\mathfrak{sl}_i(2)$. The odd part $(\mathfrak{g}_\sigma)_\bar{1}$ of \mathfrak{g}_σ is (isomorphic to) $\square_1 \boxtimes \square_2 \boxtimes \square_3$, with its natural structure of $(\mathfrak{sl}_1(2) \oplus \mathfrak{sl}_2(2) \oplus \mathfrak{sl}_3(2))$ -module. We describe a \mathbb{C} -basis of it with the following shorthand notation: for every $\epsilon_1, \epsilon_2, \epsilon_3 \in \{+, -\}$, we set $v_{\epsilon_1, \epsilon_2, \epsilon_3} := |\epsilon_1\rangle \otimes |\epsilon_2\rangle \otimes |\epsilon_3\rangle \in \square_1 \boxtimes \square_2 \boxtimes \square_3$.

Step (c): We define a projection $\psi : \square^{\otimes 2} \cong S^2\square \oplus \wedge^2\square \longrightarrow \wedge^2\square \cong \mathbb{C}$ by

$$|\pm\rangle \otimes |\pm\rangle \mapsto 0, \quad |\pm\rangle \otimes |\mp\rangle \mapsto \pm 2^{-1}.$$

Then for $\sigma' \in \mathbb{C}$, we define the linear map $p : \square^{\otimes 2} \cong S^2\square \oplus \wedge^2\square \longrightarrow S^2\square \cong \mathfrak{sl}_2$ by

$$p(u, v).w := \sigma'(\psi(v, w).u - \psi(w, u).v) \quad \forall u, v, w \in \square$$

that more explicitly reads

$$|+\rangle \otimes |+\rangle \mapsto \sigma'e, \quad |\pm\rangle \otimes |\mp\rangle \mapsto -2^{-1}\sigma'h, \quad |-\rangle \otimes |-\rangle \mapsto -\sigma'f$$

with $\{e, h, f\}$ being the standard \mathfrak{sl}_2 -triple, i.e., $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.

Now, for each triple $\sigma := (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{C}^3$ and each $i \in \{1, 2, 3\}$, let $p_i : \square_i^{\otimes 2} \longrightarrow \mathfrak{sl}_i(2)$ be the above map with scalar factor $\sigma' := -2\sigma_i \in \mathbb{C}$. The Lie superbracket $[\]$ on $\mathfrak{g}_\bar{1} \times \mathfrak{g}_\bar{1}$ can be expressed as

$$[\otimes_{i=1}^3 u_i, \otimes_{i=1}^3 v_i] = \sum_{\tau \in \mathfrak{S}_3} \psi(u_{\tau(1)}, v_{\tau(1)})\psi(u_{\tau(2)}, v_{\tau(2)})p_{\tau(3)}(u_{\tau(3)}, v_{\tau(3)}).$$

Tidying everything up, we can define a bracket on $\mathfrak{g}_\sigma = (\mathfrak{g}_\sigma)_\bar{0} \oplus (\mathfrak{g}_\sigma)_\bar{1}$ by the formula

$$[x + v, y + w] := [x, y] + x.w - y.v + [v, w] \quad \forall x, y \in (\mathfrak{g}_\sigma)_\bar{0}, \quad v, w \in (\mathfrak{g}_\sigma)_\bar{1}.$$

The following proposition resumes the outcome of this construction (see also Theorem 4.1 later on for what happens for singular values of σ):

Proposition 3.1 (cf. [15, Chapter II, Section 4.5, p. 135]). *Let $\sigma \in \mathbb{C}^3$.*

- (a) *The bracket given above defines a structure of Lie superalgebra on \mathfrak{g}_σ if and only if the given $\sigma \in \mathbb{C}^3$ satisfies the condition $\sigma_1 + \sigma_2 + \sigma_3 = 0$, that is, $\sigma \in V := \mathbb{C}^3 \cap \{ \sum_i \sigma_i = 0 \}$.*
- (b) *Let $\sigma \in V$. Then the Lie superalgebra \mathfrak{g}_σ is simple if and only if $\sigma \in V^\times$.*
- (c) *Let $\sigma', \sigma'' \in V$. Then the Lie superalgebras $\mathfrak{g}_{\sigma'}$ and $\mathfrak{g}_{\sigma''}$ are isomorphic if and only if there exists $\tau \in \mathfrak{S}_3$ such that σ'' and $\tau.\sigma'$ are proportional.*

Thus, the isomorphism classes of our \mathfrak{g}_σ 's are in bijection with the orbits of the \mathfrak{S}_3 -action onto $\mathbb{P}(V) \cup \{\mathbf{0} := (0, 0, 0)\} \cong \mathbb{P}_\mathbb{C}^1 \cup \{*\}$, a complex projective line plus an extra point.

3.1.2. The multiplicative table of \mathfrak{g}_σ . For later use, we record hereafter the *complete multiplication table* of the Lie superalgebra \mathfrak{g}_σ – for every $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$ – with respect to the \mathbb{C} -basis $\{h_i, e_j, f_j, v_{\epsilon_1, \epsilon_2, \epsilon_3} \mid i, j \in \{1, 2, 3\}, \epsilon_1, \epsilon_2, \epsilon_3 \in \{+, -\}\}$ given from scratch. In addition, in the formulas below we also take into account the following *coroots*:

$$\begin{aligned} h_{\beta_1} &:= +2^{-1}\sigma_1 h_1 - 2^{-1}\sigma_2 h_2 - 2^{-1}\sigma_3 h_3, \\ h_{\beta_2} &:= -2^{-1}\sigma_1 h_1 + 2^{-1}\sigma_2 h_2 - 2^{-1}\sigma_3 h_3, \\ h_{\beta_3} &:= -2^{-1}\sigma_1 h_1 - 2^{-1}\sigma_2 h_2 + 2^{-1}\sigma_3 h_3, \\ h_\theta &:= +2^{-1}\sigma_1 h_1 + 2^{-1}\sigma_2 h_2 + 2^{-1}\sigma_3 h_3. \end{aligned}$$

In short, the table is the following ($\forall i, j \in \{1, 2, 3\}, \epsilon_1, \epsilon_2, \epsilon_3 \in \{+, -\}$):

$$\begin{aligned} [h_i, h_j] &= 0, & [e_i, e_j] &= 0, & [f_i, f_j] &= 0, \\ [h_i, e_j] &= 2\delta_{ij}e_j, & [h_i, f_j] &= -2\delta_{ij}f_j, & [e_i, f_j] &= \delta_{ij}h_j, \\ [e_i, v_{\epsilon_1, \epsilon_2, \epsilon_3}] &= \delta_{\epsilon_i, -} v_{(-1)^{\delta_{1,i}\epsilon_1}, (-1)^{\delta_{2,i}\epsilon_2}, (-1)^{\delta_{3,i}\epsilon_3}}, \\ [h_i, v_{\epsilon_1, \epsilon_2, \epsilon_3}] &= \epsilon_i v_{\epsilon_1, \epsilon_2, \epsilon_3}, \\ [f_i, v_{\epsilon_1, \epsilon_2, \epsilon_3}] &= \delta_{\epsilon_i, +} v_{(-1)^{\delta_{1,i}\epsilon_1}, (-1)^{\delta_{2,i}\epsilon_2}, (-1)^{\delta_{3,i}\epsilon_3}}, \\ [v_{+, +, +}, v_{+, -, -}] &= -\sigma_1 e_1, & [v_{+, +, +}, v_{-, +, -}] &= -\sigma_2 e_2, \\ [v_{+, +, +}, v_{-, -, +}] &= -\sigma_3 e_3, & [v_{+, +, -, v_{+, -, +}}] &= +\sigma_1 e_1, \\ [v_{+, +, -, v_{-, +, +}}] &= +\sigma_2 e_2, & [v_{+, -, +}, v_{-, +, +}] &= +\sigma_3 e_3, \\ [v_{+, +, +}, v_{-, -, -}] &= +2^{-1}\sigma_1 h_1 + 2^{-1}\sigma_2 h_2 + 2^{-1}\sigma_3 h_3 = h_\theta, \\ [v_{+, +, -, v_{-, -, +}}] &= -2^{-1}\sigma_1 h_1 - 2^{-1}\sigma_2 h_2 + 2^{-1}\sigma_3 h_3 = h_{\beta_3}, \\ [v_{+, -, +}, v_{-, +, -}] &= -2^{-1}\sigma_1 h_1 + 2^{-1}\sigma_2 h_2 - 2^{-1}\sigma_3 h_3 = h_{\beta_2}, \\ [v_{-, +, +}, v_{+, -, -}] &= +2^{-1}\sigma_1 h_1 - 2^{-1}\sigma_2 h_2 - 2^{-1}\sigma_3 h_3 = h_{\beta_1}, \\ [v_{-, +, +}, v_{-, -, -}] &= +\sigma_1 f_1, & [v_{+, -, +}, v_{-, -, -}] &= +\sigma_2 f_2, \\ [v_{+, +, -, v_{-, -, -}}] &= +\sigma_3 f_3, & [v_{-, +, -, v_{-, -, +}}] &= -\sigma_1 f_1, \\ [v_{+, -, -, v_{-, -, +}}] &= -\sigma_2 f_2, & [v_{+, -, -, v_{-, +, -}}] &= -\sigma_3 f_3. \end{aligned}$$

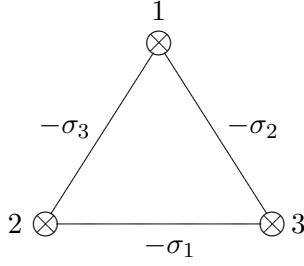
3.2 Kac' realization

We show now how the Lie superalgebras \mathfrak{g}_σ of Section 3.1, for *non-singular values* $\sigma \in V^\times$, can be also realized as contragredient Lie superalgebras (via Kac' method).

3.2.1. First construction. For any $\sigma \in V$, we consider the Cartan matrix

$$A_\sigma = (a_{i,j})_{i=1,2,3}^{j=1,2,3} = \begin{pmatrix} 0 & -\sigma_3 & -\sigma_2 \\ -\sigma_3 & 0 & -\sigma_1 \\ -\sigma_2 & -\sigma_1 & 0 \end{pmatrix}$$

(see [3] for a far-reaching analysis of what else is associated with such a Cartan matrix). We associate to it a Dynkin diagram for which the nodes are defined as in [12, Section 2.5.5], and we join any two vertices i and j with an edge labeled by a_{ij} : the resulting diagram then is



Following Kac, we consider a realization $(\mathfrak{h}, \Pi^\vee = \{H_{\beta_i}\}_{i=1,2,3}, \Pi = \{\beta_i\}_{i=1,2,3})$ of A_σ , that is

- 1) \mathfrak{h} is a \mathbb{C} -vector space,
- 2) Π^\vee is the set of *simple coroots*, a basis of \mathfrak{h} ,
- 3) Π is the set of *simple roots*, a basis of \mathfrak{h}^* ,
- 4) $\beta_j(H_{\beta_i}) = a_{i,j}$ for all $1 \leq i, j \leq 3$.

The (contragredient) Lie superalgebra $\mathfrak{g}(A_\sigma)$ is, by definition, the simple Lie superalgebra defined as follows. First, let $\tilde{\mathfrak{g}}(A_\sigma)$ be the Lie superalgebra with the nine generators $H_{\beta_i}, X_{\pm\beta_i}$ ($i = 1, 2, 3$), and relations (for $1 \leq i, j \leq 3$)

$$\begin{aligned} [H_{\beta_i}, H_{\beta_j}] &= 0, & [H_{\beta_i}, X_{\pm\beta_j}] &= \pm\beta_j(H_{\beta_i})X_{\pm\beta_j}, \\ [X_{\beta_i}, X_{-\beta_j}] &= \delta_{i,j}H_{\beta_i}, & [X_{\pm\beta_i}, X_{\pm\beta_i}] &= 0 \end{aligned}$$

with parity $|H_{\beta_i}| = \bar{0}$ and $|X_{\pm\beta_i}| = \bar{1}$ for all i . Then one considers the maximal homogeneous ideal I_σ of $\tilde{\mathfrak{g}}(A)$ which meets trivially the \mathbb{C} -span of the generators, and finally defines $\mathfrak{g}(A_\sigma) := \tilde{\mathfrak{g}}(A_\sigma)/I_\sigma$.

A straightforward (and easy) analysis shows that $\mathfrak{g}(A_\sigma)$ is finite-dimensional. Then, by general results (cf. [12, Section 2.5.1]), there exists an epimorphism $\Phi_\sigma: \mathfrak{g}_\sigma \twoheadrightarrow \mathfrak{g}(A_\sigma)$ of Lie superalgebras uniquely determined by

$$\begin{aligned} h_{\beta_1} &\mapsto H_{\beta_1}, & h_{\beta_2} &\mapsto H_{\beta_2}, & h_{\beta_3} &\mapsto H_{\beta_3}, \\ v_{-,+,+} &\mapsto X_{+\beta_1}, & v_{+,-,+} &\mapsto X_{+\beta_2}, & v_{+,-,-} &\mapsto X_{+\beta_3}, \\ v_{+,-,-} &\mapsto X_{-\beta_1}, & v_{-,-,-} &\mapsto X_{-\beta_2}, & v_{-,-,+} &\mapsto X_{-\beta_3}. \end{aligned}$$

When σ is non-singular, that is $\sigma \in V^\times$, this epimorphism Φ_σ is actually an isomorphism, so that $\mathfrak{osp}(4, 2; \sigma) =: \mathfrak{g}_\sigma \cong \mathfrak{g}(A_\sigma)$. One can see it in two ways: first, since \mathfrak{g}_σ is simple for $\sigma \in V^\times$, the kernel of Φ_σ is then necessarily trivial; second, direct inspection shows that for $\sigma \in V^\times$ the ideal I_σ has the ‘‘correct’’ codimension in $\tilde{\mathfrak{g}}(A_\sigma)$ so that Φ_σ be injective.

We assume now, for the rest of the present subsection, that $\sigma \in V^\times$ (non-singular case).

Then the set Δ^+ of positive roots of $\mathfrak{g}(A_\sigma)$ has the following description:

$$\Delta^+ = \{\beta_1, \beta_2, \beta_3, \beta_1 + \beta_2, \beta_2 + \beta_3, \beta_3 + \beta_1, \beta_1 + \beta_2 + \beta_3\}.$$

This can be seen as a consequence of $\mathfrak{g}_\sigma \cong \mathfrak{g}(A_\sigma)$, or more directly by inspection (namely, describing I_σ explicitly). The set of roots then is $\Delta = \Delta^+ \cup \Delta^-$, with $\Delta^- := -\Delta^+$.

The dual \mathfrak{h}^* of the Cartan subalgebra has the following description. Let $\{\varepsilon_i\}_{i=1,2,3} \subset \mathfrak{h}^*$ be an orthogonal basis normalized by the conditions $(\varepsilon_i, \varepsilon_i) = -\frac{1}{2}\sigma_i$ ($i = 1, 2, 3$); then one can verify that $(\beta_i, \beta_j) = -\sigma_k$ with $\{i, j, k\} = \{1, 2, 3\}$, where the simple roots are

$$\beta_1 = -\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \quad \beta_2 = \varepsilon_1 - \varepsilon_2 + \varepsilon_3, \quad \beta_3 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3.$$

Now let Δ_0^+ and Δ_1^+ be the set of even (resp. odd) positive roots. One has

$$\Delta_0^+ = \{\beta_1 + \beta_2, \beta_2 + \beta_3, \beta_3 + \beta_1\} = \{2\varepsilon_i \mid 1 \leq i \leq 3\}, \quad \Delta_1^+ = \{\beta_1, \beta_2, \beta_3, \theta\},$$

where $\theta := \beta_1 + \beta_2 + \beta_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ is the highest root.

We introduce now further root vectors and coroots, defined by

$$\begin{aligned} X_{2\varepsilon_1} &:= [X_{\beta_2}, X_{\beta_3}], & X_{2\varepsilon_2} &:= [X_{\beta_3}, X_{\beta_1}], & X_{2\varepsilon_3} &:= [X_{\beta_1}, X_{\beta_2}], \\ X_{-2\varepsilon_1} &:= -[X_{-\beta_2}, X_{-\beta_3}], & X_{-2\varepsilon_2} &:= -[X_{-\beta_3}, X_{-\beta_1}], & X_{-2\varepsilon_3} &:= -[X_{-\beta_1}, X_{-\beta_2}], \\ H_{2\varepsilon_1} &:= -(H_{\beta_2} + H_{\beta_3}), & H_{2\varepsilon_2} &:= -(H_{\beta_3} + H_{\beta_1}), & H_{2\varepsilon_3} &:= -(H_{\beta_1} + H_{\beta_2}). \end{aligned}$$

It can be checked that, for $i, j \in \{1, 2, 3\}$, one has

$$[X_{2\varepsilon_i}, X_{-2\varepsilon_j}] = \sigma_i \delta_{i,j} H_{2\varepsilon_i}, \quad [H_{2\varepsilon_i}, X_{\pm 2\varepsilon_j}] = \pm 2\sigma_i \delta_{i,j} X_{\pm 2\varepsilon_j}, \quad (3.1)$$

which implies that each $\mathfrak{a}_i := \mathbb{C}X_{2\varepsilon_i} \oplus \mathbb{C}H_{2\varepsilon_i} \oplus \mathbb{C}X_{-2\varepsilon_i}$ (for $1 \leq i \leq 3$) is a Lie sub-(super)algebra, with $[\mathfrak{a}_j, \mathfrak{a}_k] = 0$ for $j \neq k$, and \mathfrak{a}_i is isomorphic to $\mathfrak{sl}(2)$ since $\sigma_i \neq 0$. In particular, the even part of the Lie superalgebra $\mathfrak{g}(A_\sigma)$ is nothing but $\mathfrak{g}(A_\sigma)_0 = \bigoplus_{i=1}^3 \mathfrak{a}_i$.

For $i = 1, 2, 3$ we set $X_\theta^i = [X_{2\varepsilon_i}, X_{\beta_i}] \in \mathfrak{g}(A_\sigma)_\theta$ and $X_{-\theta}^i = [X_{-2\varepsilon_i}, X_{-\beta_i}] \in \mathfrak{g}(A_\sigma)_{-\theta}$; then the following identities hold:

$$\sum_{i=1}^3 X_{\pm\theta}^i = 0, \quad [X_\theta^j, X_{-\theta}^k] = -\sigma_j \sigma_k \sum_{i=1}^3 H_{\beta_i}. \quad (3.2)$$

These formulas imply that there exists $X_\theta \in \mathfrak{g}(A_\sigma)_\theta$ and $X_{-\theta} \in \mathfrak{g}(A_\sigma)_{-\theta}$ such that

$$X_\theta^i = \sigma_i X_\theta, \quad X_{-\theta}^i = \sigma_i X_{-\theta}, \quad (3.3)$$

for any $1 \leq i \leq 3$. Hence, setting $H_\theta = -(H_{\beta_1} + H_{\beta_2} + H_{\beta_3})$, one has also $[X_\theta, X_{-\theta}] = H_\theta$. Moreover, it also follows that $H_{2\varepsilon_1} + H_{2\varepsilon_2} + H_{2\varepsilon_3} = 2H_\theta$. Eventually, from all this we see that the odd part $\mathfrak{g}(A_\sigma)_1$ of the Lie superalgebra $\mathfrak{g}(A_\sigma)$ is the \mathbb{C} -span of $\{X_{\pm\beta_i}\}_{i=1,2,3} \cup \{X_{\pm\theta}\}$.

Finally, from the previous description we see that the isomorphism $\mathfrak{osp}(4, 2; \sigma) \cong \mathfrak{g}(A_\sigma)$ can be described on basis elements by

$$\begin{aligned} h_i &\mapsto \sigma_i^{-1} H_{2\varepsilon_i}, & e_i &\mapsto \sigma_i^{-1} X_{2\varepsilon_i}, & f_i &\mapsto \sigma_i^{-1} X_{-2\varepsilon_i}, & i &= 1, 2, 3, \\ v_{-,+,+} &\mapsto X_{\beta_1}, & v_{+,-,+} &\mapsto X_{\beta_2}, & v_{+,+,-} &\mapsto X_{\beta_3}, & v_{+,+,+} &\mapsto X_\theta, \\ v_{+,-,-} &\mapsto X_{-\beta_1}, & v_{-,-,+} &\mapsto X_{-\beta_2}, & v_{-,-,-} &\mapsto X_{-\beta_3}, & v_{-,-,-} &\mapsto X_{-\theta}. \end{aligned} \quad (3.4)$$

Note that in first line of (3.4) the non-singularity of $\sigma \in V^\times$ plays a key role!

3.2.2. Second construction. For the reader's convenience, we present now a second construction based upon a different, more familiar Cartan matrix (and associated Dynkin diagram). The link with the previous construction of Section 3.2.1 is through the application of the odd reflection with respect to the root β_2 , following V. Serganova (see [16]).

To begin with, set $\alpha_1 = \beta_2 + \beta_3$, $\alpha_2 = -\beta_2$, $\alpha_3 = \beta_1 + \beta_2$. Then $\Pi' := \{\alpha_i\}_{i=1,2,3}$ is another set of simple roots of $\mathfrak{g}(A_\sigma)$, which is not Weyl-group conjugate to Π ; the corresponding set of coroots $(\Pi')^\vee = \{h_{\alpha_i}\}_{i=1,2,3}$ should be taken as $h_{\alpha_1} = H_{2\varepsilon_1}$, $h_{\alpha_2} = H_{\beta_2}$, $h_{\alpha_3} = H_{2\varepsilon_3}$. With such a choice, the associated Cartan matrix $A'_\sigma := (a'_{i,j} := \alpha_j(h_{\alpha_i}))_{i=1,2,3}^{j=1,2,3}$ is given by

$$A'_\sigma = \begin{pmatrix} 2\sigma_1 & -\sigma_1 & 0 \\ -\sigma_1 & 0 & -\sigma_3 \\ 0 & -\sigma_3 & 2\sigma_3 \end{pmatrix} = \sigma_1 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -\frac{\sigma_3}{\sigma_1} \\ 0 & -\frac{\sigma_3}{\sigma_1} & 2\frac{\sigma_3}{\sigma_1} \end{pmatrix},$$

where the second equality is available only if $\sigma_1 \neq 0$. In particular, for $\sigma_1 \neq 0$ and $\sigma_3 \neq 0$, our original $\mathfrak{g}(A_\sigma)$ can be also defined via the Cartan matrix

$$A'_a = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & a \\ 0 & -1 & 2 \end{pmatrix}$$

with $a := \frac{\sigma_3}{\sigma_1}$. When $a \neq -1$, this in turn corresponds – following Kac’ conventions, up to renumbering the vertices (cf. [12, Section 2.5]) – to the simple Lie superalgebra attached to the following *distinguished* (i.e., with just one odd vertex) Dynkin diagram of type $D(2, 1; a)$

$$\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \\ \circ & \text{---} \otimes \text{---} & \circ \\ & \underset{1}{\circ} & \underset{a}{\circ} \end{array}$$

(instead, $a = -1$ corresponds to $\sigma_2 = 0$, that is a *singular* case). Kac’ results tell us that, for all $a \in \mathbb{C} \setminus \{-1, 0\}$, the set of positive roots with respect to Π' is given by

$$\Delta'^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3\},$$

while the coroots can be expressed as

$$\begin{aligned} h_{\alpha_1} &= H_{\beta_2 + \beta_3} = -(H_{\beta_2} + H_{\beta_3}), & h_{\alpha_2} &= H_{-\beta_2} = H_{\beta_2}, \\ h_{\alpha_3} &= H_{\beta_2 + \beta_1} = -(H_{\beta_2} + H_{\beta_1}), \\ h_{\alpha_1 + \alpha_2} &= -H_{\beta_3} = h_{\alpha_1} + h_{\alpha_2}, & h_{\alpha_3 + \alpha_2} &= -H_{\beta_1} = h_{\alpha_3} + h_{\alpha_2}, \\ h_{\alpha_1 + \alpha_2 + \alpha_3} &= H_{\beta_1 + \beta_2 + \beta_3} = -H_{\beta_1} - H_{\beta_2} - H_{\beta_3} = h_{\alpha_1} + h_{\alpha_2} + h_{\alpha_3}, \\ h_{\alpha_1 + 2\alpha_2 + \alpha_3} &= H_{\beta_1 + \beta_3} = -H_{\beta_1} - H_{\beta_3} = h_{\alpha_1} + 2h_{\alpha_2} + h_{\alpha_3}. \end{aligned}$$

3.2.3. The singular case. We saw in Section 3.2.1 that for every non-singular parameter $\sigma \in V^\times$, we have a Lie superalgebra isomorphism $\Phi_\sigma: \mathfrak{osp}(4, 2; \sigma) = \mathfrak{g}_\sigma \xrightarrow{\cong} \mathfrak{g}(A_\sigma)$. For singular values $\sigma \in V \setminus V^\times$, instead, the epimorphism Φ_σ is no longer an isomorphism: indeed, in this case, the Lie ideal I_σ in $\tilde{\mathfrak{g}}(A_\sigma)$ is bigger than in the non-singular case, for instance (as a straightforward calculation shows), we have

$$[X_{\beta_i}, X_{\beta_j}] \in I_\sigma \iff \sigma_k = 0 \quad \forall \{i, j, k\} = \{1, 2, 3\}.$$

Similarly, the contragredient Lie superalgebra $\mathfrak{g}(A'_a)$ of Section 3.2.2 can be defined also for the “singular value” $a = -1$ (corresponding to $\sigma_1 + \sigma_3 = 0$, which is equivalent to $\sigma_2 = 0$). However, in this case the corresponding Lie ideal I'_a in $\tilde{\mathfrak{g}}(A'_a)$ is bigger, as we have, for instance,

$$[[[X_{\alpha_1}, X_{\alpha_2}], X_{\alpha_3}], X_{\alpha_2}] \in I'_a \iff a = -1,$$

therefore $\mathfrak{g}(A'_{a=-1})$ has strictly smaller dimension than $\mathfrak{osp}(4, 2; \sigma)$ for $\sigma := (1, 0, -1)$, whereas $\mathfrak{g}(A'_a) \cong \mathfrak{g}_\sigma := \mathfrak{osp}(4, 2; \sigma)$ via Φ_σ for all $\sigma = (1, a + 1, a) \neq (1, 0, -1)$.

This shows that, in a sense, describing our objects $\mathfrak{g}_\sigma = \mathfrak{osp}(4, 2; \sigma)$ of Section 3.1 as contragredient Lie superalgebras is problematic, so to say, at singular values of σ , in that the contragredient construction yields not the outcome we are looking for.

3.3 Bases of \mathfrak{g}_σ

In this subsection, for any given $\sigma \in V^\times$ we sort out three special bases of \mathfrak{g}_σ . Later on (in Section 4) we use them to construct three different families, indexed by V , of Lie superalgebras: by construction these families will coincide on all non-singular parameters $\sigma \in V^\times$ but will actually differ instead on singular values $\sigma \in V \setminus V^\times$.

3.3.1. First basis. Let $B := \{H_{2\varepsilon_1}, H_{2\varepsilon_2}, H_{2\varepsilon_3}\} \cup \{X_\alpha\}_{\alpha \in \Delta}$ and $B_+ := B \cup \{H_\theta\}$ be the subsets of \mathfrak{g}_σ whose elements are defined by

$$\begin{aligned} H_{2\varepsilon_i} &:= \sigma_i h_i, & X_{2\varepsilon_i} &:= \sigma_i e_i, & X_{-2\varepsilon_i} &:= \sigma_i f_i, & i &= 1, 2, 3, & H_\theta &:= 2^{-1} \sum_{i=1}^3 \sigma_i h_i, \\ X_{\beta_1} &:= v_{-,+,+}, & X_{\beta_2} &:= v_{+,-,+}, & X_{\beta_3} &:= v_{+,+,-}, & X_\theta &:= v_{+,+,+}, \\ X_{-\beta_1} &:= v_{+,-,-}, & X_{-\beta_2} &:= v_{-,+,-}, & X_{-\beta_3} &:= v_{-,-,+}, & X_{-\theta} &:= v_{-,-,-}, \end{aligned} \quad (3.5)$$

(cf. Section 3.1.1); the analysis in Section 3.1.1 tells us that, for every $\sigma \in V^\times$, the set B is a \mathbb{C} -basis of \mathfrak{g}_σ , hence B_+ is a spanning set. Again from Section 3.1 – in particular Section 3.1.2 – for the Lie brackets among the elements of B_+ we find the following multiplication table

$$\begin{aligned} [H_{2\varepsilon_i}, H_{2\varepsilon_j}] &= 0, & [H_{2\varepsilon_i}, X_{\pm 2\varepsilon_j}] &= \pm 2\delta_{i,j} \sigma_i X_{\pm 2\varepsilon_j}, \\ [X_{2\varepsilon_i}, X_{2\varepsilon_j}] &= 0, & [X_{-2\varepsilon_i}, X_{-2\varepsilon_j}] &= 0, & [X_{2\varepsilon_i}, X_{-2\varepsilon_j}] &= \delta_{i,j} \sigma_i H_{2\varepsilon_i}, \\ [H_{2\varepsilon_i}, X_{\pm \beta_j}] &= \pm (-1)^{\delta_{i,j}} \sigma_i X_{\pm \beta_j}, & [H_{2\varepsilon_i}, X_{\pm \theta}] &= \pm \sigma_i X_{\pm \theta}, \\ [H_\theta, X_{\pm 2\varepsilon_i}] &= \pm \sigma_i X_{2\varepsilon_i}, & [H_\theta, X_{\pm \beta_i}] &= \mp \sigma_i X_{\pm \beta_i}, & [H_\theta, X_{\pm \theta}] &= 0, \\ [X_{2\varepsilon_i}, X_{\beta_j}] &= \delta_{i,j} \sigma_i X_\theta, & [X_{2\varepsilon_i}, X_{-\beta_j}] &= (1 - \delta_{i,j}) \sigma_i X_{\beta_k}, \\ [X_{-2\varepsilon_i}, X_{\beta_j}] &= (1 - \delta_{i,j}) \sigma_i X_{-\beta_k}, & [X_{-2\varepsilon_i}, X_{-\beta_j}] &= \delta_{i,j} \sigma_i X_{-\theta}, \\ [X_{2\varepsilon_i}, X_\theta] &= 0, & [X_{2\varepsilon_i}, X_{-\theta}] &= \sigma_i X_{-\beta_i}, & [X_{-2\varepsilon_i}, X_\theta] &= \sigma_i X_{\beta_i}, & [X_{-2\varepsilon_i}, X_{-\theta}] &= 0, \\ [X_{\beta_i}, X_{\beta_j}] &= (1 - \delta_{i,j}) X_{2\varepsilon_k}, & [X_{-\beta_i}, X_{-\beta_j}] &= -(1 - \delta_{i,j}) X_{-2\varepsilon_k}, \\ [X_{\beta_i}, X_{-\beta_j}] &= \delta_{i,j} (H_{2\varepsilon_i} - H_\theta), \\ [X_{\beta_i}, X_\theta] &= 0, & [X_{\beta_i}, X_{-\theta}] &= X_{-2\varepsilon_i}, & [X_{-\beta_i}, X_\theta] &= -X_{2\varepsilon_i}, & [X_{-\beta_i}, X_{-\theta}] &= 0, \\ [X_\theta, X_\theta] &= 0, & [X_\theta, X_{-\theta}] &= H_\theta, & [X_{-\theta}, X_{-\theta}] &= 0 \end{aligned}$$

for all $i, j \in \{1, 2, 3\}$, with $k \in \{1, 2, 3\} \setminus \{i, j\}$.

3.3.2. Second basis. Let now $B' := \{H'_{2\varepsilon_1}, H'_{2\varepsilon_2}, H'_{2\varepsilon_3}\} \cup \{X'_\alpha\}_{\alpha \in \Delta}$ and $B'_+ := B' \cup \{H'_\theta\}$ be the subsets of \mathfrak{g}_σ with elements

$$\begin{aligned} H'_{2\varepsilon_i} &:= h_i, & X'_{+2\varepsilon_i} &:= e_i, & X'_{-2\varepsilon_i} &:= f_i, & i &= 1, 2, 3, & H'_\theta &:= 2^{-1} \sum_{i=1}^3 \sigma_i h_i, \\ X'_{\beta_1} &:= v_{-,+,+}, & X'_{\beta_2} &:= v_{+,-,+}, & X'_{\beta_3} &:= v_{+,+,-}, & X'_\theta &:= v_{+,+,+}, \\ X'_{-\beta_1} &:= v_{+,-,-}, & X'_{-\beta_2} &:= v_{-,+,-}, & X'_{-\beta_3} &:= v_{-,-,+}, & X'_{-\theta} &:= v_{-,-,-}, \end{aligned} \quad (3.6)$$

(cf. Section 3.1.1). Again from Section 3.1.1 we see that, for every $\sigma \in V$ (including also the singular locus), B' is a \mathbb{C} -basis of \mathfrak{g}_σ , so B'_+ is a spanning set: indeed, B' is nothing but a different notation for the natural, built-in \mathbb{C} -basis of \mathfrak{g}_σ in Kaplansky's realization (cf. Section 3.1.1). Then from Section 3.1.2 we get the following multiplication table for Lie brackets among elements of B'_+

$$\begin{aligned} [H'_{2\varepsilon_i}, H'_{2\varepsilon_j}] &= 0, & [H'_{2\varepsilon_i}, X'_{\pm 2\varepsilon_j}] &= \pm 2\delta_{i,j} X'_{\pm 2\varepsilon_j}, \\ [X'_{2\varepsilon_i}, X'_{2\varepsilon_j}] &= 0, & [X'_{-2\varepsilon_i}, X'_{-2\varepsilon_j}] &= 0, & [X'_{2\varepsilon_i}, X'_{-2\varepsilon_j}] &= \delta_{i,j} H'_{2\varepsilon_i}, \\ [H'_{2\varepsilon_i}, X'_{\pm \beta_j}] &= \pm (-1)^{\delta_{i,j}} X'_{\pm \beta_j}, & [H'_{2\varepsilon_i}, X'_{\pm \theta}] &= \pm X'_{\pm \theta}, \\ [H'_\theta, X'_{\pm 2\varepsilon_i}] &= \pm \sigma_i X'_{2\varepsilon_i}, & [H'_\theta, X'_{\pm \beta_i}] &= \mp \sigma_i X'_{\pm \beta_i}, & [H'_\theta, X'_{\pm \theta}] &= 0, \\ [X'_{2\varepsilon_i}, X'_{\beta_j}] &= \delta_{i,j} X'_\theta, & [X'_{2\varepsilon_i}, X'_{-\beta_j}] &= (1 - \delta_{i,j}) X'_{\beta_k}, \\ [X'_{-2\varepsilon_i}, X'_{\beta_j}] &= (1 - \delta_{i,j}) X'_{-\beta_k}, & [X'_{-2\varepsilon_i}, X'_{-\beta_j}] &= \delta_{i,j} X'_{-\theta}, \end{aligned}$$

$$\begin{aligned}
[X'_{2\varepsilon_i}, X'_\theta] &= 0, & [X'_{2\varepsilon_i}, X'_{-\theta}] &= X'_{-\beta_i}, & [X'_{-2\varepsilon_i}, X'_\theta] &= X'_{\beta_i}, & [X'_{-2\varepsilon_i}, X'_{-\theta}] &= 0, \\
[X'_{\beta_i}, X'_{\beta_j}] &= (1 - \delta_{i,j})\sigma_i X'_{2\varepsilon_k}, & [X'_{-\beta_i}, X'_{-\beta_j}] &= -(1 - \delta_{i,j})\sigma_i X'_{-2\varepsilon_k}, \\
[X'_{\beta_i}, X'_{-\beta_j}] &= \delta_{i,j}(\sigma_i H'_{2\varepsilon_i} - H'_\theta), \\
[X'_{\beta_i}, X'_\theta] &= 0, & [X'_{\beta_i}, X'_{-\theta}] &= \sigma_i X'_{-2\varepsilon_i}, & [X'_{-\beta_i}, X'_\theta] &= -\sigma_i X'_{2\varepsilon_i}, & [X'_{-\beta_i}, X'_{-\theta}] &= 0, \\
[X'_\theta, X'_\theta] &= 0, & [X'_\theta, X'_{-\theta}] &= H'_\theta, & [X'_{-\theta}, X'_{-\theta}] &= 0,
\end{aligned}$$

for all $i, j \in \{1, 2, 3\}$, with $k \in \{1, 2, 3\} \setminus \{i, j\}$.

3.3.3. Third basis. Let now $\sigma \in V^\times$ (generic case again!). We fix as third basis (and spanning set) of \mathfrak{g}_σ a suitable blending of the two ones in Sections 3.3.1 and 3.3.2 above. Namely, we set $B'' := \{H'_{2\varepsilon_1}, H'_{2\varepsilon_2}, H'_{2\varepsilon_3}\} \cup \{X_\alpha\}_{\alpha \in \Delta}$ and $B''_+ := B'' \cup \{H'_\theta\}$, with elements defined by (3.6) and (3.5). Then B'' is another \mathbb{C} -basis, and B''_+ another spanning set, of \mathfrak{g}_σ .

For later use we record hereafter the complete table of Lie brackets among elements of B''_+ , which can be argued at once from those for B_+ and B'_+ in Sections 3.5 and 3.6, respectively:

$$\begin{aligned}
[H'_{2\varepsilon_i}, H'_{2\varepsilon_j}] &= 0, & [H'_{2\varepsilon_i}, X_{\pm 2\varepsilon_j}] &= \pm 2\delta_{i,j} X_{\pm 2\varepsilon_j}, \\
[X_{2\varepsilon_i}, X_{2\varepsilon_j}] &= 0, & [X_{-2\varepsilon_i}, X_{-2\varepsilon_j}] &= 0, & [X_{2\varepsilon_i}, X_{-2\varepsilon_j}] &= \delta_{i,j} \sigma_i^2 H'_{2\varepsilon_i}, \\
[H'_{2\varepsilon_i}, X_{\pm \beta_j}] &= \pm (-1)^{\delta_{i,j}} X_{\pm \beta_j}, & [H'_{2\varepsilon_i}, X_{\pm \theta}] &= \pm X_{\pm \theta}, \\
[H'_\theta, X_{\pm 2\varepsilon_i}] &= \pm \sigma_i X_{2\varepsilon_i}, & [H'_\theta, X_{\pm \beta_i}] &= \mp \sigma_i X_{\pm \beta_i}, & [H'_\theta, X_{\pm \theta}] &= 0, \\
[X_{2\varepsilon_i}, X_{\beta_j}] &= \delta_{i,j} \sigma_i X_\theta, & [X_{2\varepsilon_i}, X_{-\beta_j}] &= (1 - \delta_{i,j}) \sigma_i X_{\beta_k}, \\
[X_{-2\varepsilon_i}, X_{\beta_j}] &= (1 - \delta_{i,j}) \sigma_i X_{-\beta_k}, & [X_{-2\varepsilon_i}, X_{-\beta_j}] &= \delta_{i,j} \sigma_i X_{-\theta}, \\
[X_{2\varepsilon_i}, X_\theta] &= 0, & [X_{2\varepsilon_i}, X_{-\theta}] &= \sigma_i X_{-\beta_i}, & [X_{-2\varepsilon_i}, X_\theta] &= \sigma_i X_{\beta_i}, \\
[X_{-2\varepsilon_i}, X_{-\theta}] &= 0, & [X_{\beta_i}, X_{\beta_j}] &= (1 - \delta_{i,j}) X_{2\varepsilon_k}, \\
[X_{-\beta_i}, X_{-\beta_j}] &= -(1 - \delta_{i,j}) X_{-2\varepsilon_k}, & [X_{\beta_i}, X_{-\beta_j}] &= \delta_{i,j} (\sigma_i H'_{2\varepsilon_i} - H'_\theta), \\
[X_{\beta_i}, X_\theta] &= 0, & [X_{\beta_i}, X_{-\theta}] &= X_{-2\varepsilon_i}, & [X_{-\beta_i}, X_\theta] &= -X_{2\varepsilon_i}, & [X_{-\beta_i}, X_{-\theta}] &= 0, \\
[X_\theta, X_\theta] &= 0, & [X_\theta, X_{-\theta}] &= H'_\theta, & [X_{-\theta}, X_{-\theta}] &= 0,
\end{aligned}$$

for all $i, j \in \{1, 2, 3\}$, with $k \in \{1, 2, 3\} \setminus \{i, j\}$.

4 Integral forms & degenerations for Lie superalgebras of type $D(2, 1; \sigma)$

Let \mathfrak{l} be any Lie (super)algebra over a field \mathbb{K} , and R any subring of \mathbb{K} . By *integral form of \mathfrak{l} over R* , or *(integral) R -form of \mathfrak{l}* , we mean by definition any Lie R -sub(super)algebra \mathfrak{t}_R of \mathfrak{l} whose scalar extension to \mathbb{K} is \mathfrak{l} itself: in other words $\mathbb{K} \otimes_R \mathfrak{t}_R \cong \mathfrak{l}$ as Lie (super)algebras over \mathbb{K} . In this subsection we introduce five particular integral forms of $\mathfrak{l} = \mathfrak{g}_\sigma$, and study some remarkable specializations of them. Let $\Delta := \Delta^+ \cup (-\Delta^+)$ be the root system of \mathfrak{g}_σ .

From now on, for any $\sigma := (\sigma_1, \sigma_2, \sigma_3) \in V := \{\sigma \in \mathbb{C}^3 \mid \sum_{i=1}^3 \sigma_i = 0\}$ we denote by $\mathbb{Z}[\sigma]$ the (unital) subring of \mathbb{C} generated by $\{\sigma_1, \sigma_2, \sigma_3\}$.

We warn the reader that the choice of a $\mathbb{Z}[\sigma]$ -form becomes very important when one considers a singular degeneration: one cannot speak instead of *the* singular degeneration, in that any degeneration actually depends not only on the specific specialization value taken by σ but also on the previous choice of a specific $\mathbb{Z}[\sigma]$ -form, that must be fixed in advance. Some specific features of this phenomenon are presented in Theorems 4.1, 4.2, 4.3 etc.

4.1 First family: the Lie superalgebras $\mathfrak{g}(\sigma)$

4.1.1. Construction of the $\mathfrak{g}(\sigma)$'s. For any $\sigma \in V$ (cf. Section 3.1), let $\mathfrak{g}_z(\sigma)$ be the Lie superalgebra over $\mathbb{Z}[\sigma]$ defined as follows. As a $\mathbb{Z}[\sigma]$ -module, $\mathfrak{g}_z(\sigma)$ is spanned by the formal set of $\mathbb{Z}[\sigma]$ -linear generators $B_{\mathfrak{g}} := \{H_{2\varepsilon_1}, H_{2\varepsilon_2}, H_{2\varepsilon_3}, H_{\theta}\} \cup \{X_{\alpha}\}_{\alpha \in \Delta}$ subject only to the single relation $H_{2\varepsilon_1} + H_{2\varepsilon_2} + H_{2\varepsilon_3} = 2H_{\theta}$. Then it follows, in particular, that $\mathfrak{g}_z(\sigma)$ is clearly a free $\mathbb{Z}[\sigma]$ -module, with basis $B_{\mathfrak{g}} \setminus \{H_{2\varepsilon_i}\}$ for any $1 \leq i \leq 3$. The Lie superalgebra structure of $\mathfrak{g}_z(\sigma)$ is defined by the formulas in Section 3.3.1, now taken as definitions of the Lie brackets among the (linear) generators of $\mathfrak{g}_z(\sigma)$ itself. Overall, all these $\mathfrak{g}_z(\sigma)$'s form a family of Lie superalgebras indexed by the points of the complex plane V . Moreover, taking $\mathfrak{g}(\sigma) := \mathbb{C} \otimes_{\mathbb{Z}[\sigma]} \mathfrak{g}_z(\sigma)$ for all $\sigma \in V$ we find a more regular situation, as now these (extended) Lie superalgebras $\mathfrak{g}(\sigma)$ all share \mathbb{C} as their common ground ring. Moreover,

$$\text{in the non-singular case, i.e., for } \sigma_i \in V^{\times}, \text{ we have } \mathfrak{g}(\sigma) \cong \mathfrak{g}_{\sigma} \quad (4.1)$$

by Section 3.3.1 and the very definition of $\mathfrak{g}(\sigma)$ itself. Indeed, the analysis in Section 3.3.1 – describing a \mathbb{C} -basis and its multiplication table for \mathfrak{g}_{σ} with $\sigma \in V^{\times}$ – prove that *for all* $\sigma \in V^{\times}$, the Lie superalgebra $\mathfrak{g}_z(\sigma)$ over $\mathbb{Z}[\sigma]$ identifies to an integral $\mathbb{Z}[\sigma]$ -form of \mathfrak{g}_{σ} . In order to formalize the description of the family $\{\mathfrak{g}(\sigma)\}_{\sigma \in V}$, we proceed as follows. Let $\mathbb{Z}[\mathbf{x}] := \mathbb{Z}[V] \cong \mathbb{Z}[x_1, x_2, x_3]/(x_1 + x_2 + x_3)$ be the ring of global sections of the \mathbb{Z} -scheme associated with V . In the construction of $\mathfrak{g}_z(\sigma)$, formally replace σ to \mathbf{x} (hence the x_i 's to the σ_i 's): this makes sense, provides a meaningful definition of a Lie superalgebra over $\mathbb{Z}[\mathbf{x}]$, denoted by $\mathfrak{g}_z(\mathbf{x})$, and then $\mathfrak{g}(\mathbf{x}) := \mathbb{C}[\mathbf{x}] \otimes_{\mathbb{Z}[\mathbf{x}]} \mathfrak{g}_z(\mathbf{x})$ by scalar extension, which is a Lie superalgebra over $\mathbb{C}[\mathbf{x}] := \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{x}]$. Definitions imply that, for any $\sigma \in V$, we have a Lie $\mathbb{Z}[\sigma]$ -superalgebra isomorphism

$$\mathfrak{g}_z(\sigma) \cong \mathbb{Z}[\sigma] \otimes_{\mathbb{Z}[\mathbf{x}]} \mathfrak{g}_z(\mathbf{x})$$

– through the ring isomorphism $\mathbb{Z}[\sigma] \cong \mathbb{Z}[\mathbf{x}]/(x_i - \sigma_i)_{i=1,2,3}$ – and similarly

$$\mathfrak{g}(\sigma) \cong \mathbb{C} \otimes_{\mathbb{C}[\mathbf{x}]} \mathfrak{g}(\mathbf{x})$$

as Lie \mathbb{C} -superalgebras, through the ring isomorphism $\mathbb{C} \cong \mathbb{C}[\mathbf{x}]/(x_i - \sigma_i)_{i=1,2,3}$. In geometrical language, all this can be formulated as follows. The Lie superalgebra $\mathfrak{g}(\mathbf{x})$ – being a free, finite rank $\mathbb{C}[\mathbf{x}]$ -module – defines a coherent sheaf $\mathcal{L}_{\mathfrak{g}_{\mathbb{C}[\mathbf{x}]}}$ of Lie superalgebras over $\text{Spec}(\mathbb{C}[\mathbf{x}])$. Moreover, there exists a unique fibre bundle over $\text{Spec}(\mathbb{C}[\mathbf{x}])$, say $\mathbb{L}_{\mathfrak{g}_{\mathbb{C}[\mathbf{x}]}}$, whose sheaf of sections is exactly $\mathcal{L}_{\mathfrak{g}_{\mathbb{C}[\mathbf{x}]}}$. This fibre bundle can be thought of as a (total) deformation space over the base space $\text{Spec}(\mathbb{C}[\mathbf{x}])$, in which every fibre can be seen as a “deformation” of any other one, and also any single fibre can be seen as a degeneration of the original Lie superalgebra $\mathfrak{g}(\mathbf{x})$. Moreover, the fibres of $\mathbb{L}_{\mathfrak{g}_{\mathbb{C}[\mathbf{x}]}}$ on $\text{Spec}(\mathbb{C}[\mathbf{x}]) = V \cup \{\star\}$ are, by definition, given by $(\mathbb{L}_{\mathfrak{g}_{\mathbb{C}[\mathbf{x}]}})_{\sigma} = \mathbb{C} \otimes_{\mathbb{C}[\mathbf{x}]} \mathfrak{g}(\mathbf{x}) \cong \mathfrak{g}(\sigma)$

for any *closed point* $\sigma \in V \subseteq \text{Spec}(\mathbb{C}[\mathbf{x}])$, while for the *generic point* $\star \in \text{Spec}(\mathbb{C}[\mathbf{x}])$ we have $(\mathbb{L}_{\mathfrak{g}_{\mathbb{C}[\mathbf{x}]}})_{\star} = \mathbb{C}(\mathbf{x}) \otimes_{\mathbb{C}[\mathbf{x}]} \mathfrak{g}(\mathbf{x}) (=:\mathfrak{g}_{\mathbb{C}(\mathbf{x})})$. Finally, it follows from our construction that these sheaf

and fibre bundle do admit an action of $\mathbb{C}^{\times} \times \mathfrak{S}_3$, that on the base space $\text{Spec}(\mathbb{C}[\mathbf{x}]) = V \cup \{\star\}$ simply fixes $\{\star\}$ and is the standard $(\mathbb{C}^{\times} \times \mathfrak{S}_3)$ -action on V . In the next result we describe the structure of these fibres $(\mathbb{L}_{\mathfrak{g}_{\mathbb{C}[\mathbf{x}]}})_{\sigma} \cong \mathfrak{g}(\sigma)$ for all $\sigma \in V$. The outcome is that in the “regular” locus V^{\times} they are simple (as Lie superalgebras), while in the “singular” locus $V \setminus V^{\times}$ they are not, and we can describe explicitly their structure.

Theorem 4.1. *Let $\sigma \in V$ as above, and set $\mathfrak{a}_i := \mathbb{C}X_{2\varepsilon_i} \oplus \mathbb{C}H_{2\varepsilon_i} \oplus \mathbb{C}X_{-2\varepsilon_i}$ with $X_{2\varepsilon_i}$, $H_{2\varepsilon_i}$ and $X_{-2\varepsilon_i}$ as defined in Section 3.3.1, for all $i = 1, 2, 3$.*

- (1) If $\sigma \in V^\times$, then the Lie superalgebra $\mathfrak{g}(\sigma)$ is simple.
- (2) If $\sigma \in V \setminus V^\times$, with $\sigma_i = 0$ and $\sigma_j \neq 0 \neq \sigma_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then \mathfrak{a}_i is a central Lie ideal of $\mathfrak{g}(\sigma)$, isomorphic to $\mathbb{C}^{3|0}$, and $\mathfrak{g}(\sigma)$ is the universal central extension of $\mathfrak{psl}(2|2)$ by \mathfrak{a}_i (cf. [10, Theorem 4.7]); in other words, there exists a short exact sequence of Lie superalgebras

$$0 \longrightarrow \mathbb{C}^{3|0} \cong \mathfrak{a}_i \longrightarrow \mathfrak{g}(\sigma) \longrightarrow \mathfrak{psl}(2|2) \longrightarrow 0.$$

A parallel result also holds true when working with $\mathfrak{g}_z(\sigma)$ over the ground ring $\mathbb{Z}[\sigma]$.

- (3) If $\sigma = \mathbf{0}$ ($\in V \setminus V^\times$), i.e., $\sigma_h = 0$ for all $h \in \{1, 2, 3\}$, then $\mathfrak{g}(\mathbf{0})_{\bar{0}} \cong \mathbb{C}^{9|0}$ is the center of $\mathfrak{g}(\mathbf{0})$, and the quotient $\mathfrak{g}(\mathbf{0})/\mathfrak{g}(\mathbf{0})_{\bar{0}} \cong \mathbb{C}^{0|8}$ is Abelian; in particular, $\mathfrak{g}(\mathbf{0})$ is a non-trivial, non-Abelian central extension of $\mathbb{C}^{0|8}$ by $\mathbb{C}^{9|0}$, i.e., there exists a short exact sequence of Lie superalgebras, with non-Abelian middle term,

$$0 \longrightarrow \mathbb{C}^{9|0} \cong \mathfrak{g}(\mathbf{0})_{\bar{0}} \longrightarrow \mathfrak{g}(\mathbf{0}) \longrightarrow \mathbb{C}^{0|8} \longrightarrow 0.$$

A parallel result holds true when working with $\mathfrak{g}_z(\mathbf{0})$ over the ground ring $\mathbb{Z}[\mathbf{0}] = \mathbb{Z}$.

Proof. Part (1) is a direct consequence of (4.1) and Proposition 3.1. Claim (2) instead follows at once by direct inspection of the formulas in Section 3.3.1. For instance, reading the lines in the first and second line in the table of formulas for Lie brackets therein, we see that \mathfrak{a}_i is Abelian when $\sigma_i = 0$, so that $\mathfrak{a}_i \cong \mathbb{C}^3$ as claimed. Moreover, the formulas from the third to the seventh in the same table tell us also that all brackets of the generators of \mathfrak{a}_i with all other generators turn to zero when $\sigma_i = 0$: thus \mathfrak{a}_i is central in $\mathfrak{g}(\sigma)$ – hence, in particular, it is a Lie ideal – as claimed. Similarly, a direct verification (setting $\sigma_i = 0$ in those formulas) shows that the quotient $\mathfrak{g}(\sigma)/\mathfrak{a}_i$ is isomorphic to $\mathfrak{psl}(2|2)$.

Claim (3) follows again by straightforward analysis of the table of formulas in Section 3.3.1. Indeed, the first two lines in the table describes the structure of the Lie algebra $\mathfrak{g}(\sigma)_{\bar{0}}$: when $\sigma = \mathbf{0}$ they simply tell that this structure is trivial, that is $\mathfrak{g}(\mathbf{0})$ is Abelian, hence isomorphic to \mathbb{C}^9 with trivial Lie bracket. Moreover, the lines from third to seventh in the same table describe the adjoint action of $\mathfrak{g}(\sigma)_{\bar{0}}$ onto $\mathfrak{g}(\sigma)_{\bar{1}}$: when $\sigma = \mathbf{0}$, all the Lie brackets therein turn to zero, so the $\mathfrak{g}(\mathbf{0})_{\bar{0}}$ -action is trivial, which means exactly that $\mathfrak{g}(\mathbf{0})_{\bar{0}}$ is central. Finally, the lines from eighth to eleventh describe the Lie brackets among linear generators of $\mathfrak{g}(\sigma)_{\bar{1}}$: all these brackets are independent of σ , and prove that none of these generators is central. Therefore we can conclude that the center of $\mathfrak{g}(\mathbf{0})$ is exactly $\mathfrak{g}(\mathbf{0})_{\bar{0}}$.

As $\mathfrak{g}(\mathbf{0})_{\bar{0}}$ is the center of $\mathfrak{g}(\mathbf{0})$, the space $\mathfrak{g}(\mathbf{0})/\mathfrak{g}(\mathbf{0})_{\bar{0}}$ bears the quotient Lie superalgebra structure, entirely odd. Since $\mathfrak{g}(\mathbf{0}) = \mathfrak{g}(\mathbf{0})_{\bar{0}} \oplus \mathfrak{g}(\mathbf{0})_{\bar{1}}$, this Lie structure is automatically trivial (no need of looking at formulas whatsoever ...) because $[\mathfrak{g}(\mathbf{0})_{\bar{0}}, \mathfrak{g}(\mathbf{0})_{\bar{0}}] \subseteq \mathfrak{g}(\mathbf{0})_{\bar{0}}$. Therefore $\mathfrak{g}(\mathbf{0})/\mathfrak{g}(\mathbf{0})_{\bar{0}}$ is Abelian, and isomorphic to $\mathbb{C}^{0|8}$ because $\dim_{\mathbb{C}}(\mathfrak{g}(\mathbf{0})_{\bar{1}}) = 8$.

Finally, we stress the point that $\mathfrak{g}(\mathbf{0})$ has *non-trivial* structure, as the lines from eighth to eleventh (in the table) display non-zero Lie brackets among some of its generators. ■

4.2 Second family: the Lie superalgebras $\mathfrak{g}'(\sigma)$

4.2.1. Construction of the $\mathfrak{g}'(\sigma)$'s. For any $\sigma \in V$ (cf. Section 3.1), we define the Lie superalgebra $\mathfrak{g}'_z(\sigma)$ over $\mathbb{Z}[\sigma]$ as follows. As a $\mathbb{Z}[\sigma]$ -module, $\mathfrak{g}'_z(\sigma)$ is spanned by the formal set of $\mathbb{Z}[\sigma]$ -linear generators $B'_g := \{H'_{2\varepsilon_1}, H'_{2\varepsilon_2}, H'_{2\varepsilon_3}, H'_\theta\} \cup \{X'_\alpha\}_{\alpha \in \Delta}$ subject only to the single relation $\sigma_1 H'_{2\varepsilon_1} + \sigma_2 H'_{2\varepsilon_2} + \sigma_3 H'_{2\varepsilon_3} = 2H'_\theta$. The Lie superalgebra structure of $\mathfrak{g}'_z(\sigma)$ is defined by the formulas in Section 3.3.2, which now we read as definitions of the Lie brackets among the (linear) generators of $\mathfrak{g}'_z(\sigma)$ itself.

Altogether, the $\mathfrak{g}'_{\mathbb{Z}}(\sigma)$'s form a family of Lie superalgebras indexed by the points of V . Then taking $\mathfrak{g}'(\sigma) := \mathbb{C} \otimes_{\mathbb{Z}[\sigma]} \mathfrak{g}'_{\mathbb{Z}}(\sigma)$ for all $\sigma \in V$ we find a more regular situation, as these Lie superalgebras $\mathfrak{g}'(\sigma)$ now share \mathbb{C} as their common ground ring. In fact, this is just another manner of presenting the family of Kaplansky's Lie superalgebras \mathfrak{g}_{σ} , in that

$$\text{for all } \sigma_i \in V, \text{ we have } \mathfrak{g}'(\sigma) \cong \mathfrak{g}_{\sigma} \quad (4.2)$$

by Sections 3.1.1 and 3.3.2 – cf. (3.6) – and the very construction of $\mathfrak{g}'(\sigma)$. In fact, the analysis in Section 3.3.2 (describing a \mathbb{C} -basis of \mathfrak{g}_{σ} , for any $\sigma \in V$, and its multiplication table) prove that *for all* $\sigma \in V$, the Lie $\mathbb{Z}[\sigma]$ -superalgebra $\mathfrak{g}'_{\mathbb{Z}}(\sigma)$ identifies to an integral $\mathbb{Z}[\sigma]$ -form of \mathfrak{g}_{σ} . We can formalize the description of the family $\{\mathfrak{g}'(\sigma)\}_{\sigma \in V}$ proceeding like in Section 4.1.1; in particular we keep the same notation, such as $\mathbb{Z}[\mathbf{x}] := \mathbb{Z}[V] \cong \mathbb{Z}[x_1, x_2, x_3]/(x_1 + x_2 + x_3)$, etc. In the construction of $\mathfrak{g}'_{\mathbb{Z}}(\sigma)$, formally replace \mathbf{x} to σ : this yields a meaningful definition of a Lie superalgebra over $\mathbb{Z}[\mathbf{x}]$, denoted by $\mathfrak{g}'_{\mathbb{Z}}(\mathbf{x})$, and also $\mathfrak{g}'(\mathbf{x}) := \mathbb{C}[\mathbf{x}] \otimes_{\mathbb{Z}[\mathbf{x}]} \mathfrak{g}'_{\mathbb{Z}}(\mathbf{x})$ by scalar extension. Now definitions imply that, for any $\sigma \in V$, we have a Lie $\mathbb{Z}[\sigma]$ -superalgebra isomorphism

$$\mathfrak{g}'_{\mathbb{Z}}(\sigma) \cong \mathbb{Z}[\sigma] \otimes_{\mathbb{Z}[\mathbf{x}]} \mathfrak{g}'_{\mathbb{Z}}(\mathbf{x})$$

– through the ring isomorphism $\mathbb{Z}[\sigma] \cong \mathbb{Z}[\mathbf{x}]/(x_i - \sigma_i)_{i=1,2,3}$ – and similarly

$$\mathfrak{g}'(\sigma) \cong \mathbb{C} \otimes_{\mathbb{C}[\mathbf{x}]} \mathfrak{g}'(\mathbf{x})$$

as Lie \mathbb{C} -superalgebras, through the ring isomorphism $\mathbb{C} \cong \mathbb{C}[\mathbf{x}]/(x_i - \sigma_i)_{i=1,2,3}$.

One can argue similarly as in Section 4.1.1 to have a geometric picture of the above description: this amounts to literally replacing $\mathfrak{g}(\sigma)$ with $\mathfrak{g}'(\sigma)$, which eventually provide a coherent sheaf $\mathcal{L}_{\mathfrak{g}'_{\mathbb{C}[\mathbf{x}]}}$ of complex Lie superalgebras over V , with a $(\mathbb{C}^{\times} \times \mathfrak{S}_3)$ -action on it, and a corresponding fibre bundle $\mathbb{L}_{\mathfrak{g}'_{\mathbb{C}[\mathbf{x}]}}$, whose fibres are the $\mathfrak{g}'(\sigma)$'s; details are left to the reader. In the next result we describe these Lie superalgebras $\mathfrak{g}'(\sigma)$ ($\cong \mathfrak{g}_{\sigma}$), for all $\sigma \in V$: like for the $\mathfrak{g}(\sigma)$'s, the outcome is again that in the “regular” locus V^{\times} they are simple, while in the “singular” locus $V \setminus V^{\times}$ we can describe explicitly their non-simple structure.

Theorem 4.2. *Given $\sigma \in V$, let $\mathfrak{a}'_i := \mathbb{C}X'_{2\varepsilon_i} \oplus \mathbb{C}H'_{2\varepsilon_i} \oplus \mathbb{C}X'_{-2\varepsilon_i}$, for all $i \in \{1, 2, 3\}$.*

- (1) *If $\sigma \in V^{\times}$, then the Lie superalgebra $\mathfrak{g}'(\sigma)$ is simple.*
- (2) *If $\sigma \in V \setminus V^{\times}$, with $\sigma_i = 0$ and $\sigma_j \neq 0 \neq \sigma_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then if*

$$\mathfrak{b}'_i := \left(\sum_{\alpha \neq \pm 2\varepsilon_i} \mathbb{C}X'_{\alpha} \right) \oplus \left(\sum_{j \neq i} \mathbb{C}H'_{2\varepsilon_j} \right)$$

we have $\mathfrak{b}'_i \trianglelefteq \mathfrak{g}'(\sigma)$ (a Lie ideal), $\mathfrak{a}'_i \leq \mathfrak{g}'(\sigma)$ (a Lie subsuperalgebra), and there exist isomorphisms $\mathfrak{b}'_i \cong \mathfrak{psl}(2|2)$, $\mathfrak{a}'_i \cong \mathfrak{sl}(2)$ and $\mathfrak{g}'(\sigma) \cong \mathfrak{sl}(2) \ltimes \mathfrak{psl}(2|2)$ – a semidirect product of Lie superalgebras. In short, there exists a split short exact sequence

$$0 \longrightarrow \mathfrak{psl}(2|2) \cong \mathfrak{b}'_i \longrightarrow \mathfrak{g}'(\sigma) \xleftarrow{\quad\quad\quad} \mathfrak{a}'_i \cong \mathfrak{sl}(2) \longrightarrow 0.$$

A parallel result also holds true when dealing with $\mathfrak{g}'_{\mathbb{Z}}(\sigma)$ over the ground ring $\mathbb{Z}[\sigma]$.

- (3) *If $\sigma = \mathbf{0}$ ($\in V \setminus V^{\times}$), i.e., $\sigma_h = 0$ for all $h \in \{1, 2, 3\}$, then $\mathfrak{g}'(\mathbf{0})_{\bar{0}} \cong \mathfrak{sl}(2)^{\oplus 3}$ as Lie (super)algebras, the Lie (super)bracket is trivial on $\mathfrak{g}'(\mathbf{0})_{\bar{1}}$ and $\mathfrak{g}'(\mathbf{0})_{\bar{1}} \cong \square^{\boxtimes 3}$ as modules over $\mathfrak{g}'(\mathbf{0})_{\bar{0}} \cong \mathfrak{sl}(2)^{\oplus 3}$. Finally, we have*

$$\mathfrak{g}'(\mathbf{0}) \cong \mathfrak{g}'(\mathbf{0})_{\bar{0}} \ltimes \mathfrak{g}'(\mathbf{0})_{\bar{1}} \cong \mathfrak{sl}(2)^{\oplus 3} \ltimes \square^{\boxtimes 3}$$

– a semidirect product of Lie superalgebras. In other words, there is a split short exact sequence

$$0 \longrightarrow \square^{\boxtimes 3} \cong \mathfrak{g}'(\mathbf{0})_{\bar{1}} \longrightarrow \mathfrak{g}'(\mathbf{0}) \xrightarrow{\leftarrow \text{-----} \rightarrow} \mathfrak{g}'(\mathbf{0})_{\bar{0}} \cong \mathfrak{sl}(2)^{\oplus 3} \longrightarrow 0.$$

A parallel result holds true when working with $\mathfrak{g}_{\mathbb{Z}}(\mathbf{0})$ over the ground ring $\mathbb{Z}[\mathbf{0}] = \mathbb{Z}$.

Proof. Part (1) is a direct consequence of (4.2) and Proposition 3.1. Claim (2) instead follows easily from direct inspection of the formulas in Section 3.3.2. Indeed, $\mathfrak{a}'_i \cong \mathfrak{sl}(2)$ follows from the first two lines of the table of formulas for Lie brackets in Section 3.3.2, which also show that \mathfrak{a}'_i is a Lie subsuperalgebra of $\mathfrak{g}'(\sigma)$ – for all $\sigma \in V$, indeed.

Similarly, those formulas show that \mathfrak{b}'_i is stable by the adjoint \mathfrak{a}'_i -action exactly if and only if $\sigma_i = 0$: in fact, the critical point is that $H'_\theta = 2^{-1} \sum_{k=1}^3 \sigma_k H'_{2\varepsilon_k} \in \mathfrak{b}'_i \iff \sigma_i = 0$, and also $[X'_{2\varepsilon_i}, H'_\theta] = \mp \sigma_i X'_{2\varepsilon_i} \in \mathfrak{b}'_i \iff \sigma_i = 0$.

Moreover, the same formulas altogether show also that \mathfrak{b}'_i is a Lie subsuperalgebra if and only if $\sigma_i = 0$ – for instance, because $[X'_{\beta_j}, X'_{\beta_k}] = (1 - \delta_{j,k})\sigma_i X'_{2\varepsilon_i} \in \mathfrak{b}'_i \iff \sigma_i = 0$. Then, when $\sigma_i = 0$, looking closely at the specific form of the Lie (sub)superalgebra \mathfrak{b}'_i these formulas also show that $\mathfrak{b}'_i \cong \mathfrak{psl}(2|2)$ as claimed.

Finally, as \mathfrak{a}'_i is a direct sum complement of \mathfrak{b}'_i in $\mathfrak{g}'(\sigma)$, since \mathfrak{b}'_i is a Lie subsuperalgebra and is \mathfrak{a}'_i -stable (when $\sigma_i = 0$) we conclude that it is also a Lie ideal, q.e.d.

Claim (3) follows again by straightforward analysis of the table of formulas in Section 3.3.2 – just a matter of sheer bookkeeping. ■

4.3 Third family: the Lie superalgebras $\mathfrak{g}''(\sigma)$

4.3.1. Construction of the $\mathfrak{g}''(\sigma)$'s. For any $\sigma \in V$ (cf. Section 3.1), we define the Lie superalgebra $\mathfrak{g}''_{\mathbb{Z}}(\sigma)$ over $\mathbb{Z}[\sigma]$ as follows. By definition, $\mathfrak{g}''_{\mathbb{Z}}(\sigma)$ is the $\mathbb{Z}[\sigma]$ -module spanned by the set of formal (linear) generators $B_{\mathfrak{g}''} := \{H'_{2\varepsilon_1}, H'_{2\varepsilon_2}, H'_{2\varepsilon_3}, H'_\theta\} \cup \{X_\alpha\}_{\alpha \in \Delta}$ subject only to the single relation $\sigma_1 H'_{2\varepsilon_1} + \sigma_2 H'_{2\varepsilon_2} + \sigma_3 H'_{2\varepsilon_3} = 2H'_\theta$. The Lie superalgebra structure of $\mathfrak{g}''_{\mathbb{Z}}(\sigma)$ is defined by the formulas in Section 3.3.3, which now must be read as definitions for Lie brackets among the (linear) generators of $\mathfrak{g}''_{\mathbb{Z}}(\sigma)$.

The $\mathfrak{g}''_{\mathbb{Z}}(\sigma)$'s altogether form a family of Lie superalgebras indexed by the points of V . Setting $\mathfrak{g}''(\sigma) := \mathbb{C} \otimes_{\mathbb{Z}[\sigma]} \mathfrak{g}''_{\mathbb{Z}}(\sigma)$ for all $\sigma \in V$ we find a more regular situation, in that these Lie superalgebras $\mathfrak{g}''(\sigma)$ share \mathbb{C} as their common ground ring. In addition

$$\text{in the non-singular case, i.e., for } \sigma_i \in V^\times, \text{ we have } \mathfrak{g}''(\sigma) \cong \mathfrak{g}_\sigma \tag{4.3}$$

by Section 3.3.3 and the very definition of $\mathfrak{g}''(\sigma)$ itself. Indeed, from Section 3.3.3 – where a \mathbb{C} -basis and its multiplication table for \mathfrak{g}_σ , with $\sigma \in V^\times$, are described – we see that for all $\sigma \in V^\times$, the Lie superalgebra $\mathfrak{g}''_{\mathbb{Z}}(\sigma)$ over $\mathbb{Z}[\sigma]$ identifies to an integral $\mathbb{Z}[\sigma]$ -form of \mathfrak{g}_σ .

Keeping notation as before, we can describe the family $\{\mathfrak{g}''(\sigma)\}_{\sigma \in V}$ in a formal way, taking its “version over $\mathbb{Z}[\mathbf{x}]$ ”, denoted by $\mathfrak{g}''_{\mathbb{Z}}(\mathbf{x})$ – just replacing the complex parameters $(\sigma_1, \sigma_2, \sigma_3) =: \sigma$ with a triple of formal parameters $(x_1, x_2, x_3) =: \mathbf{x}$ adding to zero – and its complex-based counterpart $\mathfrak{g}''(\mathbf{x}) := \mathbb{C}[\mathbf{x}] \otimes_{\mathbb{Z}[\mathbf{x}]} \mathfrak{g}''_{\mathbb{Z}}(\mathbf{x})$. Then the very construction implies that, for any $\sigma \in V$, one has a Lie $\mathbb{Z}[\sigma]$ -superalgebra isomorphism

$$\mathfrak{g}''_{\mathbb{Z}}(\sigma) \cong \mathbb{Z}[\sigma] \otimes_{\mathbb{Z}[\mathbf{x}]} \mathfrak{g}''_{\mathbb{Z}}(\mathbf{x})$$

– through $\mathbb{Z}[\sigma] \cong \mathbb{Z}[\mathbf{x}]/(x_i - \sigma_i)_{i=1,2,3}$ – and similarly

$$\mathfrak{g}''(\sigma) \cong \mathbb{C} \otimes_{\mathbb{C}[\mathbf{x}]} \mathfrak{g}''(\mathbf{x})$$

as Lie \mathbb{C} -superalgebras, through $\mathbb{C} \cong \mathbb{C}[\mathbf{x}]/(x_i - \sigma_i)_{i=1,2,3}$. Finally, the reader can easily mimick what is done in Section 4.1.1 and find a geometric description of the family of the $\mathfrak{g}''(\sigma)$'s.

Like in Section 4.1.1, we can re-cast all this in geometrical terms, defining a coherent sheaf $\mathcal{L}_{\mathfrak{g}''_{\mathbb{C}[\mathbf{x}]}}$ of complex Lie superalgebras over V , with a $(\mathbb{C}^\times \times \mathfrak{S}_3)$ -action on it, and a corresponding fibre bundle $\mathbb{L}_{\mathfrak{g}''_{\mathbb{C}[\mathbf{x}]}}$ with the $\mathfrak{g}''(\sigma)$'s as fibres; the reader can easily fill in the details.

In the next result we describe these Lie superalgebras $\mathfrak{g}''(\sigma)$, for all $\sigma \in V$: like for the previous two families, the outcome is that in the ‘‘regular’’ locus V^\times they are simple, while in the ‘‘singular’’ locus $V \setminus V^\times$ we can describe explicitly their non-simple structure.

Theorem 4.3. *Let $\sigma \in V$, and set $\mathfrak{c}''_i := (\mathbb{C}X_{2\varepsilon_i} \oplus \mathbb{C}X_{-2\varepsilon_i})$, $\overline{\mathfrak{b}}''_i := \mathfrak{g}''(\sigma)/\mathfrak{c}''_i$ – a subsuperspace and a quotient superspace of $\mathfrak{g}''(\sigma)$, in general – for all $i \in \{1, 2, 3\}$.*

- (1) *If $\sigma \in V^\times$, then the Lie superalgebra $\mathfrak{g}''(\sigma)$ is simple.*
- (2) *If $\sigma \in V \setminus V^\times$, with $\sigma_i = 0$ and $\sigma_j \neq 0 \neq \sigma_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then $\mathfrak{c}''_i := (\mathbb{C}X_{2\varepsilon_i} \oplus \mathbb{C}X_{-2\varepsilon_i})$ is an Abelian Lie ideal of $\mathfrak{g}''(\sigma)$, hence $\overline{\mathfrak{b}}''_i$ is a quotient Lie superalgebra of $\mathfrak{g}''(\sigma)$; therefore, there exists a short exact sequence*

$$0 \longrightarrow \mathfrak{c}''_i := (\mathbb{C}X_{2\varepsilon_i} \oplus \mathbb{C}X_{-2\varepsilon_i}) \longrightarrow \mathfrak{g}''(\sigma) \longrightarrow \overline{\mathfrak{b}}''_i \longrightarrow 0.$$

Furthermore, setting $\overline{\mathfrak{d}}''_i := (\bigoplus_{j \neq i} (\overline{\mathbb{C}H'_{2\varepsilon_j}} \oplus \overline{\mathbb{C}X_{+2\varepsilon_j}} \oplus \overline{\mathbb{C}X_{-2\varepsilon_j}})) \oplus (\bigoplus_{\gamma \in \Delta_{\bar{1}}} \overline{\mathbb{C}X_\gamma})$ – in the quotient Lie superalgebra $\overline{\mathfrak{b}}''_i$ – we have that $\overline{\mathfrak{d}}''_i$ is a Lie ideal and $\overline{\mathbb{C}H'_{2\varepsilon_i}}$ is a Lie subsuperalgebra of $\overline{\mathfrak{b}}''_i$, with $\overline{\mathfrak{d}}''_i \cong \mathfrak{psl}(2|2)$, $\overline{\mathbb{C}H'_{2\varepsilon_i}} \cong \mathbb{C}$, and $\overline{\mathfrak{b}}''_i \cong (\overline{\mathbb{C}H'_{2\varepsilon_i}}) \ltimes \overline{\mathfrak{d}}''_i$ – a semidirect product of Lie superalgebras. In short, there exists a split short exact sequence

$$0 \longrightarrow \mathfrak{psl}(2|2) \cong \overline{\mathfrak{d}}''_i \longrightarrow \overline{\mathfrak{b}}''_i \xleftarrow{\text{-----}} \overline{\mathbb{C}H'_{2\varepsilon_i}} \cong \mathbb{C} \longrightarrow 0.$$

A parallel result holds true when working with $\mathfrak{g}''_Z(\sigma)$ over the ground ring $\mathbb{Z}[\sigma]$.

- (3) *If $\sigma = \mathbf{0} (\in V \setminus V^\times)$, i.e., $\sigma_h = 0$ for all $h \in \{1, 2, 3\}$, then $\mathfrak{c}'' := \bigoplus_{i=1}^3 \mathfrak{c}''_i$ is an Abelian Lie ideal, hence $\overline{\mathfrak{b}}'' := \mathfrak{g}''(\sigma)/\mathfrak{c}''$ is a quotient Lie superalgebra (of $\mathfrak{g}''(\sigma)$); therefore, there exists a short exact sequence*

$$0 \longrightarrow \mathfrak{c}'' := \bigoplus_{i=1}^3 \mathfrak{c}''_i \longrightarrow \mathfrak{g}''(\mathbf{0}) \longrightarrow \overline{\mathfrak{b}}'' \longrightarrow 0.$$

Moreover, there exists a second, split short exact sequence

$$0 \longrightarrow \bigoplus_{\alpha \in \Delta_{\bar{1}}} \overline{\mathbb{C}X_\alpha} \longrightarrow \overline{\mathfrak{b}}'' \xleftarrow{\text{-----}} \bigoplus_{i=1}^3 \overline{\mathbb{C}H'_{2\varepsilon_i}} \longrightarrow 0,$$

where $\bigoplus_{\alpha \in \Delta_{\bar{1}}} \overline{\mathbb{C}X_\alpha} \trianglelefteq \overline{\mathfrak{b}}''$ is an Abelian Lie ideal and $\bigoplus_{i=1}^3 \overline{\mathbb{C}H'_{2\varepsilon_i}} \leq \overline{\mathfrak{b}}''$ is an Abelian Lie subsuperalgebra (of $\overline{\mathfrak{b}}''$), so that $\overline{\mathfrak{b}}'' \cong (\bigoplus_{i=1}^3 \overline{\mathbb{C}H'_{2\varepsilon_i}}) \ltimes (\bigoplus_{\alpha \in \Delta_{\bar{1}}} \overline{\mathbb{C}X_\alpha})$ – a semidirect product of Lie superalgebras.

A parallel result holds true when working with $\mathfrak{g}''(\mathbf{0})$ over the ground ring $\mathbb{Z}[\mathbf{0}] = \mathbb{Z}$.

Proof. Claim (1) is a direct consequence of (4.3) along with Proposition 3.1. Claims (2) and (3), like for the previous, parallel results, both follow as direct outcome of the formulas for Lie brackets among linear generators of $\mathfrak{g}''(\sigma)$, which we read from Section 3.3.3.

Indeed, for claim (2) we notice that $[X_{2\varepsilon_i}, X_{-2\varepsilon_j}] = \delta_{i,j} \sigma_i^2 H'_{2\varepsilon_i}$ is zero when $\sigma_i = 0$, so that \mathfrak{c}''_i is then an Abelian Lie subsuperalgebra. Moreover, \mathfrak{c}''_i is stable for the adjoint action by elements of $\{H_{2\varepsilon_1}, H_{2\varepsilon_2}, H_{2\varepsilon_3}, H_\theta\}$ by construction (this holds true for every $\sigma \in V$ indeed). Finally, \mathfrak{c}''_i is also stable for the adjoint action by odd root vectors such as $X_{\pm\beta_j}$ and $X_{\pm\theta}$ because the

formulas in fifth, sixth and seventh line of the table in Section 3.3.3 all give zero Lie brackets when $\sigma_i = 0$. Overall, this means that \mathfrak{c}_i'' is an Abelian Lie ideal of $\mathfrak{g}''(\sigma)$.

The claim about $\overline{\mathfrak{b}}'' := \mathfrak{g}''(\sigma)/\mathfrak{c}''$ and the short exact sequence now are obvious consequences of \mathfrak{c}_i'' being a Lie ideal.

As to the rest of claim (1), one sees again that everything follows from straightforward bookkeeping, nothing more. For claim (3) one has again to carry out a similar analysis. For instance, $\mathfrak{c}'' := \bigoplus_{i=1}^3 \mathfrak{c}_i''$ is an Abelian Lie ideal because of claim (2) and the fact that the \mathfrak{c}_i'' 's commute with each other.

Something less obvious only occurs with the analysis of $\bigoplus_{\alpha \in \Delta_{\overline{1}}} \mathbb{C}\overline{X}_\alpha$. Indeed, the fact that this is an Abelian Lie ideal of $\overline{\mathfrak{b}}''$ follows once more from the formulas in Section 3.3.3 but one also has to pay attention to some detail.

Indeed, a first bunch of Lie brackets among odd root vectors which are non-zero in $\mathfrak{g}(\mathbf{0})$ but are trivial instead in its subquotient $\bigoplus_{\alpha \in \Delta_{\overline{1}}} \mathbb{C}\overline{X}_\alpha$ are the following:

$$\begin{aligned} [\overline{X}_{\beta_i}, \overline{X}_{\beta_j}] &= (1 - \delta_{i,j})\overline{X}_{2\varepsilon_k} = -\overline{0}, & [\overline{X}_{-\beta_i}, \overline{X}_{-\beta_j}] &= -(1 - \delta_{i,j})\overline{X}_{-2\varepsilon_k} = -\overline{0}, \\ [\overline{X}_{\beta_i}, \overline{X}_{-\theta}] &= \overline{X}_{-2\varepsilon_i} = -\overline{0}, & [\overline{X}_{-\beta_i}, \overline{X}_\theta] &= -\overline{X}_{2\varepsilon_i} = -\overline{0}. \end{aligned}$$

Second, the remaining non-obvious Lie brackets are described by the two formulas

$$[X_{\beta_i}, X_{-\beta_j}] = \delta_{i,j}(\sigma_i H'_{2\varepsilon_i} - H'_\theta), \quad [X_\theta, X_{-\theta}] = H'_\theta$$

but the relation $\sigma_1 H'_{2\varepsilon_1} + \sigma_2 H'_{2\varepsilon_2} + \sigma_3 H'_{2\varepsilon_3} = 2H'_\theta$ in $\mathfrak{g}''(\sigma)$ reads now $H'_\theta = 0$ in $\mathfrak{g}''(\mathbf{0})$, hence from the last formulas above we get $[\overline{X}_{\beta_i}, \overline{X}_{-\beta_j}] = \overline{0}$ and $[\overline{X}_\theta, \overline{X}_{-\theta}] = \overline{0}$ in $\overline{\mathfrak{b}}''$.

All other parts of claim (3) follow equally from a similar analysis – a sheer matter of bookkeeping – so we leave them to the reader. \blacksquare

4.4 Degenerations from contractions: the $\widehat{\mathfrak{g}}(\sigma)$'s and the $\widehat{\mathfrak{g}}'(\sigma)$'s

We finish our study of remarkable integral forms of \mathfrak{g}_σ by introducing some further ones, that all are obtained through a general construction; when specializing these forms, one obtains again degenerations, now of the kind that is often referred to as “contraction” (see, e.g., [5]). We start with a very general construction. Let R be a (commutative, unital) ring, and let \mathcal{A} be an “algebra” (not necessarily associative, nor unitary), in some category of R -bimodules, for some “product” denoted by “ \cdot ”: we assume in addition that

$$\mathcal{A} = \mathcal{F} \oplus \mathcal{C} \quad \text{with} \quad \mathcal{F} \cdot \mathcal{F} \subseteq \mathcal{F}, \quad \mathcal{F} \cdot \mathcal{C} \subseteq \mathcal{C}, \quad \mathcal{C} \cdot \mathcal{F} \subseteq \mathcal{C}, \quad \mathcal{C} \cdot \mathcal{C} \subseteq \mathcal{F} \quad (4.4)$$

Choose now τ be a non-unit in R , and correspondingly consider in \mathcal{A} the R -submodules

$$\mathcal{F}_\tau := \mathcal{F}, \quad \mathcal{C}_\tau := \tau\mathcal{C}, \quad \mathcal{A}_\tau := \mathcal{F}_\tau + \mathcal{C}_\tau = \mathcal{F} \oplus (\tau\mathcal{C}). \quad (4.5)$$

Fix also a (strict) ideal $I \triangleleft R$; then set $R_I := R/I$ for the corresponding quotient ring, and use notation $\mathcal{A}_{\tau,I} := \mathcal{A}_\tau / I\mathcal{A}_\tau \cong (R/I) \otimes_R \mathcal{A}_\tau$, $\mathcal{F}_{\tau,I} := \mathcal{F}_\tau / I\mathcal{F}_\tau \cong (R/I) \otimes_R \mathcal{F}$ and $\mathcal{C}_{\tau,I} := \mathcal{C}_\tau / I\mathcal{C}_\tau = (\tau\mathcal{C}) / (I\tau\mathcal{C}) \cong (R/I) \otimes_R \mathcal{C}_\tau \cong (R/I) \otimes_R (\tau\mathcal{C})$. By construction we have $\mathcal{A}_{\tau,I} \cong \mathcal{F}_{\tau,I} \oplus \mathcal{C}_{\tau,I}$ as an R_I -module; moreover,

$$\mathcal{F}_{\tau,I} \cdot \mathcal{F}_{\tau,I} \subseteq \mathcal{F}_{\tau,I}, \quad \mathcal{F}_{\tau,I} \cdot \mathcal{C}_{\tau,I} \subseteq \mathcal{C}_{\tau,I}, \quad \mathcal{C}_{\tau,I} \cdot \mathcal{F}_{\tau,I} \subseteq \mathcal{C}_{\tau,I}, \quad \mathcal{C}_{\tau,I} \cdot \mathcal{C}_{\tau,I} \subseteq \overline{\tau}^2 \mathcal{F}_{\tau,I},$$

where the last identity comes from $\mathcal{C}_\tau \cdot \mathcal{C}_\tau = \tau^2(\mathcal{C} \cdot \mathcal{C}) \subseteq \tau^2\mathcal{F} = \tau^2\mathcal{F}_\tau$ and we write $\overline{\tau} := (\tau \bmod I) \in R/I$. In particular, if $\tau \in I$, then $\mathcal{C}_{\tau,I} \cdot \mathcal{C}_{\tau,I} = \{0\}$ and we get

$$\mathcal{A}_{\tau,I} = \mathcal{F}_{\tau,I} \ltimes \mathcal{C}_{\tau,I}, \quad (4.6)$$

where $\mathcal{C}_{\tau, I}$ bears the $\mathcal{F}_{\tau, I}$ -bimodule structure induced from \mathcal{A} and is given a trivial product, so that it sits inside $\mathcal{A}_{\tau, I}$ as a two-sided Abelian ideal, with (4.6) being a semidirect product splitting. In fact, $\mathcal{A}_{\tau, I}$ is what is called a “central extension of $\mathcal{F}_{\tau, I}$ by $\mathcal{C}_{\tau, I}$ ”.

In short, for $\tau \in I$ this process leads us from the initial object \mathcal{A} , that splits into $\mathcal{A} = \mathcal{F} \oplus \mathcal{C}$ as R -module, to the final object $\mathcal{A}_{\tau, I} = \mathcal{F}_{\tau, I} \ltimes \mathcal{C}_{\tau, I}$, now split as a semidirect product. Following [5, Section 2 and references therein], we shall refer to this process as “contraction”, and also refer to $\mathcal{A}_{\tau, I}$ as to a “contraction of \mathcal{A} ”. Note, however, that these are contractions of a very special type, in that only the odd part is “contracted”: in the general theory of contractions of Lie superalgebras, instead, one has to do with a richer variety of objects – cf., e.g., [17]. For our purpose however we do not need the general theory in its full extent. We apply now the above contraction procedure to a couple of integral forms of \mathfrak{g}_σ . First consider the case $\mathcal{A} := \mathfrak{g}_z(\mathbf{x})$, $\mathcal{F} := \mathfrak{g}_z(\mathbf{x})_{\bar{0}}$ and $\mathcal{C} := \mathfrak{g}_z(\mathbf{x})_{\bar{1}}$; here the ground ring is $R := \mathbb{Z}[\mathbf{x}]$, and we choose in it $\tau := x_1x_2x_3$ and the ideal $I = I_\sigma$ generated by $x_1 - \sigma_1$, $x_2 - \sigma_2$ and $x_3 - \sigma_3$. In this case, the “blown-up” Lie superalgebras in (4.5) reads $\mathfrak{g}_z(\mathbf{x})_\tau = \mathfrak{g}_z(\mathbf{x})_{\bar{0}} \oplus (\tau \mathfrak{g}_z(\mathbf{x})_{\bar{1}})$, that we write also with the simpler notation $\widehat{\mathfrak{g}}_z(\mathbf{x}) := \mathfrak{g}_z(\mathbf{x})_\tau$; similarly we write $\widehat{\mathfrak{g}}_z(\sigma) := \mathfrak{g}_z(\mathbf{x})_{\tau, I_\sigma}$. Note that each $\widehat{\mathfrak{g}}_z(\sigma)$ for *non-singular* $\sigma \in V^\times$ is yet another $\mathbb{Z}[\sigma]$ -integral form of our initial complex Lie superalgebra \mathfrak{g}_σ . Similarly occurs if we work over \mathbb{C} , i.e., when we consider $\mathcal{A} := \mathfrak{g}(\mathbf{x})$, $\mathcal{F} := \mathfrak{g}(\mathbf{x})_{\bar{0}}$, $\mathcal{C} := \mathfrak{g}(\mathbf{x})_{\bar{1}}$ and the blown-up algebra $\mathfrak{g}(\mathbf{x})_\tau$ with ground ring $R := \mathbb{C}[\mathbf{x}]$, and the contraction $\widehat{\mathfrak{g}}(\sigma) := \mathfrak{g}(\mathbf{x})_{\tau, I_\sigma}$ over \mathbb{C} . This gives (as in Section 4.1.1) a new coherent sheaf of complex Lie superalgebras over V , say $\mathcal{L}_{\widehat{\mathfrak{g}}_{\mathbb{C}[\mathbf{x}]}}$, with a $(\mathbb{C}^\times \times \mathfrak{S}_3)$ -action on it, and an associated fibre bundle $\mathbb{L}_{\widehat{\mathfrak{g}}_{\mathbb{C}[\mathbf{x}]}}$ with the $\widehat{\mathfrak{g}}(\sigma)$'s as fibres; details are left to the reader. Next result describes the structure of the $\widehat{\mathfrak{g}}(\sigma)$'s.

Theorem 4.4. *Let $\sigma \in V$, $i \in \{1, 2, 3\}$, $\widehat{\mathfrak{a}}_i := \mathbb{C}X_{2\varepsilon_i} \oplus \mathbb{C}H_{2\varepsilon_i} \oplus \mathbb{C}X_{-2\varepsilon_i}$ ($= \mathfrak{a}_i$ of Section 3.2.1).*

- (1) *If $\sigma \in V^\times$, then the Lie superalgebra $\widehat{\mathfrak{g}}(\sigma)$ is simple.*
- (2) *If $\sigma \in V \setminus V^\times$, with $\sigma_i = 0$ and $\sigma_j \neq 0 \neq \sigma_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then $\widehat{\mathfrak{a}}_i \leq \widehat{\mathfrak{g}}$ is a central Lie ideal in $\widehat{\mathfrak{g}}(\sigma)$, with $\widehat{\mathfrak{a}}_i \cong \mathbb{C}^{3|0}$, while $\widehat{\mathfrak{a}}_j \cong \widehat{\mathfrak{a}}_k \cong \mathfrak{sl}(2)$ for $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$. Moreover, we have a semidirect product splitting*

$$\widehat{\mathfrak{g}}(\sigma) \cong \widehat{\mathfrak{g}}(\sigma)_{\bar{0}} \ltimes \widehat{\mathfrak{g}}(\sigma)_{\bar{1}}$$

with $\widehat{\mathfrak{g}}(\sigma)_{\bar{0}} = \bigoplus_{\ell=1}^3 \widehat{\mathfrak{a}}_\ell \cong \mathbb{C}^{3|0} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ while $\widehat{\mathfrak{g}}(\sigma)_{\bar{1}}$ is endowed with trivial Lie bracket and $\widehat{\mathfrak{g}}(\sigma)_{\bar{1}} \cong (\blacksquare \oplus \blacksquare) \boxtimes \square \boxtimes \square$ – where \blacksquare is the trivial representation – as a module over $\widehat{\mathfrak{g}}(\sigma)_{\bar{0}} \cong \mathbb{C}^{3|0} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$; so there exists a split short exact sequence of Lie superalgebras

$$0 \longrightarrow \widehat{\mathfrak{g}}(\sigma)_{\bar{1}} \cong (\blacksquare \oplus \blacksquare) \boxtimes \square \boxtimes \square \longrightarrow \widehat{\mathfrak{g}}(\sigma) \xleftarrow{\quad} \widehat{\mathfrak{g}}(\sigma)_{\bar{0}} \cong \mathbb{C}^{3|0} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \longrightarrow 0.$$

A parallel result also holds true when working with $\widehat{\mathfrak{g}}_z(\sigma)$ over the ground ring $\mathbb{Z}[\sigma]$.

- (3) *If $\sigma = \mathbf{0} (\in V \setminus V^\times)$, i.e., $\sigma_h = 0$ for all $h \in \{1, 2, 3\}$, then $\widehat{\mathfrak{g}}(\mathbf{0})$ is the Abelian complex Lie superalgebra of superdimension $9|8$, that is $\widehat{\mathfrak{g}}(\mathbf{0}) \cong \mathbb{C}^{9|8}$ with trivial bracket.*

A parallel result holds true when working with $\widehat{\mathfrak{g}}_z(\mathbf{0})$ over the ground ring $\mathbb{Z}[\mathbf{0}] = \mathbb{Z}$.

Proof. The claim follows directly from Theorem 4.1 once we take also into account the fact that the $\widehat{\mathfrak{g}}(\sigma)$'s are specializations of $\mathfrak{g}(\mathbf{x})_\tau$, and for singular values $\sigma \in V^\times$ any such specialization is a contraction of $\widehat{\mathfrak{g}}(\sigma)$, of the form $\widehat{\mathfrak{g}}(\sigma) = \mathfrak{g}(\mathbf{x})_{\tau, I}$ for the element $\tau := x_1x_2x_3$ and the ideal $I := \{(x_i - \sigma_i)_{i=1,2,3}\}$. Otherwise, one can deduce the statement directly from the explicit formulas for (linear) generators of $\widehat{\mathfrak{g}}(\mathbf{x})$: indeed, the latter are easily obtained as slight modification – taking into account that odd generators must be “rescaled” by the coefficient $\tau := x_1x_2x_3$ – of the similar formulas in Section 3.3.1 for $\mathfrak{g}(\sigma)$, which read as formulas for $\mathfrak{g}(\mathbf{x})$ just switching the σ_ℓ 's into x_ℓ 's. \blacksquare

As a second instance, we consider the case $\mathcal{A} := \mathfrak{g}'_{\mathbb{Z}}(\mathbf{x})$, $\mathcal{F} := \mathfrak{g}'_{\mathbb{Z}}(\mathbf{x})_{\bar{0}}$ and $\mathcal{C} := \mathfrak{g}'_{\mathbb{Z}}(\mathbf{x})_{\bar{1}}$; the ground ring is again $R := \mathbb{Z}[\mathbf{x}]$, and again we choose in it $\tau := x_1x_2x_3$ and the ideal I generated by $x_1 - \sigma_1$, $x_2 - \sigma_2$ and $x_3 - \sigma_3$. In this second case, we have again a “blown-up” Lie superalgebra as in (4.5), that now reads $\mathfrak{g}'_{\mathbb{Z}}(\mathbf{x})_{\tau} = \mathfrak{g}'_{\mathbb{Z}}(\mathbf{x})_{\bar{0}} \oplus (\tau \mathfrak{g}'_{\mathbb{Z}}(\mathbf{x})_{\bar{1}})$, for which we use the simpler notation $\widehat{\mathfrak{g}}'_{\mathbb{Z}}(\mathbf{x}) := \mathfrak{g}'_{\mathbb{Z}}(\mathbf{x})_{\tau}$; similarly we write also $\widehat{\mathfrak{g}}'_{\mathbb{Z}}(\sigma) := \mathfrak{g}'_{\mathbb{Z}}(\mathbf{x})_{\tau, I_{\sigma}}$. Again, each $\widehat{\mathfrak{g}}'_{\mathbb{Z}}(\sigma)$ for *non-singular* $\sigma \in V^{\times}$ is another $\mathbb{Z}[\sigma]$ -integral form of the complex Lie superalgebra \mathfrak{g}_{σ} we started with. Similarly, working over \mathbb{C} we consider $\mathcal{A} := \mathfrak{g}'(\mathbf{x})$, $\mathcal{F} := \mathfrak{g}'(\mathbf{x})_{\bar{0}}$, $\mathcal{C} := \mathfrak{g}'(\mathbf{x})_{\bar{1}}$ and the blown-up algebra $\mathfrak{g}'(\mathbf{x})_{\tau}$ with ground ring $R := \mathbb{C}[\mathbf{x}]$, and the contraction $\widehat{\mathfrak{g}}'(\sigma) := \mathfrak{g}'(\mathbf{x})_{\tau, I_{\sigma}}$ over \mathbb{C} . This provides one more coherent sheaf of complex Lie superalgebras over V , denoted $\mathcal{L}_{\widehat{\mathfrak{g}}'_{\mathbb{C}[\mathbf{x}]}}$, with a $(\mathbb{C}^{\times} \times \mathfrak{S}_3)$ -action on it, and an associated fibre bundle $\mathbb{L}_{\widehat{\mathfrak{g}}'_{\mathbb{C}[\mathbf{x}]}}$ having the $\widehat{\mathfrak{g}}'(\sigma)$'s as fibres (just like in Section 4.1.1: details are left to the reader). Next result describes the structure of these fibres $\widehat{\mathfrak{g}}'(\sigma)$'s:

Theorem 4.5. *Let $\sigma \in V$, and $\widehat{\mathfrak{a}}'_i := \mathbb{C}X'_{2\varepsilon_i} \oplus \mathbb{C}H'_{2\varepsilon_i} \oplus \mathbb{C}X'_{-2\varepsilon_i}$ for all $i = 1, 2, 3$.*

- (1) *If $\sigma \in V^{\times}$, then the Lie superalgebra $\widehat{\mathfrak{g}}'(\sigma)$ is simple.*
- (2) *If $\sigma \in V \setminus V^{\times}$, then $\widehat{\mathfrak{g}}'(\sigma)_{\bar{0}} \cong \mathfrak{sl}(2)^{\oplus 3}$ as Lie superalgebras, $\widehat{\mathfrak{g}}'(\sigma)_{\bar{1}} \cong \boxtimes_{i=1}^3 \square_i$ as modules over $\widehat{\mathfrak{g}}'(\sigma)_{\bar{0}} \cong \mathfrak{sl}(2)^{\oplus 3}$ and the Lie bracket is trivial on $\widehat{\mathfrak{g}}'(\sigma)_{\bar{1}}$; finally, we have semidirect product splittings*

$$\widehat{\mathfrak{g}}'(\sigma) \cong \widehat{\mathfrak{g}}'(\sigma)_{\bar{0}} \ltimes \widehat{\mathfrak{g}}'(\sigma)_{\bar{1}} \cong \mathfrak{sl}(2)^{\oplus 3} \ltimes (\boxtimes_{i=1}^3 \square_i).$$

In other words, there exists a split short exact sequence

$$0 \longrightarrow \square^{\boxtimes 3} \cong \widehat{\mathfrak{g}}'(\sigma)_{\bar{1}} \longrightarrow \widehat{\mathfrak{g}}'(\sigma) \xleftarrow{\quad} \widehat{\mathfrak{g}}'(\sigma)_{\bar{0}} \cong \mathfrak{sl}(2)^{\oplus 3} \longrightarrow 0.$$

A parallel result also holds true when working with $\widehat{\mathfrak{g}}'_{\mathbb{Z}}(\sigma)$ over the ground ring $\mathbb{Z}[\sigma]$.

Proof. One can easily deduce the claim from Theorem 4.2 along with the fact that each $\widehat{\mathfrak{g}}'(\sigma)$ is a specialization of $\mathfrak{g}'(\mathbf{x})_{\tau}$, and in particular for singular values $\sigma \in V^{\times}$ any such specialization is indeed a *contraction* of $\widehat{\mathfrak{g}}'(\sigma)$, namely of the form $\widehat{\mathfrak{g}}'(\sigma) = \mathfrak{g}'(\mathbf{x})_{\tau, I}$ for the element $\tau := x_1x_2x_3$ and the ideal $I := \{(x_i - \sigma_i)_{i=1,2,3}\}$. As alternative method, one can obtain the statement by means of a direct analysis of the explicit formulas for (linear) generators of $\widehat{\mathfrak{g}}'(\mathbf{x})$: in fact, one easily obtains such formulas as slight modifications – taking into account the “rescaling” of odd generators by the coefficient $\tau := x_1x_2x_3$ – of the formulas in Section 3.3.2. We leave details to the interested reader. \blacksquare

Remark 4.6. We considered five families of Lie superalgebras, denoted by $\{\mathfrak{g}(\sigma)\}_{\sigma \in V}$, $\{\mathfrak{g}'(\sigma)\}_{\sigma \in V}$, $\{\mathfrak{g}''(\sigma)\}_{\sigma \in V}$, $\{\widehat{\mathfrak{g}}(\sigma)\}_{\sigma \in V}$ and $\{\widehat{\mathfrak{g}}'(\sigma)\}_{\sigma \in V}$, all being indexed by the points of the complex plane V . Now, our analysis shows that these five families share most of their elements, namely all those indexed by “general points” $\sigma \in V^{\times} := V \cap (\mathbb{C}^{\times})^3$. On the other hand, the five families are drastically different at all points in the “singular locus” $\mathcal{S} := V \setminus V^{\times} = V \cap (\bigcup_{i=1,2,3} \{\sigma_i = 0\})$. In other words, the five sheaves $\mathcal{L}_{\mathfrak{g}_{\mathbb{C}[\mathbf{x}]}}$, $\mathcal{L}_{\mathfrak{g}'_{\mathbb{C}[\mathbf{x}]}}$, $\mathcal{L}_{\mathfrak{g}''_{\mathbb{C}[\mathbf{x}]}}$, $\mathcal{L}_{\widehat{\mathfrak{g}}_{\mathbb{C}[\mathbf{x}]}}$ and $\mathcal{L}_{\widehat{\mathfrak{g}}'_{\mathbb{C}[\mathbf{x}]}}$ of Lie superalgebras over $\text{Spec}(\mathbb{C}[\mathbf{x}]) \cong V \cup \{\star\} (\cong \mathbb{A}_{\mathbb{C}}^2 \cup \{\star\})$ share the same stalks on all “general” points (i.e., those outside \mathcal{S}), and have different stalks instead on “singular” points (i.e., those in \mathcal{S}). Likewise, the five fibre bundles $\mathbb{L}_{\mathfrak{g}_{\mathbb{C}[\mathbf{x}]}}$, $\mathbb{L}_{\mathfrak{g}'_{\mathbb{C}[\mathbf{x}]}}$, $\mathbb{L}_{\mathfrak{g}''_{\mathbb{C}[\mathbf{x}]}}$, $\mathbb{L}_{\widehat{\mathfrak{g}}_{\mathbb{C}[\mathbf{x}]}}$ and $\mathbb{L}_{\widehat{\mathfrak{g}}'_{\mathbb{C}[\mathbf{x}]}}$ over $\text{Spec}(\mathbb{C}[\mathbf{x}])$ share the same fibres on all general points and have different fibres on singular points. Let us also stress that the second family $\{\mathfrak{g}'(\sigma)\}_{\sigma \in V}$ is just Kaplansky’s one, as $\mathfrak{g}'(\sigma) \cong \mathfrak{g}_{\sigma}$ – cf. Section 4.2. The outcome of the previous discussion is, loosely speaking, that *our construction provides five different “completions” of the family $\{\mathfrak{g}_{\sigma}\}_{\sigma \in V \setminus \mathcal{S}}$ of simple Lie superalgebras, by adding – in five different ways – some new non-simple extra elements on top of each point of*

the “singular locus” \mathcal{S} . In particular, this shows that *it makes no sense to speak of “taking the limit for σ going to \mathcal{S} ” of the simple Lie superalgebras \mathfrak{g}_σ (for $\sigma \in V$)*, unless one states exactly what is the *total* family – namely, indexed over *all* of V – of Lie superalgebras one has chosen to complete the family $\{\mathfrak{g}_\sigma\}_{\sigma \in V \setminus \mathcal{S}}$. Indeed, as our results show, depending on such a choice one finds very different, non-isomorphic “limits”.

5 Lie supergroups of type $D(2, 1; \sigma)$: presentations and degenerations

In this section, we introduce (complex) Lie supergroups of type $D(2, 1; \sigma)$, basing on the five families of Lie superalgebras introduced in Section 4 and following the approach of Section 2.4.3. For simplicity, we formulate everything over \mathbb{C} , but the reader may see some subtleties to discuss about the Chevalley groups over a $\mathbb{Z}[\sigma]$ -algebra. The latter had been discussed in [6] and [7] for some basis, i.e., for one particular choice of $\mathbb{Z}[\sigma]$ -integral form (though with slightly different formalism); in the present case everything works similarly, up to paying attention to the σ -dependence of the commutation relations of the $\mathbb{Z}[\sigma]$ -form one chooses (cf. Section 4). The details are left to the reader.

5.1 First family: the Lie supergroups \mathbf{G}_σ

Given $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$, let $\mathfrak{g} = \mathfrak{g}(\sigma)$ be the complex Lie superalgebra associated with σ as in Section 4.1, and $\mathfrak{g}_{\bar{0}}$ its even part. We recall that \mathfrak{g} is spanned over \mathbb{C} by $\{H_{2\varepsilon_1}, H_{2\varepsilon_2}, H_{2\varepsilon_3}, H_\theta\} \cup \{X_\alpha\}_\Delta$. Like in Section 4.1, we set $\mathfrak{a}_i := \mathbb{C}X_{2\varepsilon_i} \oplus \mathbb{C}H_{2\varepsilon_i} \oplus \mathbb{C}X_{-2\varepsilon_i}$ for each i – all these being Lie subalgebras of $\mathfrak{g}(\sigma)$, with $\mathfrak{g}(\sigma)_{\bar{0}} = \bigoplus_{i=1}^3 \mathfrak{a}_i$. When $\sigma_i \neq 0$, the Lie algebra \mathfrak{a}_i is isomorphic to $\mathfrak{sl}(2)$: an explicit isomorphism is realized by mapping $X_{2\varepsilon_i} \mapsto \sigma_i e$, $H_{2\varepsilon_i} \mapsto \sigma_i h$ and $X_{-2\varepsilon_i} \mapsto \sigma_i f$, where $\{e, h, f\}$ is the standard basis $\mathfrak{sl}(2)$. When $\sigma_i = 0$ instead, $\mathfrak{a}_i \cong \mathbb{C}^{\oplus 3}$ becomes the 3-dimensional Abelian Lie algebra.

Let us now set $A_i := \mathrm{SL}_2$ if $\sigma_i \neq 0$ and $A_i := \mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}$ if $\sigma_i = 0$, and define $G := \times_{i=1}^3 A_i$ – a complex Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}(\sigma)_{\bar{0}}$. One sees that the adjoint action of $\mathfrak{g}(\sigma)_{\bar{0}}$ onto $\mathfrak{g}(\sigma)$ integrates to a Lie group action of G onto $\mathfrak{g}(\sigma)$ again, so that the pair $\mathcal{P}_\sigma := (G, \mathfrak{g}(\sigma))$ – endowed with that action – is a super Harish-Chandra pair (cf. Section 2.4.1); note that its dependence on σ lies within all its constituents: the structure of G , the Lie superalgebra $\mathfrak{g}(\sigma)$, and the action of the former onto the latter.

Finally, we let

$$\mathbf{G}_\sigma := \mathbf{G}_{\mathcal{P}_\sigma}$$

be the complex Lie supergroup associated with the super Harish-Chandra pair \mathcal{P}_σ through the category equivalence given in Section 2.4.3.

5.1.1. A presentation of \mathbf{G}_σ . We shall now provide an explicit presentation by generators and relations for the supergroups \mathbf{G}_σ , i.e., for the abstract groups $\mathbf{G}_\sigma(A)$, $A \in (\mathbf{Wsalg})$.

To begin with, inside each subgroup A_i we consider the elements

$$x_{2\varepsilon_i}(c) := \exp(cX_{2\varepsilon_i}), \quad h_{2\varepsilon_i}(c) := \exp(cH_{2\varepsilon_i}), \quad x_{-2\varepsilon_i}(c) := \exp(cX_{-2\varepsilon_i})$$

for every $c \in \mathbb{C}$; then $\Gamma_i := \{x_{2\varepsilon_i}(c), h_{2\varepsilon_i}(c), x_{-2\varepsilon_i}(c)\}_{c \in \mathbb{C}}$ is a generating set for A_i .

We define also elements $h_\theta(c) := \exp(cH_\theta)$ for all $c \in \mathbb{C}$: then the commutation relations $[H_{2\varepsilon_r}, H_{2\varepsilon_s}] = 0$ and $H_{2\varepsilon_1} + H_{2\varepsilon_2} + H_{2\varepsilon_3} = 2H_\theta$ inside $\mathfrak{g}(\sigma)$ together imply the group relations $h_{2\varepsilon_1}(c)h_{2\varepsilon_2}(c)h_{2\varepsilon_3}(c) = h_\theta(c)^2$ for all $c \in \mathbb{C}$.

The complex Lie group G is clearly generated by

$$\Gamma_0 := \{x_{2\varepsilon_i}(c), h_{2\varepsilon_i}(c), h_\theta(c), x_{-2\varepsilon_i}(c)\}_{c \in \mathbb{C}}^{i \in \{1,2,3\}}$$

(the $h_\theta(c)$'s might be dropped, but we prefer to add them too as generators).

In addition, when we consider G as a (totally even) supergroup and we look at it as a group-valued functor $G: (\mathbf{Wsalg}) \rightarrow (\mathbf{grps})$, the abstract group $G(A)$ of its A -points – for $A \in (\mathbf{Wsalg})$ – is generated by the set

$$\Gamma_0(A) := \{x_{2\varepsilon_i}(a), h_{2\varepsilon_i}(a), h_\theta(a), x_{-2\varepsilon_i}(a)\}_{a \in A_0}^{i \in \{1,2,3\}}. \quad (5.1)$$

Note that here the generators do make sense – as operators in $\mathrm{GL}(A \otimes \mathfrak{g}(\sigma))$, but also formally – since $A = \mathbb{C} \oplus \mathfrak{N}(A)$ (cf. Section 2.1), so each $a \in A$ reads as $a = c + n_a$ for some $c \in \mathbb{C}$ and a nilpotent $n_a \in \mathfrak{N}(A)$, hence $\exp(aX_{2\varepsilon_i}) = \exp(cX_{2\varepsilon_i})\exp(n_aX_{2\varepsilon_i})$, etc., are all well-defined.

Following the recipe in Section 2.4.3, in order to generate the group $\mathbf{G}_\sigma(A) := \mathbf{G}_{\mathfrak{p}_\sigma}(A)$, beside the subgroup $G(A)$ we need also all the elements of the form $(1 + \eta_i Y_i)$ with $(i, \eta_i) \in I \times A_{\bar{1}}$ – cf. Section 2.4.3 – where now the \mathbb{C} -basis $\{Y_i\}_{i \in I}$ of $\mathfrak{g}_{\bar{1}}$ is $\{Y_i\}_{i \in I} = \{X_{\pm\theta}, X_{\pm\beta_i}\}_{i=1,2,3}$. Therefore, we introduce notation $x_{\pm\theta}(\eta) := (1 + \eta X_{\pm\theta})$, $x_{\pm\beta_i}(\eta) := (1 + \eta X_{\pm\beta_i})$ for all $\eta \in A_{\bar{1}}$, $i \in \{1, 2, 3\}$, and we consider the set $\Gamma_{\bar{1}}(A) := \{x_{\pm\theta}(\eta), x_{\pm\beta_i}(\eta) \mid \eta \in A_{\bar{1}}\}$.

Now, taking into account that $G(A)$ is generated by $\Gamma_0(A)$, we can modify the set of relations given in Section 2.4.3 by letting $g \in G(A)$ range inside the set $\Gamma_0(A)$: then we can find the following *full set of relations* (where hereafter we freely use notation $e^Z := \exp(Z)$):

$$\begin{aligned} 1_G &= 1, & g' \cdot g'' &= g' \cdot_G g'', & \forall g', g'' \in G(A), \\ h_{2\varepsilon_i}(a)x_{\pm\beta_j}(\eta)h_{2\varepsilon_i}(a)^{-1} &= x_{\pm\beta_j}(e^{\pm(-1)^{-\delta_{i,j}}\sigma_i a}\eta), \\ h_{2\varepsilon_i}(a)x_{\pm\theta}(\eta)h_{2\varepsilon_i}(a)^{-1} &= x_{\pm\theta}(e^{\pm\sigma_i a}\eta), \\ h_\theta(a)x_{\pm\beta_i}(\eta)h_\theta(a)^{-1} &= x_{\pm\beta_i}(e^{\mp\sigma_i a}\eta), & h_\theta(a)x_{\pm\theta}(\eta)h_\theta(a)^{-1} &= x_{\pm\theta}(\eta), \\ x_{2\varepsilon_i}(a)x_{\beta_j}(\eta)x_{2\varepsilon_i}(a)^{-1} &= x_{\beta_j}(\eta)x_\theta(\delta_{i,j}\sigma_i a\eta), \\ x_{2\varepsilon_i}(a)x_{-\beta_j}(\eta)x_{2\varepsilon_i}(a)^{-1} &= x_{-\beta_j}(\eta)x_{\beta_k}((1 - \delta_{i,j})\sigma_i a\eta), \\ x_{-2\varepsilon_i}(a)x_{\beta_j}(\eta)x_{-2\varepsilon_i}(a)^{-1} &= x_{\beta_j}(\eta)x_{-\beta_k}((1 - \delta_{i,j})\sigma_i a\eta), \\ x_{-2\varepsilon_i}(a)x_{-\beta_j}(\eta)x_{-2\varepsilon_i}(a)^{-1} &= x_{-\beta_j}(\eta)x_{-\theta}(\delta_{i,j}\sigma_i a\eta), \\ x_{2\varepsilon_i}(a)x_\theta(\eta)x_{2\varepsilon_i}(a)^{-1} &= x_\theta(\eta), \\ x_{2\varepsilon_i}(a)x_{-\theta}(\eta)x_{2\varepsilon_i}(a)^{-1} &= x_{-\theta}(\eta)x_{-\beta_i}(\sigma_i a\eta), \\ x_{-2\varepsilon_i}(a)x_\theta(\eta)x_{-2\varepsilon_i}(a)^{-1} &= x_\theta(\eta)x_{\beta_i}(\sigma_i a\eta), \\ x_{-2\varepsilon_i}(a)x_{-\theta}(\eta)x_{-2\varepsilon_i}(a)^{-1} &= x_{-\theta}(\eta) \\ x_{\beta_i}(\eta_i)x_{\beta_j}(\eta'_j) &= x_{2\varepsilon_k}((1 - \delta_{i,j})\eta'_j\eta_i)x_{\beta_j}(\eta'_j)x_{\beta_i}(\eta_i), \\ x_{-\beta_i}(\eta_i)x_{-\beta_j}(\eta'_j) &= x_{-2\varepsilon_k}(-(1 - \delta_{i,j})\eta'_j\eta_i)x_{-\beta_j}(\eta'_j)x_{-\beta_i}(\eta_i), \\ x_{\beta_i}(\eta_i)x_{-\beta_j}(\eta'_j) &= h_{2\varepsilon_i}(\delta_{i,j}\eta'_j\eta_i)h_\theta(-\delta_{i,j}\eta'_j\eta_i)x_{-\beta_j}(\eta'_j)x_{\beta_i}(\eta_i), \\ x_{\beta_i}(\eta_i)x_\theta(\eta) &= x_\theta(\eta)x_{\beta_i}(\eta_i), & x_{\beta_i}(\eta_i)x_{-\theta}(\eta) &= x_{-2\varepsilon_i}(\eta\eta_i)x_{-\theta}(\eta)x_{\beta_i}(\eta_i), \\ x_{-\beta_i}(\eta_i)x_\theta(\eta) &= x_{2\varepsilon_i}(-\eta\eta_i)x_\theta(\eta)x_{-\beta_i}(\eta_i), & x_{-\beta_i}(\eta_i)x_{-\theta}(\eta) &= x_{-\theta}(\eta)x_{-\beta_i}(\eta_i), \\ x_\theta(\eta_+)x_{-\theta}(\eta_-) &= h_\theta(\eta_-\eta_+)x_{-\theta}(\eta_-)x_\theta(\eta_+), \\ x_{\pm\beta_i}(\eta')x_{\pm\beta_i}(\eta'') &= x_{\pm\beta_i}(\eta' + \eta''), & x_{\pm\theta}(\eta')x_{\pm\theta}(\eta'') &= x_{\pm\theta}(\eta' + \eta'') \end{aligned}$$

with $\{i, j, k\} \in \{1, 2, 3\}$.

5.1.2. Singular specializations of the supergroup(s) \mathbf{G}_σ . From the very construction of the supergroups \mathbf{G}_σ , we get that

\mathbf{G}_σ is simple (as a Lie supergroup) for all $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V^\times$,

where we recall that a Lie supergroup is said to be *simple* if it has no non-trivial normal closed connected Lie sub-supergroup. This follows from the presentation of \mathbf{G}_σ in Section 5.1.1 above, or it can be seen as a direct consequence of the relation $\text{Lie}(\mathbf{G}_\sigma) = \mathfrak{g}(\sigma) = \mathfrak{g}_\sigma$ and of Proposition 3.1.

On the other hand, the situation is different at “singular values” of the parameter σ : the following records the whole situation.

Theorem 5.1. *Let $\sigma \in V$ as usual, and keep notation as above.*

- (1) *If $\sigma \in V^\times$, then the Lie supergroup \mathbf{G}_σ is simple.*
- (2) *If $\sigma \in V \setminus V^\times$, with $\sigma_i = 0$ and $\sigma_j \neq 0 \neq \sigma_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then A_i is a central subgroup of \mathbf{G}_σ , isomorphic to $\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}$, and \mathbf{G}_σ is the universal central extension of $\mathbb{P}\text{SL}(2|2)$ by A_i ; in other words, there exists a short exact sequence of Lie supergroups*

$$\mathbf{1} \longrightarrow \mathbb{C} \times \mathbb{C}^\times \times \mathbb{C} \cong A_i \longrightarrow \mathbf{G}_\sigma \longrightarrow \mathbb{P}\text{SL}(2|2) \longrightarrow \mathbf{1}.$$

- (3) *If $\sigma = \mathbf{0} (\in V \setminus V^\times)$, i.e., $\sigma_h = 0$ for all $h \in \{1, 2, 3\}$, then $(\mathbf{G}_\sigma)_{\text{rd}} \cong (\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C})^{\times 3}$ is the center of \mathbf{G}_σ , and the quotient $\mathbf{G}_\sigma / (\mathbf{G}_\sigma)_{\text{rd}} \cong \mathbb{C}^8$ is Abelian; in particular, \mathbf{G}_σ is a central extension of \mathbb{C}^8 by $(\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C})^{\times 3}$, i.e., there exists a short exact sequence of Lie supergroups, with non-Abelian middle term,*

$$\mathbf{1} \longrightarrow (\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C})^{\times 3} \cong (\mathbf{G}_\sigma)_{\text{rd}} \longrightarrow \mathbf{G}_\sigma \longrightarrow \mathbb{C}^8 \longrightarrow \mathbf{1}.$$

Proof. The claim follows directly from the presentation of \mathbf{G}_σ given in Section 5.1.1 above, or also from the relation $\text{Lie}(\mathbf{G}_\sigma) = \mathfrak{g}(\sigma)$ along with Theorem 4.1. \blacksquare

5.2 Second family: the Lie supergroups \mathbf{G}'_σ

Given $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$, let $\mathfrak{g}' := \mathfrak{g}'(\sigma)$ be the complex Lie superalgebra associated with σ as in Section 4.2.1, and let \mathfrak{g}'_0 be its even part. Fix the \mathbb{C} -basis $\{X'_{2\varepsilon_i}, H'_{2\varepsilon_i}, X'_{-2\varepsilon_i}\}_{i=1,2,3}$ of \mathfrak{g}'_0 as in Section 3.3, and set $\mathfrak{a}'_i := \mathbb{C}X'_{2\varepsilon_i} \oplus \mathbb{C}H'_{2\varepsilon_i} \oplus \mathbb{C}X'_{-2\varepsilon_i}$ for each i : each one of these is a Lie subalgebra of \mathfrak{g}'_0 , with $\mathfrak{g}'_0 = \mathfrak{a}'_1 \oplus \mathfrak{a}'_2 \oplus \mathfrak{a}'_3$. Moreover, each Lie algebra \mathfrak{a}'_i is isomorphic to $\mathfrak{sl}(2)$, an explicit isomorphism being given by $X'_{2\varepsilon_i} \mapsto e$, $H'_{2\varepsilon_i} \mapsto h$ and $X'_{-2\varepsilon_i} \mapsto f$, where $\{e, h, f\}$ is the standard basis $\mathfrak{sl}(2)$. It follows that \mathfrak{g}'_0 is isomorphic to $\mathfrak{sl}(2)^{\oplus 3}$. For each $i \in \{1, 2, 3\}$, let A'_i be a copy of SL_2 , and set $G' := A'_1 \times A'_2 \times A'_3$. By the previous analysis, $\text{Lie}(G')$ is isomorphic to \mathfrak{g}'_0 and the $\text{Lie}(G')$ -action lifts to a holomorphic G' -action on \mathfrak{g}' again: in fact, one easily sees that this action is faithful too. With this action, $\mathcal{P}'_\sigma := (G', \mathfrak{g}')$ is a super Harish-Chandra pair (cf. Section 2.4.1), which overall depends on \mathcal{P}'_σ (although G' alone does not). Finally, we define

$$\mathbf{G}'_\sigma := \mathbf{G}_{\mathcal{P}'_\sigma}$$

to be the complex Lie supergroup associated with the super Harish-Chandra pair \mathcal{P}'_σ via the equivalence of categories given in Section 2.4.3.

5.2.1. A presentation of \mathbf{G}'_σ . The supergroups \mathbf{G}'_σ can be described in concrete terms via an explicit presentation by generators and relations of all the abstract groups $\mathbf{G}'_\sigma(A)$, with A ranging in **(Wsalg)**. To this end, we first consider the Lie group $G' = A'_1 \times A'_2 \times A'_3$ with $A'_i \cong \text{SL}_2$. Letting

exp: $\mathfrak{g}'_0 \cong \text{Lie}(G') \longrightarrow G'$ be the exponential map, we consider $x'_{2\varepsilon_i}(c) := \exp(cX'_{2\varepsilon_i})$, $h'_{2\varepsilon_i}(c) := \exp(cH'_{2\varepsilon_i})$, $x'_{-2\varepsilon_i}(c) := \exp(cX'_{-2\varepsilon_i})$ and $h'_\theta(c) := \exp(cH'_\theta)$ for all $c \in \mathbb{C}$. Note that the commutation relations $[H'_{2\varepsilon_r}, H'_{2\varepsilon_s}] = 0$ along with $\sigma_1 H'_{2\varepsilon_1} + \sigma_2 H'_{2\varepsilon_2} + \sigma_3 H'_{2\varepsilon_3} = 2H'_\theta$ inside $\mathfrak{g}'(\sigma)_{\mathbb{C}}$ together imply, inside G'_+ , the group relations $h'_{2\varepsilon_1}(\sigma_1 c)h'_{2\varepsilon_2}(\sigma_2 c)h'_{2\varepsilon_3}(\sigma_3 c) = h'_\theta(c)^2$ for all $c \in \mathbb{C}$. The complex Lie group G' is clearly generated by the set

$$\Gamma'_0 := \{x'_{2\varepsilon_i}(c), h'_{2\varepsilon_i}(c), h'_\theta(c), x'_{-2\varepsilon_i}(c) \mid c \in \mathbb{C}\}$$

(where the $h'_\theta(c)$'s might be discarded, but we prefer to keep them). Then, looking at G' as a (totally even) supergroup thought of as a group-valued functor $G': (\mathbf{Wsalg}) \longrightarrow (\mathbf{grps})$, each abstract group $G'(A)$ of its A -points – for $A \in (\mathbf{Wsalg})$ – is generated by the set

$$\Gamma'_0(A) := \{x'_{2\varepsilon_i}(a), h'_{2\varepsilon_i}(a), h'_\theta(a), x'_{-2\varepsilon_i}(a) \mid a \in A_{\bar{0}}\}. \quad (5.2)$$

Following Section 2.4.3, we need as generators of $\mathbf{G}'_\sigma(A) := \mathbf{G}_{\rho'_\sigma}(A)$ all the elements of $G'(A)$ and all those of the form $x'_{\pm\theta}(\eta) := (1 + \eta X'_{\pm\theta})$ or $x'_{\pm\beta_i}(\eta) := (1 + \eta X'_{\pm\beta_i})$ with $\eta \in A_{\bar{1}}$ and $i \in \{1, 2, 3\}$ – since now we fix $\{Y'_i\}_{i \in I} = \{X'_{\pm\theta}, X'_{\pm\beta_i}\}_{i=1,2,3}$ as our \mathbb{C} -basis of $\mathfrak{g}'_{\bar{1}}$; we denote the set of all the latter by $\Gamma'_1(A) := \{x'_{\pm\theta}(\eta), x'_{\pm\beta_i}(\eta) \mid \eta \in A_{\bar{1}}\}$.

Implementing the recipe in Section 2.4.3, and recalling that $G'(A)$ is generated by $\Gamma'_0(A)$, we can now slightly modify the relations presented in Section 2.4.3 and consider instead the following, alternative *full set of relations* among the generators of $\mathbf{G}'_\sigma(A)$:

$$\begin{aligned} 1_G &= 1, & g' \cdot g'' &= g' \cdot_{G'} g'', & \forall g', g'' \in G'(A), \\ h'_{2\varepsilon_i}(a)x'_{\pm\beta_j}(\eta)h'_{2\varepsilon_i}(a)^{-1} &= x'_{\pm\beta_j}(e^{\pm(-1)^{-\delta_{i,j}}a}\eta), \\ h'_{2\varepsilon_i}(a)x'_{\pm\theta}(\eta)h'_{2\varepsilon_i}(a)^{-1} &= x'_{\pm\theta}(e^{\pm a}\eta), \\ h'_\theta(a)x'_{\pm\beta_i}(\eta)h'_\theta(a)^{-1} &= x'_{\pm\beta_i}(e^{\mp\sigma_i a}\eta), & h'_\theta(a)x'_{\pm\theta}(\eta)h'_\theta(a)^{-1} &= x'_{\pm\theta}(\eta), \\ x'_{2\varepsilon_i}(a)x'_{\beta_j}(\eta)x'_{2\varepsilon_i}(a)^{-1} &= x'_{\beta_j}(\eta)x'_\theta(\delta_{i,j}a\eta), \\ x'_{2\varepsilon_i}(a)x'_{-\beta_j}(\eta)x'_{2\varepsilon_i}(a)^{-1} &= x'_{-\beta_j}(\eta)x'_{\beta_k}((1 - \delta_{i,j})a\eta), \\ x'_{-2\varepsilon_i}(a)x'_{\beta_j}(\eta)x'_{-2\varepsilon_i}(a)^{-1} &= x'_{\beta_j}(\eta)x'_{-\beta_k}((1 - \delta_{i,j})a\eta), \\ x'_{-2\varepsilon_i}(a)x'_{-\beta_j}(\eta)x'_{-2\varepsilon_i}(a)^{-1} &= x'_{-\beta_j}(\eta)x'_{-\theta}(\delta_{i,j}a\eta), \\ x'_{2\varepsilon_i}(a)x'_\theta(\eta)x'_{2\varepsilon_i}(a)^{-1} &= x'_\theta(\eta), \\ x'_{2\varepsilon_i}(a)x'_{-\theta}(\eta)x'_{2\varepsilon_i}(a)^{-1} &= x'_{-\theta}(\eta)x'_{-\beta_i}(a\eta), \\ x'_{-2\varepsilon_i}(a)x'_\theta(\eta)x'_{-2\varepsilon_i}(a)^{-1} &= x'_\theta(\eta)x'_{\beta_i}(a\eta), \\ x'_{-2\varepsilon_i}(a)x'_{-\theta}(\eta)x'_{-2\varepsilon_i}(a)^{-1} &= x'_{-\theta}(\eta), \\ x'_{\beta_i}(\eta_i)x'_{\beta_j}(\eta'_j) &= x'_{2\varepsilon_k}(+(1 - \delta_{i,j})\sigma_i\eta'_j\eta_i)x'_{\beta_j}(\eta'_j)x'_{\beta_i}(\eta_i), \\ x'_{-\beta_i}(\eta_i)x'_{-\beta_j}(\eta'_j) &= x'_{-2\varepsilon_k}(-(1 - \delta_{i,j})\sigma_i\eta'_j\eta_i)x'_{-\beta_j}(\eta'_j)x'_{-\beta_i}(\eta_i), \\ x'_{\beta_i}(\eta_i)x'_{-\beta_j}(\eta'_j) &= h'_{2\varepsilon_i}(\delta_{i,j}\sigma_i\eta'_j\eta_i)h'_\theta(-\delta_{i,j}\eta'_j\eta_i)x'_{-\beta_j}(\eta'_j)x'_{\beta_i}(\eta_i), \\ x'_{\beta_i}(\eta_i)x'_\theta(\eta) &= x'_\theta(\eta)x'_{\beta_i}(\eta_i), & x'_{\beta_i}(\eta_i)x'_{-\theta}(\eta) &= x'_{-2\varepsilon_i}(+\sigma_i\eta\eta_i)x'_{-\theta}(\eta)x'_{+\beta_i}(\eta_i), \\ x'_{-\beta_i}(\eta_i)x'_\theta(\eta) &= x'_{2\varepsilon_i}(-\sigma_i\eta\eta_i)x'_\theta(\eta)x'_{-\beta_i}(\eta_i), & x'_{-\beta_i}(\eta_i)x'_{-\theta}(\eta) &= x'_{-\theta}(\eta)x'_{-\beta_i}(\eta_i), \\ x'_\theta(\eta_+)x'_{-\theta}(\eta_-) &= v h'_\theta(\eta_-\eta_+)x'_{-\theta}(\eta_-)x'_\theta(\eta_+), \\ x'_{\pm\beta_i}(\eta')x'_{\pm\beta_i}(\eta'') &= x'_{\pm\beta_i}(\eta' + \eta''), & x'_{\pm\theta}(\eta')x'_{\pm\theta}(\eta'') &= x'_{\pm\theta}(\eta' + \eta'') \end{aligned}$$

with $\{i, j, k\} = \{1, 2, 3\}$.

5.2.2. Singular specializations of the supergroup(s) \mathbf{G}'_σ . By construction, for the supergroups \mathbf{G}'_σ we have that

\mathbf{G}'_σ is simple (as a Lie supergroup) for all $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V^\times$.

Indeed, this follows from the presentation of \mathbf{G}'_σ in Section 5.2.1 above, but also as a fallout of the relation $\text{Lie}(\mathbf{G}'_\sigma) = \mathfrak{g}'(\sigma)_\mathbb{C} = \mathfrak{g}_\sigma$ along with Proposition 3.1. The situation is different at “singular values” of the parameter σ ; the complete result is

Theorem 5.2. *Given $\sigma \in V$, keep notation as above.*

- (1) *If $\sigma \in V^\times$, then the Lie supergroup \mathbf{G}'_σ is simple.*
- (2) *If $\sigma \in V \setminus V^\times$, with $\sigma_i = 0$ and $\sigma_j \neq 0 \neq \sigma_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then letting \mathbf{B}'_i be the Lie subsupergroup of \mathbf{G}'_σ defined on every $A \in (\mathbf{W}\text{salg})$ by*

$$\mathbf{B}'_i(A) := \langle \{h'_{2\varepsilon_i}(a), x'_{\pm\alpha}(b), x'_{\pm\beta}(\eta)\}_{a \in A_0, b \in A_{\bar{0}}, \eta \in A_{\bar{1}}}^{t \neq i, \alpha \in \Delta_{\bar{0}} \setminus \{2\varepsilon_i\}, \beta \in \Delta_{\bar{1}}}\rangle$$

we have $\mathbf{B}'_i \trianglelefteq \mathbf{G}'_\sigma$ (a normal Lie subsupergroup), $A'_i \leq \mathbf{G}'_\sigma$ (a Lie subsupergroup), and there exist isomorphisms $\mathbf{B}'_i \cong \mathbb{P}\text{SL}(2|2)$, $A'_i \cong \text{SL}_2$ and $\mathbf{G}'_\sigma \cong \text{SL}_2 \times \mathbb{P}\text{SL}(2|2)$ – a semidirect product of Lie supergroups. In short, there exists a split short exact sequence

$$\mathbf{1} \longrightarrow \mathbb{P}\text{SL}(2|2) \cong \mathbf{B}'_i \longrightarrow \mathbf{G}'_\sigma \overset{\longleftarrow}{\longrightarrow} A'_i \cong \text{SL}_2 \longrightarrow \mathbf{1}.$$

- (3) *If $\sigma = \mathbf{0} (\in V \setminus V^\times)$, i.e., $\sigma_h = 0$ for all $h \in \{1, 2, 3\}$, then letting $(\mathbf{G}'_\sigma)_{\bar{1}}$ be the Lie subsupergroup of \mathbf{G}'_σ defined on every $A \in (\mathbf{W}\text{salg})$ by*

$$(\mathbf{G}'_\sigma)_{\bar{1}}(A) := \langle \{x'_\alpha(\eta)\}_{\eta \in A_{\bar{1}}}^{\alpha \in \Delta_{\bar{1}}}\rangle$$

we have $(\mathbf{G}'_\sigma)_{\text{rd}} \cong \text{SL}_2^3$ and $(\mathbf{G}'_\sigma)_{\bar{1}} \cong \mathbb{C}^8$ as Lie (super)groups, $(\mathbf{G}'_\sigma)_{\bar{1}} \cong \boxtimes_{i=1}^3 \square_i \cong \square^{\boxtimes 3}$ ($\cong \mathbb{C}^8$) as modules over $(\mathbf{G}'_\sigma)_{\bar{0}} \cong \text{SL}_2^3$ – where $\square_i \cong \square := \mathbb{C}|+ \oplus \mathbb{C}|-$ is the tautological 2-dimensional module over the i -th copy $\text{SL}_2^{(i)}$ of SL_2 (for $i = 1, 2, 3$). Finally, we have

$$\mathbf{G}'_\sigma \cong (\mathbf{G}'_\sigma)_{\text{rd}} \times (\mathbf{G}'_\sigma)_{\bar{1}} \cong \text{SL}_2^3 \times (\boxtimes_{i=1}^3 \square_i) \cong \text{SL}_2^3 \times \square^{\boxtimes 3}$$

– a semidirect product of Lie supergroups.

In other words, there is a split short exact sequence

$$\mathbf{1} \longrightarrow \square^{\boxtimes 3} \cong (\mathbf{G}'_\sigma)_{\bar{1}} \longrightarrow \mathbf{G}'_\sigma \overset{\longleftarrow}{\longrightarrow} (\mathbf{G}'_\sigma)_{\text{rd}} \cong \text{SL}_2^{\times 3} \longrightarrow \mathbf{1}.$$

Proof. Like for Theorem 5.1, the present claim can be obtained from the presentation of \mathbf{G}'_σ in Section 5.2.1, or otherwise from the relation $\text{Lie}(\mathbf{G}'_\sigma) = \mathfrak{g}'(\sigma)$ along with Theorem 4.2. \blacksquare

5.3 Third family: the Lie supergroups \mathbf{G}''_σ

Given $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$, let $\mathfrak{g}'' := \mathfrak{g}''(\sigma)$ be the complex Lie superalgebra associated with σ as in Section 4.3.1, and let \mathfrak{g}''_0 be its even part. Fix the elements $X_\alpha, H'_{2\varepsilon_i}, H'_\theta$ (with $\alpha \in \Delta$, $i \in \{1, 2, 3\}$) of \mathfrak{g} as in Section 3.3.3 and set $\mathfrak{a}''_i := \mathbb{C}X_{2\varepsilon_i} \oplus \mathbb{C}H'_{2\varepsilon_i} \oplus \mathbb{C}X_{-2\varepsilon_i}$ for each i : the latter are Lie subalgebras of \mathfrak{g}''_0 such that $\mathfrak{g}''_0 = \mathfrak{a}''_1 \oplus \mathfrak{a}''_2 \oplus \mathfrak{a}''_3$. Moreover, every \mathfrak{a}''_i is isomorphic to $\mathfrak{sl}(2)$ when $\sigma_i \neq 0$ – an explicit isomorphism being given by $X_{2\varepsilon_i} \mapsto \sigma_i e$, $H'_{2\varepsilon_i} \mapsto h$ and $X_{-2\varepsilon_i} \mapsto \sigma_i f$, where $\{e, h, f\}$ is the standard basis $\mathfrak{sl}(2)$ – while for $\sigma_i = 0$ it is isomorphic to the Lie subalgebra of $\mathfrak{b}_+ \oplus \mathfrak{b}_-$, with $\mathfrak{b}_+ := \mathbb{C}e + \mathbb{C}h$ and $\mathfrak{b}_- := \mathbb{C}h + \mathbb{C}f$ being the standard Borel subalgebras inside $\mathfrak{sl}(2)$, with \mathbb{C} -basis $\{(e, 0), (h, h), (0, f)\}$.

Let B_{\pm} be the Borel subgroup of SL_2 of all upper, resp. lower, triangular matrices, and let S be the subgroup of $B_+ \times B_-$ whose elements are all the pairs of matrices (X_+, X_-) such that the diagonal parts of X_+ and of X_- are the same. For each $i \in \{1, 2, 3\}$, let A_i'' (depending on σ_i) respectively be a copy of SL_2 if $\sigma_i \neq 0$ and a copy of S otherwise; then set $G'' := A_1'' \times A_2'' \times A_3''$. The adjoint action of $\mathfrak{g}_0'' \cong \mathrm{Lie}(G'')$ on \mathfrak{g}'' lifts to a holomorphic G'' -action on \mathfrak{g}'' , which is faithful again; then the pair $\mathcal{P}_{\sigma}'' := (G'', \mathfrak{g}'')$ with this action is a super Harish-Chandra pair, in the sense of Section 2.4.1. At last, we can define

$$\mathbf{G}_{\sigma}'' := \mathbf{G}_{\mathcal{P}_{\sigma}''}$$

as being the complex Lie supergroup associated with the super Harish-Chandra pair \mathcal{P}_{σ}'' through the equivalence of categories given in Section 2.4.3.

5.3.1. A presentation of \mathbf{G}_{σ}'' . In order to describe the supergroups \mathbf{G}_{σ}'' , we aim now for an explicit presentation by generators and relations of the abstract groups $\mathbf{G}_{\sigma}''(A)$, for all $A \in (\mathbf{Wsalg})$. To start with, let $G'' = A_1'' \times A_2'' \times A_3''$ be the complex Lie group considered above, and let $\exp: \mathfrak{g}_0'' \cong \mathrm{Lie}(G'') \longrightarrow G''$ be the exponential map: then consider $x_{2\varepsilon_i}(c) := \exp(cX_{2\varepsilon_i})$, $h'_{2\varepsilon_i}(c) := \exp(cH'_{2\varepsilon_i})$, $x_{-2\varepsilon_i}(c) := \exp(cX_{-2\varepsilon_i})$ and $h'_{\theta}(c) := \exp(cH'_{\theta})$ for all $c \in \mathbb{C}$. It is clear that G'' is generated by the set

$$\Gamma_0'' := \{x_{2\varepsilon_i}(c), h'_{2\varepsilon_i}(c), h'_{\theta}(c), x_{-2\varepsilon_i}(c) \mid c \in \mathbb{C}\}$$

(actually the $h'_{\theta}(c)$'s might be discarded, but we choose to keep them); therefore, looking at G'' as a supergroup, thought of as a group-valued functor $G'': (\mathbf{Wsalg}) \longrightarrow (\mathbf{grps})$, every abstract group $G''(A)$ of its A -points – for $A \in (\mathbf{Wsalg})$ – is generated by the set

$$\Gamma_0''(A) := \{x_{2\varepsilon_i}(a), h'_{2\varepsilon_i}(a), h'_{\theta}(a), x_{-2\varepsilon_i}(a) \mid a \in A_0\}. \quad (5.3)$$

According to Section 2.4.3, the group $\mathbf{G}_{\sigma}''(A) := \mathbf{G}_{\mathcal{P}_{\sigma}''}(A)$ is generated by $G''(A)$ and all elements of the form $x_{\pm\theta}(\eta) := (1 + \eta X_{\pm\theta})$ or $x_{\pm\beta_i}(\eta) := (1 + \eta X_{\pm\beta_i})$ with $\eta \in A_{\bar{1}}$ and $i \in \{1, 2, 3\}$ – as now the chosen \mathbb{C} -basis of \mathfrak{g}_1'' is $\{Y'_i\}_{i \in I} = \{X_{\pm\theta}, X_{\pm\beta_i}\}_{i=1,2,3}$; the set of all the latter is denoted $\Gamma_1''(A) := \{x_{\pm\theta}(\eta), x_{\pm\beta_i}(\eta) \mid \eta \in A_{\bar{1}}\}$ – coinciding with $\Gamma_1''(A)$ in Section 5.2.1.

From the recipe in Section 2.4.3, and the fact that $G''(A)$ is generated by $\Gamma_0''(A)$, with a slight modification of the relations in Section 2.4.3 we can find the following *full set of relations* among generators of $\mathbf{G}_{\sigma}''(A)$ (for all $\{i, j, k\} = \{1, 2, 3\}$):

$$\begin{aligned} 1_{G''} &= 1, & g' \cdot g'' &= g' \cdot_{G''} g'', & \forall g', g'' \in G''(A), \\ h'_{2\varepsilon_i}(a)x_{\pm\beta_j}(\eta)h'_{2\varepsilon_i}(a)^{-1} &= x_{\pm\beta_j}(e^{\pm(-1)^{\delta_{i,j}}a}\eta), \\ h'_{2\varepsilon_i}(a)x_{\pm\theta}(\eta)h'_{2\varepsilon_i}(a)^{-1} &= x_{\pm\theta}(e^{\pm a}\eta), \\ h_{\theta}(a)x_{\pm\beta_i}(\eta)h_{\theta}(a)^{-1} &= x_{\pm\beta_i}(e^{\mp\sigma_i a}\eta), & h_{\theta}(a)x_{\pm\theta}(\eta)h_{\theta}(a)^{-1} &= x_{\pm\theta}(\eta), \\ x_{2\varepsilon_i}(a)x_{\beta_j}(\eta)x_{2\varepsilon_i}(a)^{-1} &= x_{\beta_j}(\eta)x_{\theta}(\delta_{i,j}\sigma_i a\eta), \\ x_{2\varepsilon_i}(a)x_{-\beta_j}(\eta)x_{2\varepsilon_i}(a)^{-1} &= x_{-\beta_j}(\eta)x_{\beta_k}((1 - \delta_{i,j})\sigma_i a\eta), \\ x_{-2\varepsilon_i}(a)x_{\beta_j}(\eta)x_{-2\varepsilon_i}(a)^{-1} &= x_{\beta_j}(\eta)x_{-\beta_k}((1 - \delta_{i,j})\sigma_i a\eta), \\ x_{-2\varepsilon_i}(a)x_{-\beta_j}(\eta)x_{-2\varepsilon_i}(a)^{-1} &= x_{-\beta_j}(\eta)x_{-\theta}(\delta_{i,j}\sigma_i a\eta), \\ x_{2\varepsilon_i}(a)x_{\theta}(\eta)x_{2\varepsilon_i}(a)^{-1} &= x_{\theta}(\eta), \\ x_{2\varepsilon_i}(a)x_{-\theta}(\eta)x_{2\varepsilon_i}(a)^{-1} &= x_{-\theta}(\eta)x_{-\beta_i}(\sigma_i a\eta), \\ x_{-2\varepsilon_i}(a)x_{\theta}(\eta)x_{-2\varepsilon_i}(a)^{-1} &= x_{\theta}(\eta)x_{\beta_i}(\sigma_i a\eta), \\ x_{-2\varepsilon_i}(a)x_{-\theta}(\eta)x_{-2\varepsilon_i}(a)^{-1} &= x_{-\theta}(\eta), \end{aligned}$$

$$\begin{aligned}
x_{\beta_i}(\eta_i)x_{\beta_j}(\eta'_j) &= x_{2\varepsilon_k}((1 - \delta_{i,j})\eta'_j\eta_i)x_{\beta_j}(\eta'_j)x_{\beta_i}(\eta_i), \\
x_{-\beta_i}(\eta_i)x_{-\beta_j}(\eta'_j) &= x_{-2\varepsilon_k}(-(1 - \delta_{i,j})\eta'_j\eta_i)x_{-\beta_j}(\eta'_j)x_{-\beta_i}(\eta_i), \\
x_{\beta_i}(\eta_i)x_{-\beta_j}(\eta'_j) &= h'_{2\varepsilon_i}(\delta_{i,j}\sigma_i\eta'_j\eta_i)h'_\theta(-\delta_{i,j}\eta'_j\eta_i)x_{-\beta_j}(\eta'_j)x_{\beta_i}(\eta_i), \\
x_{\beta_i}(\eta_i)x_\theta(\eta) &= x_\theta(\eta)x_{\beta_i}(\eta_i), \quad x_{\beta_i}(\eta_i)x_{-\theta}(\eta) = x_{-2\varepsilon_i}(\eta\eta_i)x_{-\theta}(\eta)x_{+\beta_i}(\eta_i), \\
x_{-\beta_i}(\eta_i)x_\theta(\eta) &= x_{2\varepsilon_i}(-\eta\eta_i)x_\theta(\eta)x_{-\beta_i}(\eta_i), \quad x_{-\beta_i}(\eta_i)x_{-\theta}(\eta) = x_{-\theta}(\eta)x_{-\beta_i}(\eta_i), \\
x_\theta(\eta_+)x_{-\theta}(\eta_-) &= h'_\theta(\eta_-\eta_+)x_{-\theta}(\eta_-)x_\theta(\eta_+), \\
x_{\pm\beta_i}(\eta')x_{\pm\beta_i}(\eta'') &= x_{\pm\beta_i}(\eta' + \eta''), \quad x_{\pm\theta}(\eta')x_{\pm\theta}(\eta'') = x_{\pm\theta}(\eta' + \eta'').
\end{aligned}$$

5.3.2. Singular specializations of the supergroup(s) \mathbf{G}''_σ . One sees easily that for the supergroups \mathbf{G}''_σ we have

\mathbf{G}''_σ is simple (as a Lie supergroup) for all $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V^\times$.

This follows from the presentation of \mathbf{G}''_σ in Section 5.3.1 above, and also as a consequence of the relation $\text{Lie}(\mathbf{G}''_\sigma) = \mathfrak{g}''(\sigma) = \mathfrak{g}_\sigma$ along with Proposition 3.1.

Things change, instead, for “singular values” of σ : hereafter is the general result.

Theorem 5.3. *Given $\sigma \in V$, keep notation as above.*

- (1) *If $\sigma \in V^\times$, then the Lie supergroup \mathbf{G}''_σ is simple.*
- (2) *If $\sigma \in V \setminus V^\times$, with $\sigma_i = 0$ and $\sigma_j \neq 0 \neq \sigma_k$ for $\{i, j, k\} = \{1, 2, 3\}$, consider the subsupergroup \mathbf{K}''_i of \mathbf{G}''_σ which is given by*

$$\mathbf{K}''_i(A) := \langle \{x_{2\varepsilon_i}(a_+), x_{-2\varepsilon_i}(a_-)\}_{a_+, a_- \in A_0} \rangle \quad \forall A \in (\mathbf{Wsalg}).$$

Then \mathbf{K}''_i is an Abelian normal Lie subsupergroup of \mathbf{G}''_σ , hence – letting $\overline{\mathbf{B}''_i} := \mathbf{G}''_\sigma/\mathbf{K}''_i$ be the quotient supergroup – there exists a short exact sequence

$$\mathbf{1} \longrightarrow \mathbf{K}''_i \longrightarrow \mathbf{G}''_\sigma \longrightarrow \overline{\mathbf{B}''_i} \longrightarrow \mathbf{1}.$$

Furthermore, defining inside $\overline{\mathbf{B}''_i}(A)$ the two subgroups $\overline{\mathbf{H}''_i}(A) := \langle \{\overline{h'_{2\varepsilon_i}(a_i)}\}_{a_i \in A_0} \rangle$ and $\overline{\mathbf{D}''_i}(A) := \langle \{\overline{h'_{2\varepsilon_j}(a_j)}, \overline{x_{+2\varepsilon_j}(c_+)}, \overline{x_{-2\varepsilon_j}(c_-)}, \overline{x_\beta(\eta)}\}_{\substack{j \neq i, \beta \in \Delta_{\overline{1}} \\ a_j, c_\pm \in A_0, \eta \in A_{\overline{1}}}} \rangle$ – for all $A \in (\mathbf{Wsalg})$ – we overall find two Lie subsupergroups $\overline{\mathbf{D}''_i}$ and $\overline{\mathbf{H}''_i}$ of $\overline{\mathbf{B}''_i}$ such that $\overline{\mathbf{D}''_i}$ is a normal Lie subsupergroup with $\overline{\mathbf{D}''_i} \cong \mathbb{P}\text{SL}(2|2)$, $\overline{\mathbf{H}''_i} \cong \mathbb{C}^\times$, and $\overline{\mathbf{B}''_i} \cong \overline{\mathbf{H}''_i} \ltimes \overline{\mathbf{D}''_i}$ – a semidirect product of Lie supergroups. In short, there exists a split short exact sequence

$$\mathbf{1} \longrightarrow \mathbb{P}\text{SL}(2|2) \cong \overline{\mathbf{D}''_i} \longrightarrow \overline{\mathbf{B}''_i} \overset{\leftarrow}{\dashrightarrow} \overline{\mathbf{H}''_i} \cong \mathbb{C}^\times \longrightarrow \mathbf{1}.$$

- (3) *If $\sigma = \mathbf{0} (\in V \setminus V^\times)$, i.e., $\sigma_h = 0$ for all $h \in \{1, 2, 3\}$, then $\mathbf{K}'' := \prod_{i=1}^3 \mathbf{K}''_i$ is an Abelian normal Lie subsupergroup, hence $\overline{\mathbf{B}''} := \mathbf{G}''_\sigma/\mathbf{K}''$ is a quotient Lie supergroup (of \mathbf{G}''_σ); therefore, there exists a short exact sequence*

$$\mathbf{1} \longrightarrow \mathbf{K}'' := \prod_{i=1}^3 \mathbf{K}''_i \longrightarrow \mathbf{G}''_\sigma \longrightarrow \overline{\mathbf{B}''} \longrightarrow \mathbf{1}.$$

Moreover, setting $\overline{\mathbf{O}''}(A) := \langle \{\overline{x_\beta(\eta)}\}_{\beta \in \Delta_{\overline{1}}}^{\eta \in A_{\overline{1}}}\rangle$ and $\overline{\mathbf{T}''}(A) := \langle \{\overline{h'_{2\varepsilon_i}(a)}\}_{i \in \{1, 2, 3\}}^{a \in A_0} \rangle$ inside $\overline{\mathbf{B}''}(A)$ for all $A \in (\mathbf{Wsalg})$ we overall find two subsupergroups $\overline{\mathbf{O}''}$ and $\overline{\mathbf{T}''}$ of $\overline{\mathbf{B}''}$ such that $\overline{\mathbf{O}''}$ is normal Abelian, isomorphic to $\mathbb{A}_C^{0|8}$ – the (totally odd) complex affine Abelian supergroup of superdimension $(0|8)$ – and $\overline{\mathbf{T}''}$ is Abelian, isomorphic to \mathbb{T}_C^3 – the (totally even) 3-dimensional complex torus – with $\overline{\mathbf{B}''} \cong \overline{\mathbf{T}''} \ltimes \overline{\mathbf{O}''}$ – a semidirect product of Lie supergroups. In other words, there exists a second, split short exact sequence

$$\mathbf{1} \longrightarrow \overline{\mathbf{O}''} \cong \mathbb{A}_C^{0|8} \longrightarrow \overline{\mathbf{B}''} \overset{\leftarrow}{\dashrightarrow} \mathbb{T}_C^3 \cong \overline{\mathbf{T}''} \longrightarrow \mathbf{1}.$$

Proof. Like for Theorem 5.1, one can deduce the claim from the presentation of \mathbf{G}_σ'' in Section 5.3.1, or also from the relation $\text{Lie}(\mathbf{G}_\sigma'') = \mathfrak{g}''(\sigma)$ along with Theorem 4.3. \blacksquare

5.4 Lie supergroups from contractions: the family of the $\widehat{\mathbf{G}}_\sigma$'s

Given $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$, following Section 4.4 we fix the element $\tau := x_1 x_2 x_3 \in \mathbb{C}[\mathbf{x}]$ and the ideal $I = I_\sigma := (\{x_i - \sigma_i\}_{i=1,2,3})$, and we consider the corresponding complex Lie algebra $\widehat{\mathfrak{g}}(\sigma)$, with $\widehat{\mathfrak{g}}(\sigma)_\bar{0}$ and $\widehat{\mathfrak{g}}(\sigma)_\bar{1}$ as its even and odd part, respectively. With a slight abuse of notation, for any element $Z \in \widehat{\mathfrak{g}}(\mathbf{x})$ we denote again by Z its corresponding coset in $\widehat{\mathfrak{g}}(\sigma) = \mathbb{C}[\sigma] \otimes_{\mathbb{C}[\mathbf{x}]} \widehat{\mathfrak{g}}(\mathbf{x}) \cong \widehat{\mathfrak{g}}(\mathbf{x})/I_\sigma \widehat{\mathfrak{g}}(\mathbf{x})$ (see Section 4.4 for notation). By construction, $\widehat{\mathfrak{g}}(\sigma)$ admits as \mathbb{C} -basis the set

$$\widehat{B} := \{X_\alpha, H_{2\varepsilon_i} \mid \alpha \in \Delta, i \in \{1, 2, 3\}\} \cup \{\widehat{X}_\beta := \tau X_\beta \mid \beta \in \Delta_{\bar{1}}\}.$$

We consider also $\widehat{\mathfrak{a}}_i := \mathfrak{a}_i := \mathbb{C}X_{2\varepsilon_i} \oplus \mathbb{C}H_{2\varepsilon_i} \oplus \mathbb{C}X_{-2\varepsilon_i}$ for all $i = 1, 2, 3$, that all are Lie subalgebras of $\widehat{\mathfrak{g}}_\bar{0}$, with $\widehat{\mathfrak{a}}_i \cong \mathfrak{sl}_i(2)$ when $\sigma_i \neq 0$ and $\mathfrak{a}_i \cong \mathbb{C}^{\oplus 3}$ – the 3-dimensional Abelian Lie algebra – if $\sigma_i = 0$ (see also Section 5.1).

Recalling the construction of \mathbf{G}_σ in Section 5.1, for each $i \in \{1, 2, 3\}$ we set $\widehat{A}_i := A_i$ (isomorphic to either SL_2 or $\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}$ depending on $\sigma_i \neq 0$ or $\sigma_i = 0$) and $\widehat{G} := \times_{i=1}^3 \widehat{A}_i = G$, a complex Lie group such that $\text{Lie}(\widehat{G}) = \widehat{\mathfrak{g}}(\sigma)_\bar{0}$. Like in Section 5.1, the adjoint action of $\widehat{\mathfrak{g}}(\sigma)_\bar{0}$ onto $\widehat{\mathfrak{g}}(\sigma)$ integrates to a Lie group action of \widehat{G} onto $\widehat{\mathfrak{g}}(\sigma)$: endowed with this action, the pair $\widehat{\mathcal{P}}_\sigma := (\widehat{G}, \widehat{\mathfrak{g}}(\sigma))$ is a super Harish-Chandra pair (cf. Section 2.4.1). Eventually, we can define

$$\widehat{\mathbf{G}}_\sigma := \mathbf{G}_{\widehat{\mathcal{P}}_\sigma}$$

to be the complex Lie supergroup associated with $\widehat{\mathcal{P}}_\sigma$ following Section 2.4.3.

5.4.1. A presentation of $\widehat{\mathbf{G}}_\sigma$. To describe the supergroups $\widehat{\mathbf{G}}_\sigma$, we provide hereafter an explicit presentation by generators and relations of all the abstract groups $\widehat{\mathbf{G}}_\sigma(A)$, with $A \in (\mathbf{Wsalg})$. To begin with, let $\exp: \widehat{\mathfrak{g}}_\bar{0} \cong \text{Lie}(\widehat{G}) \longrightarrow \widehat{G}$ be the exponential map. Like we did in Section 5.1.1 for the supergroup \mathbf{G}_σ , inside each subgroup A_i we consider

$$x_{2\varepsilon_i}(c) := \exp(cX_{2\varepsilon_i}), \quad h_{2\varepsilon_i}(c) := \exp(cH_{2\varepsilon_i}), \quad x_{-2\varepsilon_i}(c) := \exp(cX_{-2\varepsilon_i})$$

for every $c \in \mathbb{C}$; then $\widehat{\Gamma}_i := \{x_{2\varepsilon_i}(c), h_{2\varepsilon_i}(c), x_{-2\varepsilon_i}(c)\}_{c \in \mathbb{C}}$ is a generating set for \widehat{A}_i ; also, we consider elements $h_\theta(c) := \exp(cH_\theta)$ for all $c \in \mathbb{C}$. It follows that the complex Lie group $\widehat{G} = \widehat{A}_1 \times \widehat{A}_2 \times \widehat{A}_3$ is generated by

$$\widehat{\Gamma}_\bar{0} := \{x_{2\varepsilon_i}(c), h_{2\varepsilon_i}(c), h_\theta(c), x_{-2\varepsilon_i}(c)\}_{c \in \mathbb{C}}^{i \in \{1, 2, 3\}}$$

(we could drop the $h_\theta(c)$'s, but we prefer to keep them among the generators).

When we think of \widehat{G} as a (totally even) supergroup, looking at it as a group-valued functor $\widehat{G}: (\mathbf{Wsalg}) \longrightarrow (\mathbf{grps})$, the abstract group $\widehat{G}(A)$ of its A -points – for $A \in (\mathbf{Wsalg})$ – is generated by the set

$$\widehat{\Gamma}_\bar{0}(A) := \{x_{2\varepsilon_i}(a), h_{2\varepsilon_i}(a), h_\theta(a), x_{-2\varepsilon_i}(a)\}_{a \in A_\bar{0}}^{i \in \{1, 2, 3\}}. \quad (5.4)$$

In fact, we would better stress that, by construction (cf. Section 5.1), we have an obvious isomorphism $\widehat{G} \cong G$ (see Section 5.1.1 for the definition of G) as complex Lie groups.

To generate the group $\widehat{\mathbf{G}}_\sigma(A) := \mathbf{G}_{\widehat{\mathcal{P}}_\sigma}(A)$ applying the recipe in Section 2.4.3, we fix in $\widehat{\mathfrak{g}}(\sigma)_\bar{1}$ the \mathbb{C} -basis $\{Y_i\}_{i \in I} = \{\widehat{X}_\beta := \tau X_\beta \mid \beta \in \Delta_{\bar{1}} = \{\pm\theta, \pm\beta_1, \pm\beta_2, \pm\beta_3\}\}$. Thus, besides the generating elements from $\widehat{G}(A)$, we take as generators also those of the set

$$\widehat{\Gamma}_\bar{1}(A) := \{\widehat{x}_{\pm\theta}(\eta) := (1 + \eta\widehat{X}_{\pm\theta}), \widehat{x}_{\pm\beta_i}(\eta) := (1 + \eta\widehat{X}_{\pm\beta_i})\}_{\eta \in A_{\bar{1}}}^{i \in \{1, 2, 3\}}.$$

Taking into account that $\widehat{G}(A)$ is generated by $\widehat{\Gamma}_0(A)$, we can modify the set of relations in Section 2.4.3 by letting $g_+ \in \widehat{G}(A)$ range inside the set $\widehat{\Gamma}_0(A)$: then we can find the following full set of relations (freely using notation $e^Z := \exp(Z)$):

$$\begin{aligned}
1_{\widehat{G}} &= 1, & g' \cdot g'' &= g' \cdot_{\widehat{G}} g'', & \forall g', g'' &\in \widehat{G}(A), \\
h_{2\varepsilon_i}(a)\widehat{x}_{\pm\beta_j}(\eta)h_{2\varepsilon_i}(a)^{-1} &= \widehat{x}_{\pm\beta_j}(e^{\pm(-1)^{-\delta_{i,j}}\sigma_i a}\eta), \\
h_{2\varepsilon_i}(a)\widehat{x}_{\pm\theta}(\eta)h_{2\varepsilon_i}(a)^{-1} &= \widehat{x}_{\pm\theta}(e^{\pm\sigma_i a}\eta), \\
h_\theta(a)\widehat{x}_{\pm\beta_i}(\eta)h_\theta(a)^{-1} &= \widehat{x}_{\pm\beta_i}(e^{\mp\sigma_i a}\eta), & h_\theta(a)\widehat{x}_{\pm\theta}(\eta)h_\theta(a)^{-1} &= \widehat{x}_{\pm\theta}(\eta), \\
x_{2\varepsilon_i}(a)\widehat{x}_{\beta_j}(\eta)x_{2\varepsilon_i}(a)^{-1} &= \widehat{x}_{\beta_j}(\eta)\widehat{x}_\theta(\delta_{i,j}\sigma_i a\eta), \\
x_{2\varepsilon_i}(a)\widehat{x}_{-\beta_j}(\eta)x_{2\varepsilon_i}(a)^{-1} &= \widehat{x}_{-\beta_j}(\eta)\widehat{x}_{\beta_k}((1 - \delta_{i,j})\sigma_i a\eta), \\
x_{-2\varepsilon_i}(a)\widehat{x}_{\beta_j}(\eta)x_{-2\varepsilon_i}(a)^{-1} &= \widehat{x}_{\beta_j}(\eta)\widehat{x}_{-\beta_k}((1 - \delta_{i,j})\sigma_i a\eta), \\
x_{-2\varepsilon_i}(a)\widehat{x}_{-\beta_j}(\eta)x_{-2\varepsilon_i}(a)^{-1} &= \widehat{x}_{-\beta_j}(\eta)\widehat{x}_{-\theta}(\delta_{i,j}\sigma_i a\eta), \\
x_{2\varepsilon_i}(a)\widehat{x}_\theta(\eta)x_{2\varepsilon_i}(a)^{-1} &= \widehat{x}_\theta(\eta), \\
x_{2\varepsilon_i}(a)\widehat{x}_{-\theta}(\eta)x_{2\varepsilon_i}(a)^{-1} &= \widehat{x}_{-\theta}(\eta)\widehat{x}_{-\beta_i}(\sigma_i a\eta), \\
x_{-2\varepsilon_i}(a)\widehat{x}_\theta(\eta)x_{-2\varepsilon_i}(a)^{-1} &= \widehat{x}_\theta(\eta)\widehat{x}_{\beta_i}(\sigma_i a\eta), \\
x_{-2\varepsilon_i}(a)\widehat{x}_{-\theta}(\eta)x_{-2\varepsilon_i}(a)^{-1} &= \widehat{x}_{-\theta}(\eta), \\
\widehat{x}_{\beta_i}(\eta_i)\widehat{x}_{\beta_j}(\eta'_j) &= x_{2\varepsilon_k}((1 - \delta_{i,j})\tau_\sigma^2\eta'_j\eta_i)\widehat{x}_{\beta_j}(\eta'_j)\widehat{x}_{\beta_i}(\eta_i), \\
\widehat{x}_{-\beta_i}(\eta_i)\widehat{x}_{-\beta_j}(\eta'_j) &= x_{-2\varepsilon_k}(-(1 - \delta_{i,j})\tau_\sigma^2\eta'_j\eta_i)\widehat{x}_{-\beta_j}(\eta'_j)\widehat{x}_{-\beta_i}(\eta_i), \\
\widehat{x}_{\beta_i}(\eta_i)\widehat{x}_{-\beta_j}(\eta'_j) &= h_{2\varepsilon_i}(\delta_{i,j}\tau_\sigma^2\eta'_j\eta_i)h_\theta(-\delta_{i,j}\tau_\sigma^2\eta'_j\eta_i)\widehat{x}_{-\beta_j}(\eta'_j)\widehat{x}_{\beta_i}(\eta_i), \\
\widehat{x}_{\beta_i}(\eta_i)\widehat{x}_\theta(\eta) &= \widehat{x}_\theta(\eta)\widehat{x}_{\beta_i}(\eta_i), & \widehat{x}_{\beta_i}(\eta_i)\widehat{x}_{-\theta}(\eta) &= x_{-2\varepsilon_i}(\tau_\sigma^2\eta\eta_i)\widehat{x}_{-\theta}(\eta)\widehat{x}_{\beta_i}(\eta_i), \\
\widehat{x}_{-\beta_i}(\eta_i)\widehat{x}_\theta(\eta) &= x_{2\varepsilon_i}(-\tau_\sigma^2\eta\eta_i)\widehat{x}_\theta(\eta)\widehat{x}_{-\beta_i}(\eta_i), & \widehat{x}_{-\beta_i}(\eta_i)\widehat{x}_{-\theta}(\eta) &= \widehat{x}_{-\theta}(\eta)\widehat{x}_{-\beta_i}(\eta_i), \\
\widehat{x}_\theta(\eta_+)\widehat{x}_{-\theta}(\eta_-) &= h_\theta(\tau_\sigma^2\eta_-\eta_+)\widehat{x}_{-\theta}(\eta_-)\widehat{x}_\theta(\eta_+), \\
\widehat{x}_{\pm\beta_i}(\eta')\widehat{x}_{\pm\beta_i}(\eta'') &= \widehat{x}_{\pm\beta_i}(\eta' + \eta''), & \widehat{x}_{\pm\theta}(\eta')\widehat{x}_{\pm\theta}(\eta'') &= \widehat{x}_{\pm\theta}(\eta' + \eta'')
\end{aligned}$$

with $\{i, j, k\} \in \{1, 2, 3\}$.

5.4.2. Singular specializations of the supergroup(s) $\widehat{\mathbf{G}}_\sigma$. The very construction of the supergroups $\widehat{\mathbf{G}}_\sigma$ implies that

$$\widehat{\mathbf{G}}_\sigma \text{ is simple (as a Lie supergroup) for all } \sigma = (\sigma_1, \sigma_2, \sigma_3) \in V^\times.$$

This also follows from the presentation of $\widehat{\mathbf{G}}_\sigma$ in Section 5.4.1 above, or as a direct consequence of the relation $\text{Lie}(\widehat{\mathbf{G}}_\sigma) = \widehat{\mathfrak{g}}(\sigma) = \widehat{\mathfrak{g}}_\sigma$ and of the fact that $\widehat{\mathfrak{g}}_\sigma \cong \mathfrak{g}_\sigma$ when $\sigma_i \neq 0$ for all i .

Things change instead at ‘‘singular values’’ of σ . The complete result is the following:

Theorem 5.4. *Given $\sigma \in V$, keep notation as above.*

- (1) *If $\sigma \in V^\times$, then the Lie supergroup $\widehat{\mathbf{G}}_\sigma$ is simple.*
- (2) *If $\sigma \in V \setminus V^\times$, with $\sigma_i = 0$ and $\sigma_j \neq 0 \neq \sigma_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then $\widehat{A}_i \trianglelefteq \widehat{\mathbf{G}}_\sigma$ is a central Lie subsupergroup of $\widehat{\mathbf{G}}_\sigma$, with $\widehat{A}_i \cong \mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}$, while $\widehat{A}_j \cong \widehat{A}_k \cong \text{SL}(2)$ for $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$. Also, we have a semidirect product splitting*

$$\widehat{\mathbf{G}}(\sigma) \cong \widehat{\mathbf{G}}(\sigma)_{\text{rd}} \ltimes \widehat{\mathbf{G}}(\sigma)_{\bar{1}}$$

with $\widehat{\mathbf{G}}(\sigma)_{\text{rd}} = \bigtimes_{\ell=1}^3 \widehat{A}_\ell \cong (\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}) \times \text{SL}(2) \times \text{SL}(2)$ while $\widehat{\mathbf{G}}(\sigma)_{\bar{1}}$ is the subsupergroup of $\widehat{\mathbf{G}}_\sigma$ generated by the $\widehat{x}_{\pm\theta}$'s and the $\widehat{x}_{\pm\beta_i}$'s (for all i), which is normal Abelian, isomorphic

to $\mathbb{A}_C^{0|8}$ – the (totally odd) complex affine Abelian supergroup of superdimension $(0|8)$ – and such that $\widehat{\mathbf{G}}(\sigma)_{\bar{1}} \cong (\blacksquare \oplus \blacksquare) \boxtimes \square \boxtimes \square$ – where \blacksquare is the trivial representation – as a module over $\widehat{\mathbf{G}}(\sigma)_{\text{rd}} \cong (\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}) \times \text{SL}(2) \times \text{SL}(2)$. In other words, there exists a split short exact sequence of Lie supergroups

$$\begin{aligned} \mathbf{1} &\longrightarrow (\blacksquare \oplus \blacksquare) \boxtimes \square \boxtimes \square \cong \widehat{\mathbf{G}}(\sigma)_{\bar{1}} \longrightarrow \widehat{\mathbf{G}}(\sigma) \overset{\leftarrow}{\dashrightarrow} \widehat{\mathbf{G}}(\sigma)_{\text{rd}} \\ &\cong (\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}) \times \text{SL}(2)^2 \longrightarrow \mathbf{1}. \end{aligned}$$

(3) If $\sigma = \mathbf{0} (\in V \setminus V^\times)$, i.e., $\sigma_h = 0$ for all $h \in \{1, 2, 3\}$, then $\widehat{\mathbf{G}}_\sigma$ is the Abelian Lie supergroup $\widehat{\mathbf{G}}_\sigma \cong (\mathbb{C} \times \mathbb{C}^\times \times \mathbb{C})^3 \times (\blacksquare \oplus \blacksquare)^{\boxtimes 3}$.

Proof. As for the parallel results for \mathbf{G}_σ , \mathbf{G}'_σ and \mathbf{G}''_σ , we can deduce the claim from the presentation of $\widehat{\mathbf{G}}_\sigma$ in Section 5.4.1, or from the link $\text{Lie}(\widehat{\mathbf{G}}_\sigma) = \widehat{\mathfrak{g}}(\sigma)$ along with Theorem 4.4. ■

5.5 Lie supergroups from contractions: the family of the $\widehat{\mathbf{G}}'_\sigma$'s

Given $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$, we follow again Section 4.4 and set $\tau := x_1 x_2 x_3 \in \mathbb{C}[\mathbf{x}]$ and $I = I_\sigma := (\{x_i - \sigma_i\}_{i=1,2,3})$; but now we consider the corresponding complex Lie algebra $\widehat{\mathfrak{g}}'(\sigma)$, with $\widehat{\mathfrak{g}}'(\sigma)_{\bar{0}}$ and $\widehat{\mathfrak{g}}'(\sigma)_{\bar{1}}$ as its even and odd part, respectively (and we still make use of some abuse of notation as in Section 5.4). By construction, a \mathbb{C} -basis of $\widehat{\mathfrak{g}}'(\sigma)$ is

$$\widehat{B}' := \{X'_\alpha, H'_{2\varepsilon_i} \mid \alpha \in \Delta, i \in \{1, 2, 3\}\} \cup \{\widehat{X}'_\beta := \tau X'_\beta \mid \beta \in \Delta_{\bar{1}}\}.$$

Consider also $\widehat{\mathfrak{a}}'_i := \mathfrak{a}'_i$ ($:= \mathbb{C}X'_{2\varepsilon_i} \oplus \mathbb{C}H'_{2\varepsilon_i} \oplus \mathbb{C}X'_{-2\varepsilon_i}$, cf. Section 4.2.1) for all $i = 1, 2, 3$: all these are Lie subalgebras in $\widehat{\mathfrak{g}}'(\sigma)$, isomorphic to $\mathfrak{sl}(2)$, and $\widehat{\mathfrak{g}}'(\sigma)_{\bar{0}} = \bigoplus_{i=1}^3 \widehat{\mathfrak{a}}'_i$. The faithful adjoint action of $\widehat{\mathfrak{g}}'(\sigma)_{\bar{0}}$ onto $\widehat{\mathfrak{g}}'(\sigma)$ gives a Lie algebra monomorphism $\widehat{\mathfrak{g}}'(\sigma)_{\bar{0}} \hookrightarrow \mathfrak{gl}(\widehat{\mathfrak{g}}'(\sigma))$, by which we identify $\widehat{\mathfrak{g}}'(\sigma)_{\bar{0}}$ with its image in $\mathfrak{gl}(\widehat{\mathfrak{g}}'(\sigma))$. Then $\exp: \mathfrak{gl}(\widehat{\mathfrak{g}}'(\sigma)) \rightarrow \text{GL}(\widehat{\mathfrak{g}}'(\sigma))$ yields a Lie subgroup $\widehat{G}' := \exp(\widehat{\mathfrak{g}}'(\sigma)_{\bar{0}})$ in $\text{GL}(\widehat{\mathfrak{g}}'(\sigma))$ which faithfully acts onto $\widehat{\mathfrak{g}}'(\sigma)$ and is such that $\text{Lie}(\widehat{G}') = (\widehat{\mathfrak{g}}'(\sigma))_{\bar{0}}$; finally, the pair $\widehat{\mathcal{P}}'_\sigma := (\widehat{G}', \widehat{\mathfrak{g}}'(\sigma))$ with this action is then a super Harish-Chandra pair (cf. Section 2.4.1).

As alternative method, we might also construct the super Harish-Chandra pair $\widehat{\mathcal{P}}'_\sigma$ via the same procedure, up to the obvious, minimal changes, adopted for \mathcal{P}'_σ in Section 5.2; indeed, one can also do the converse, namely use the present method to construct \mathcal{P}'_σ as well.

Once we have the super Harish-Chandra pair $\widehat{\mathcal{P}}'_\sigma$, it makes sense to define

$$\widehat{\mathbf{G}}'_\sigma := \mathbf{G}_{\widehat{\mathcal{P}}'_\sigma}$$

that is the complex Lie supergroup associated with $\widehat{\mathcal{P}}'_\sigma$ after the recipe in Section 2.4.3.

5.5.1. A presentation of $\widehat{\mathbf{G}}'_\sigma$. We shall presently describe the supergroups $\widehat{\mathbf{G}}'_\sigma$ by means of an explicit presentation by generators and relations of all the abstract groups $\widehat{\mathbf{G}}'_\sigma(A)$, for all $A \in (\mathbf{W}\text{salg})$. To begin with, let $\exp: \widehat{\mathfrak{g}}'_0 \cong \text{Lie}(\widehat{G}') \longrightarrow \widehat{G}'$ be the exponential map. Just like for the supergroup \mathbf{G}_σ in Section 5.1.1, inside each subgroup A'_i we consider

$$x'_{2\varepsilon_i}(c) := \exp(cX'_{2\varepsilon_i}), \quad h'_{2\varepsilon_i}(c) := \exp(cH'_{2\varepsilon_i}), \quad x'_{-2\varepsilon_i}(c) := \exp(cX'_{-2\varepsilon_i})$$

for every $c \in \mathbb{C}$; then $\widehat{A}'_i := \{x'_{2\varepsilon_i}(c), h'_{2\varepsilon_i}(c), x'_{-2\varepsilon_i}(c)\}_{c \in \mathbb{C}}$ is a generating set for $\widehat{A}'_i = A'_i$; also, we consider elements $h'_\theta(c) := \exp(cH'_\theta)$ for all $c \in \mathbb{C}$. It follows that the complex Lie group $\widehat{G}' = \widehat{A}'_1 \times \widehat{A}'_2 \times \widehat{A}'_3$ is generated by

$$\widehat{\Gamma}'_0 := \{x'_{2\varepsilon_i}(c), h'_{2\varepsilon_i}(c), h'_\theta(c), x'_{-2\varepsilon_i}(c)\}_{c \in \mathbb{C}}^{i \in \{1, 2, 3\}}$$

(as before, we could drop the $h'_\theta(c)$'s, but we prefer to keep them among the generators).

When thinking of \widehat{G}' as a (totally even) supergroup, considered as a group-valued functor $\widehat{G}' : (\mathbf{Wsalg}) \rightarrow (\mathbf{grps})$, the abstract group $\widehat{G}'(A)$ of its A -points – for $A \in (\mathbf{Wsalg})$ – is generated by the set

$$\widehat{\Gamma}'_0(A) := \{x'_{2\varepsilon_i}(a), h'_{2\varepsilon_i}(a), h_\theta(a), x'_{-2\varepsilon_i}(a)\}_{a \in A_0}^{i \in \{1,2,3\}}. \quad (5.5)$$

Indeed, we can also stress that, by construction (cf. Section 5.1), there exists an obvious isomorphism $\widehat{G}' \cong G'$ as complex Lie groups. Now, to generate the group $\widehat{\mathbf{G}}'_\sigma(A) := \mathbf{G}_{\widehat{\rho}'_\sigma}(A)$ following the recipe in Section 2.4.3, we fix in $(\widehat{\mathfrak{g}}'(\sigma)_c)_{\bar{1}}$ the \mathbb{C} -basis $\{Y_i\}_{i \in I} = \{\widehat{X}'_\beta := \tau X'_\beta \mid \beta \in \Delta_{\bar{1}} = \{\pm\theta, \pm\beta_1, \pm\beta_2, \pm\beta_3\}\}$. Then, beside the generating elements from $\widehat{G}'(A)$ we take as generators also those of the set

$$\widehat{\Gamma}'_1(A) := \{\widehat{x}'_{\pm\theta}(\eta) := (1 + \eta \widehat{X}'_{\pm\theta}), \widehat{x}'_{\pm\beta_i}(\eta) := (1 + \eta \widehat{X}'_{\pm\beta_i})\}_{\eta \in A_{\bar{1}}}^{i \in \{1,2,3\}}.$$

Knowing that $\widehat{G}'(A)$ is generated by $\widehat{\Gamma}'_0(A)$, we can modify the set of relations in Section 2.4.3 by letting $g \in \widehat{G}'(A)$ range inside $\widehat{\Gamma}'_0(A)$; eventually, we can find the following *full set of relations* (freely using notation $e^Z := \exp(Z)$):

$$\begin{aligned} 1_{\widehat{G}'} &= 1, & g' \cdot g'' &= g' \cdot_{\widehat{G}'} g'', & \forall g', g'' \in \widehat{G}'(A), \\ h'_{2\varepsilon_i}(a) \widehat{x}'_{\pm\beta_j}(\eta) h'_{2\varepsilon_i}(a)^{-1} &= \widehat{x}'_{\pm\beta_j}(e^{\pm(-1)^{-\delta_{i,j}} a} \eta), \\ h'_{2\varepsilon_i}(a) \widehat{x}'_{\pm\theta}(\eta) h'_{2\varepsilon_i}(a)^{-1} &= \widehat{x}'_{\pm\theta}(e^{\pm a} \eta), \\ h'_\theta(a) \widehat{x}'_{\pm\beta_i}(\eta) h'_\theta(a)^{-1} &= \widehat{x}'_{\pm\beta_i}(e^{\mp \sigma_i a} \eta), & h'_\theta(a) \widehat{x}'_{\pm\theta}(\eta) h'_\theta(a)^{-1} &= \widehat{x}'_{\pm\theta}(\eta), \\ x'_{2\varepsilon_i}(a) \widehat{x}'_{\beta_j}(\eta) x'_{2\varepsilon_i}(a)^{-1} &= \widehat{x}'_{\beta_j}(\eta) \widehat{x}'_\theta(\delta_{i,j} a \eta), \\ x'_{2\varepsilon_i}(a) \widehat{x}'_{-\beta_j}(\eta) x'_{2\varepsilon_i}(a)^{-1} &= \widehat{x}'_{-\beta_j}(\eta) \widehat{x}'_{\beta_k}((1 - \delta_{i,j}) a \eta), \\ x'_{-2\varepsilon_i}(a) \widehat{x}'_{\beta_j}(\eta) x'_{-2\varepsilon_i}(a)^{-1} &= \widehat{x}'_{\beta_j}(\eta) \widehat{x}'_{-\beta_k}((1 - \delta_{i,j}) a \eta), \\ x'_{-2\varepsilon_i}(a) \widehat{x}'_{-\beta_j}(\eta) x'_{-2\varepsilon_i}(a)^{-1} &= \widehat{x}'_{-\beta_j}(\eta) \widehat{x}'_{-\theta}(\delta_{i,j} a \eta), \\ x'_{2\varepsilon_i}(a) \widehat{x}'_\theta(\eta) x'_{2\varepsilon_i}(a)^{-1} &= \widehat{x}'_\theta(\eta), \\ x'_{2\varepsilon_i}(a) \widehat{x}'_{-\theta}(\eta) x'_{2\varepsilon_i}(a)^{-1} &= \widehat{x}'_{-\theta}(\eta) \widehat{x}'_{-\beta_i}(a \eta), \\ x'_{-2\varepsilon_i}(a) \widehat{x}'_\theta(\eta) x'_{-2\varepsilon_i}(a)^{-1} &= \widehat{x}'_\theta(\eta) \widehat{x}'_{\beta_i}(a \eta), \\ x'_{-2\varepsilon_i}(a) \widehat{x}'_{-\theta}(\eta) x'_{-2\varepsilon_i}(a)^{-1} &= \widehat{x}'_{-\theta}(\eta), \\ \widehat{x}'_{\beta_i}(\eta_i) \widehat{x}'_{\beta_j}(\eta'_j) &= x'_{2\varepsilon_k}((1 - \delta_{i,j}) \tau_\sigma^2 \sigma_i \eta'_j \eta_i) \widehat{x}'_{\beta_j}(\eta'_j) \widehat{x}'_{\beta_i}(\eta_i), \\ \widehat{x}'_{-\beta_i}(\eta_i) \widehat{x}'_{-\beta_j}(\eta'_j) &= x'_{-2\varepsilon_k}(-(1 - \delta_{i,j}) \tau_\sigma^2 \sigma_i \eta'_j \eta_i) \widehat{x}'_{-\beta_j}(\eta'_j) \widehat{x}'_{-\beta_i}(\eta_i), \\ \widehat{x}'_{\beta_i}(\eta_i) \widehat{x}'_{-\beta_j}(\eta'_j) &= h'_{2\varepsilon_i}(\delta_{i,j} \tau_\sigma^2 \sigma_i \eta'_j \eta_i) h'_\theta(-\delta_{i,j} \tau_\sigma^2 \eta'_j \eta_i) \widehat{x}'_{-\beta_j}(\eta'_j) \widehat{x}'_{\beta_i}(\eta_i), \\ \widehat{x}'_{\beta_i}(\eta_i) \widehat{x}'_\theta(\eta) &= \widehat{x}'_\theta(\eta) \widehat{x}'_{\beta_i}(\eta_i), & \widehat{x}'_{\beta_i}(\eta_i) \widehat{x}'_{-\theta}(\eta) &= x'_{-2\varepsilon_i}(\tau_\sigma^2 \sigma_i \eta_i \eta) \widehat{x}'_{-\theta}(\eta) \widehat{x}'_{+\beta_i}(\eta_i), \\ \widehat{x}'_{-\beta_i}(\eta_i) \widehat{x}'_\theta(\eta) &= x'_{2\varepsilon_i}(-\tau_\sigma^2 \sigma_i \eta_i \eta) \widehat{x}'_\theta(\eta) \widehat{x}'_{-\beta_i}(\eta_i), & \widehat{x}'_{-\beta_i}(\eta_i) \widehat{x}'_{-\theta}(\eta) &= \widehat{x}'_{-\theta}(\eta) \widehat{x}'_{-\beta_i}(\eta_i), \\ \widehat{x}'_\theta(\eta_+) \widehat{x}'_{-\theta}(\eta_-) &= h'_\theta(\tau_\sigma^2 \eta_- \eta_+) \widehat{x}'_{-\theta}(\eta_-) \widehat{x}'_\theta(\eta_+), \\ \widehat{x}'_{\pm\beta_i}(\eta') \widehat{x}'_{\pm\beta_i}(\eta'') &= \widehat{x}'_{\pm\beta_i}(\eta' + \eta''), & \widehat{x}'_{\pm\theta}(\eta') \widehat{x}'_{\pm\theta}(\eta'') &= \widehat{x}'_{\pm\theta}(\eta' + \eta'') \end{aligned}$$

with $\{i, j, k\} \in \{1, 2, 3\}$.

5.5.2. Singular specializations of the supergroup(s) $\widehat{\mathbf{G}}'_\sigma$. From the very construction of the supergroups $\widehat{\mathbf{G}}'_\sigma$ we get

$$\widehat{\mathbf{G}}'_\sigma \text{ is simple (as a Lie supergroup) for all } \sigma = (\sigma_1, \sigma_2, \sigma_3) \in V^\times.$$

This follows from the presentation of $\widehat{\mathbf{G}}'_\sigma$ in Section 5.5.1 above, but also as direct consequence of the relation $\text{Lie}(\widehat{\mathbf{G}}'_\sigma) = \widehat{\mathfrak{g}}'(\sigma) = \widehat{\mathfrak{g}}'_\sigma$ and of $\widehat{\mathfrak{g}}'_\sigma \cong \mathfrak{g}_\sigma$ when $\sigma_i \neq 0$ for all i .

At “singular values” of σ instead things are quite different; the precise claim is as follows:

Theorem 5.5. *Given $\sigma \in V$, keep notation as above.*

- (1) *If $\sigma \in V^\times$, then the Lie supergroup $\widehat{\mathbf{G}}'_\sigma$ is simple.*
- (2) *If $\sigma \in V \setminus V^\times$, let $(\widehat{\mathbf{G}}'_\sigma)_{\bar{1}}$ be the subsupergroup of $\widehat{\mathbf{G}}'_\sigma$ generated by the $\widehat{x}'_{\pm\theta}$'s and the $\widehat{x}'_{\pm\beta_i}$'s (for all i). Then $(\widehat{\mathbf{G}}'_\sigma)_{\bar{1}}$ is Abelian and normal in $\widehat{\mathbf{G}}'_\sigma$, and there exist isomorphisms $(\widehat{\mathbf{G}}'_\sigma)_{\text{rd}} \cong \text{SL}(2)^{\times 3}$ and $(\widehat{\mathbf{G}}'_\sigma)_{\bar{1}} \cong \boxtimes_{i=1}^3 \square_i$ as a $(\widehat{\mathbf{G}}'_\sigma)_{\text{rd}}$ -module; in particular, $(\widehat{\mathbf{G}}'_\sigma)_{\bar{1}}$ is Abelian; moreover, there exists an isomorphism*

$$\widehat{\mathbf{G}}'_\sigma \cong (\widehat{\mathbf{G}}'_\sigma)_{\text{rd}} \times (\widehat{\mathbf{G}}'_\sigma)_{\bar{1}} \cong \text{SL}(2)^{\times 3} \times \square^{\boxtimes 3}.$$

In other words, there exists a split short exact sequence

$$\mathbf{1} \longrightarrow \square^{\boxtimes 3} \cong (\widehat{\mathbf{G}}'_\sigma)_{\bar{1}} \longrightarrow \widehat{\mathbf{G}}'_\sigma \xrightarrow{\text{surjective}} (\widehat{\mathbf{G}}'_\sigma)_{\text{rd}} \cong \text{SL}(2)^{\times 3} \longrightarrow \mathbf{1}.$$

Proof. Much like the similar result for $\widehat{\mathbf{G}}_\sigma$, we can deduce the present claim from the presentation of $\widehat{\mathbf{G}}'_\sigma$ in Section 5.5.1, or from the relation $\text{Lie}(\widehat{\mathbf{G}}'_\sigma) = \widehat{\mathfrak{g}}'(\sigma)$ along with Theorem 4.5. ■

5.5.3. The integral case: \mathbf{G}_σ , \mathbf{G}'_σ , \mathbf{G}''_σ , $\widehat{\mathbf{G}}_\sigma$ and $\widehat{\mathbf{G}}'_\sigma$ as algebraic supergroups. In the integral case, i.e., when $\sigma \in \mathbb{Z}^3$, the Lie supergroups we have introduced above are, in fact, complex *algebraic* supergroups: indeed, this follows as a consequence of an alternative presentation of them, that makes sense if and only if $\sigma \in \mathbb{Z}^3$.

Let us look at \mathbf{G}_σ , for some fixed $\sigma \in \mathbb{Z}^3$. Consider the generating set (5.1) for the groups $\Gamma_{\bar{0}}(A)$, and for each $\alpha \in \{2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3, \theta\}$ replace the generators $h_\alpha(a) := \exp(aH_\alpha) -$ for all $a \in A_{\bar{0}}$ – therein with $\widetilde{h}_\alpha(u) -$ for all $u \in U(A_{\bar{0}})$, the group of units of $A_{\bar{0}}$. Every such $\widetilde{h}_\alpha(u)$ is the toral element in $G(A)$ whose adjoint action on \mathfrak{g}_σ is given by $\text{Ad}(\widetilde{h}_\alpha(u))(X_\gamma) = u^{\gamma(H_\alpha)} X_\gamma$ for all $\gamma \in \Delta$; note that this makes sense, since we have $\gamma(H_\alpha) \in \mathbb{Z}$ just because $\sigma \in \mathbb{Z}^3$. Now, the set

$$\widetilde{\Gamma}_{\bar{0}}(A) := \{x_{2\varepsilon_i}(a), \widetilde{h}_{2\varepsilon_i}(u), \widetilde{h}_\theta(u), x_{-2\varepsilon_i}(a) \mid i \in \{1, 2, 3\}, a \in A_{\bar{0}}, u \in U(A_{\bar{0}})\}$$

still generates $G(A)$. A moment's thought then shows that $\mathbf{G}_\sigma(A)$ can be realized as the group generated by $\widetilde{\Gamma}(A) := G(A) \cup \widetilde{\Gamma}_{\bar{1}}(A)$ with the same relations as in Section 5.1.1 up to the following changes: all relations that involve no generators of type $h_\alpha(a)$ are kept the same, while the others are replaced by the following ones (with $\{i, j, k\} \in \{1, 2, 3\}$):

$$\begin{aligned} \widetilde{h}_{2\varepsilon_i}(u)x_{\pm\beta_j}(\eta)\widetilde{h}_{2\varepsilon_i}(u)^{-1} &= x_{\pm\beta_j}(u^{\pm(-1)^{-\delta_{i,j}}\sigma_i}\eta), \\ \widetilde{h}_{2\varepsilon_i}(u)x_{\pm\theta}(\eta)\widetilde{h}_{2\varepsilon_i}(u)^{-1} &= x_{\pm\theta}(u^{\pm\sigma_i}\eta), \\ \widetilde{h}_\theta(u)x_{\pm\beta_i}(\eta)\widetilde{h}_\theta(u)^{-1} &= x_{\pm\beta_i}(u^{\mp\sigma_i}\eta), \quad \widetilde{h}_\theta(u)x_{\pm\theta}(\eta)\widetilde{h}_\theta(u)^{-1} = x_{\pm\theta}(\eta), \\ x_\theta(\eta_+)x_{-\theta}(\eta_-) &= \widetilde{h}_\theta(\eta_-\eta_+)x_{-\theta}(\eta_-)x_\theta(\eta_+). \end{aligned}$$

In fact, the key point here is that if (and only if) $\sigma \in \mathbb{Z}^3$, then all our construction does make sense in the framework of algebraic supergeometry, namely $\mathcal{P}_\sigma := (G, \mathfrak{g}(\sigma))$ is a super Harish-Chandra pair in the algebraic sense – like in [8] – and $\mathbf{G}_\sigma := \mathbf{G}_{\mathcal{P}_\sigma}$ is nothing but the corresponding algebraic supergroup associated with \mathcal{P}_σ through the algebraic version of category equivalence in Section 2.4.3 – cf. [8] again. If we present the groups $G(A)$ using $\widetilde{\Gamma}_{\bar{0}}(A)$ as generating set, we can also extend such a description – as $\sigma \in \mathbb{Z}^3$ – to a presentation of the groups $\mathbf{G}_\sigma(A)$ as above. Leaving details to the reader, the same analysis applies when we look at \mathbf{G}'_σ , \mathbf{G}''_σ , $\widehat{\mathbf{G}}_\sigma$ or $\widehat{\mathbf{G}}'_\sigma$ instead of \mathbf{G}_σ : whenever $\sigma \in \mathbb{Z}^3$, all of them are in fact complex *algebraic* supergroups.

5.5.4. A geometrical interpretation. In the previous discussion we considered five families of Lie supergroups indexed by the points of V , namely $\{\mathbf{G}_\sigma\}_{\sigma \in V}$, $\{\mathbf{G}'_\sigma\}_{\sigma \in V}$, $\{\mathbf{G}''_\sigma\}_{\sigma \in V}$, $\{\widehat{\mathbf{G}}_\sigma\}_{\sigma \in V}$ and $\{\widehat{\mathbf{G}}'_\sigma\}_{\sigma \in V}$. Our analysis shows that these families have in common all the elements indexed by “general” points, i.e., elements $\sigma \in V^\times := V \setminus \mathcal{S}$ with $\mathcal{S} := \bigcup_{i=1}^3 \{\sigma_i = 0\}$. On the other hand, these families are entirely different at all points in the “singular locus” \mathcal{S} .

In geometrical terms, each family forms a fibre space, say $\mathbb{L}_{\mathbf{G}_{\mathbb{C}[\mathbf{x}]}}$, $\mathbb{L}_{\mathbf{G}'_{\mathbb{C}[\mathbf{x}]}}$, $\mathbb{L}_{\mathbf{G}''_{\mathbb{C}[\mathbf{x}]}}$, $\mathbb{L}_{\widehat{\mathbf{G}}_{\mathbb{C}[\mathbf{x}]}}$, and $\mathbb{L}_{\widehat{\mathbf{G}}'_{\mathbb{C}[\mathbf{x}]}}$, respectively, over the base space $\text{Spec}(\mathbb{C}[\mathbf{x}]) \cong V \cup \{\star\}$ ($\cong \mathbb{A}_{\mathbb{C}}^2 \cup \{\star\}$), whose fibres are Lie supergroups. Our results show that the fibres in any two of these fibre spaces do coincide at general points – where they are *simple* Lie supergroups – and do differ instead at singular points – where they are non-simple indeed.

As an outcome, loosely speaking we can say that *our construction provides five different “completions” of the family $\{\mathbf{G}_\sigma\}_{\sigma \in V \setminus \mathcal{S}}$ of simple Lie supergroups, by adding – in five different ways – some new non-simple extra elements*. Note also that, a priori, many other such “completions”, more or less similar, could be devised: we just presented these ones as significant, interesting examples, with no claims whatsoever of being exhaustive.

Acknowledgements

The first author is partially supported by the French *Agence Nationale de la Recherche* (ANR *GeoLie project ANR-15-CE40-0012*). The second author acknowledges the MIUR *Excellence Department Project* awarded to the Department of Mathematics, University of Rome “Tor Vergata”, CUP E83C18000100006. The authors also would like to thank the anonymous referees for their useful comments and suggestions to improve the presentation of this article.

References

- [1] Balduzzi L., Carmeli C., Fiorese R., A comparison of the functors of points of supermanifolds, *J. Algebra Appl.* **12** (2013), 1250152, 41 pages, [arXiv:0902.1824](https://arxiv.org/abs/0902.1824).
- [2] Bouarroudj S., Grozman P., Leites D., Classification of finite dimensional modular Lie superalgebras with indecomposable Cartan matrix, *SIGMA* **5** (2009), 060, 63 pages, [arXiv:0710.5149](https://arxiv.org/abs/0710.5149).
- [3] Chapovalov D., Chapovalov M., Lebedev A., Leites D., The classification of almost affine (hyperbolic) Lie superalgebras, *J. Nonlinear Math. Phys.* **17** (2010), suppl. 1, 103–161, [arXiv:0906.1860](https://arxiv.org/abs/0906.1860).
- [4] Deligne P., Morgan J.W., Notes on supersymmetry (following Joseph Bernstein), in *Quantum Fields and Strings: a Course for Mathematicians*, Vols. 1, 2 (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI, 1999, 41–97.
- [5] Dooley A.H., Rice J.W., On contractions of semisimple Lie groups, *Trans. Amer. Math. Soc.* **289** (1985), 185–202.
- [6] Fiorese R., Gavarini F., Chevalley supergroups, *Mem. Amer. Math. Soc.* **215** (2012), vi+64 pages, [arXiv:0808.0785](https://arxiv.org/abs/0808.0785).
- [7] Gavarini F., Chevalley supergroups of type $D(2, 1; a)$, *Proc. Edinb. Math. Soc.* **57** (2014), 465–491, [arXiv:1006.0464](https://arxiv.org/abs/1006.0464).
- [8] Gavarini F., Global splittings and super Harish-Chandra pairs for affine supergroups, *Trans. Amer. Math. Soc.* **368** (2016), 3973–4026, [arXiv:1308.0462](https://arxiv.org/abs/1308.0462).
- [9] Gavarini F., Lie supergroups vs. super Harish-Chandra pairs: a new equivalence, [arXiv:1609.02844](https://arxiv.org/abs/1609.02844).
- [10] Iohara K., Koga Y., Central extensions of Lie superalgebras, *Comment. Math. Helv.* **76** (2001), 110–154.
- [11] Kac V.G., Classification of simple algebraic supergroups, *Russian Math. Surveys* **32** (1977), no. 3, 214–215, in Russian, available at http://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=rm&paperid=3212&option_lang=eng.
- [12] Kac V.G., Lie superalgebras, *Adv. Math.* **26** (1977), 8–96.

- [13] Kaplansky I., Graded Lie algebras I, University of Chicago Report, 1976, available at <http://www1.osu.cz/~zusmanovich/links/files/kaplansky/>.
- [14] Kaplansky I., Graded Lie algebras II, University of Chicago Report, 1976, available at <http://www1.osu.cz/~zusmanovich/links/files/kaplansky/>.
- [15] Scheunert M., The theory of Lie superalgebras. An introduction, *Lecture Notes in Math.*, Vol. 716, Springer, Berlin, 1979.
- [16] Serganova V., On generalizations of root systems, *Comm. Algebra* **24** (1996), 4281–4299.
- [17] Vaintrob A.Yu., Deformation of complex superspaces and coherent sheaves on them, *J. Sov. Math.* **51** (1990), 2140–2188.
- [18] Veisfeiler B.Ju., Kac V.G., Exponentials in Lie algebras of characteristic p , *Math. USSR Izv.* **5** (1971), 777–803.