

Duality for Knizhnik–Zamolodchikov and Dynamical Operators

Vitaly TARASOV ^{†‡} and Filipp UVAROV [†]

[†] Department of Mathematical Sciences, Indiana University – Purdue University Indianapolis, 402 North Blackford St, Indianapolis, IN 46202-3216, USA

E-mail: vtarasov@iupui.edu, filuvaro@iu.edu

[‡] St. Petersburg Branch of Steklov Mathematical Institute, Fontanka 27, St. Petersburg, 191023, Russia

E-mail: vt@pdmi.ras.ru

Received February 25, 2020, in final form April 10, 2020; Published online April 25, 2020

<https://doi.org/10.3842/SIGMA.2020.035>

Abstract. We consider the Knizhnik–Zamolodchikov and dynamical operators, both differential and difference, in the context of the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$ -duality for the space of polynomials in kn anticommuting variables. We show that the Knizhnik–Zamolodchikov and dynamical operators naturally exchange under the duality.

Key words: Knizhnik–Zamolodchikov operators; dynamical operators; the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$ -duality

2020 Mathematics Subject Classification: 17B37; 81R10; 81R50; 39A12

1 Introduction

The Knizhnik–Zamolodchikov (KZ) operators is a family of pairwise commuting differential operators acting on $U(\mathfrak{gl}_k)^{\otimes n}$ -valued functions. They play an important role in conformal field theory, representation theory, and they are closely related to the famous Gaudin Hamiltonians. The difference analogue of the KZ operators is the quantum Knizhnik–Zamolodchikov (qKZ) operators. There are rational, trigonometric, and elliptic versions of the KZ and qKZ operators. For a review and references see, for example, [2].

There exist other families of commuting differential or difference operators called the dynamical differential (DD) or dynamical difference (qDD) operators, respectively. There are rational and trigonometric versions of the DD and qDD operators as well. It is known that the rational DD operators commute with the rational KZ operators, the trigonometric DD operators commute with the rational qKZ operators, and the rational qDD operators commute with the trigonometric KZ operators, see [4, 5, 7]. The qDD operators appear as the action of the dynamical Weyl groups [3]. The DD operators are also known as the Casimir connection, see [8, 9].

Together with the KZ, DD, qKZ, and qDD operators associated with $U(\mathfrak{gl}_k)^{\otimes n}$, we will simultaneously consider similar operators associated with $U(\mathfrak{gl}_n)^{\otimes k}$ interchanging k and n . Let P_{kn} be the space of polynomials in kn commuting variables. One can define actions of $U(\mathfrak{gl}_k)^{\otimes n}$ and $U(\mathfrak{gl}_n)^{\otimes k}$ on the space P_{kn} . It is well known that the $U(\mathfrak{gl}_k)^{\otimes n}$ - and $U(\mathfrak{gl}_n)^{\otimes k}$ -actions on P_{kn} commute, which manifests the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$ -duality, see, for example, [1]. Consider the images of the KZ, DD, qKZ, qDD operators associated with $U(\mathfrak{gl}_k)^{\otimes n}$ and $U(\mathfrak{gl}_n)^{\otimes k}$ under the corresponding actions. It was proved in [6] that the images of the rational (trigonometric) KZ operators associated with $U(\mathfrak{gl}_n)^{\otimes k}$ coincide with the images of the rational (trigonometric) DD operators associated

This paper is a contribution to the Special Issue on Representation Theory and Integrable Systems in honor of Vitaly Tarasov on the 60th birthday and Alexander Varchenko on the 70th birthday. The full collection is available at <https://www.emis.de/journals/SIGMA/Tarasov-Varchenko.html>

with $U(\mathfrak{gl}_k)^{\otimes n}$. Similarly, the images of the rational qKZ operators associated with $U(\mathfrak{gl}_n)^{\otimes k}$, up to an action of a central element in \mathfrak{gl}_n , coincide with the images of the rational qDD operators associated with $U(\mathfrak{gl}_k)^{\otimes n}$. In this paper we obtain a similar duality for the case of $U(\mathfrak{gl}_k)^{\otimes n}$ - and $U(\mathfrak{gl}_n)^{\otimes k}$ -actions on the space of polynomials in kn anticommuting variables, see Theorem 4.4.

The duality for the rational and trigonometric KZ and DD operators is proved in a straightforward way. To prove the duality for the rational qKZ and qDD operators, we study the eigenvalues of the rational R -matrix and compare them to the eigenvalues of the operator $B_{12}^{(2)}(t)$, which is used in the construction of the qDD operators.

The $(\mathfrak{gl}_k, \mathfrak{gl}_n)$ -duality for classical integrable models related to Gaudin Hamiltonians and the actions of \mathfrak{gl}_k and \mathfrak{gl}_n on the space of polynomials in anticommuting variables was studied in [10, Section 3.3]. The result of [10] resembles what one can expect for Bethe algebras of Gaudin models discussed in our work. We will study those Bethe algebras in an upcoming paper.

The paper is organized as follows. In Section 2, we introduce necessary notations. In Section 3, we define the KZ, DD, qKZ, and qDD operators. In Section 4, we formulate and prove the main result.

2 Basic notation

Let e_{ab} , $a, b = 1, \dots, k$, be the standard basis of the Lie algebra \mathfrak{gl}_k : $[e_{ab}, e_{cd}] = \delta_{bc}e_{ad} - \delta_{ad}e_{cb}$. We take the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_k$ spanned by e_{11}, \dots, e_{kk} , and the nilpotent subalgebras \mathfrak{n}_+ and \mathfrak{n}_- spanned by the elements e_{ab} for $a < b$ and $a > b$ respectively. We have standard Gauss decomposition $\mathfrak{gl}_k = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$.

Let $\varepsilon_1, \dots, \varepsilon_k$ be the basis of \mathfrak{h}^* dual to e_{11}, \dots, e_{kk} : $\langle \varepsilon_a, e_{bb} \rangle = \delta_{ab}$. We identify \mathfrak{h}^* with \mathbb{C}^k mapping $l_1\varepsilon_1 + \dots + l_k\varepsilon_k$ to (l_1, \dots, l_k) . The root vectors of \mathfrak{gl}_k are e_{ab} for $a \neq b$, the corresponding root being equal to $\alpha_{ab} = \varepsilon_a - \varepsilon_b$. The roots α_{ab} for $a < b$ are positive. The simple roots are $\alpha_1, \dots, \alpha_{k-1}$: $\alpha_a = \varepsilon_a - \varepsilon_{a+1}$. Denote by ρ the half-sum of positive roots.

We choose the standard invariant bilinear form $(,)$ on \mathfrak{gl}_k : $(e_{ab}, e_{cd}) = \delta_{ad}\delta_{bc}$. It defines an isomorphism $\mathfrak{h} \rightarrow \mathfrak{h}^*$. The induced bilinear form on \mathfrak{h}^* is $(\varepsilon_a, \varepsilon_b) = \delta_{ab}$.

For a \mathfrak{gl}_k -module W and a weight $\mathbf{l} \in \mathfrak{h}^*$, let $W[\mathbf{l}]$ be the weight subspace of W of weight \mathbf{l} .

For any $\mathbf{l} = (l_1, \dots, l_k)$ with $l_a \geq l_{a+1}$, we denote by $V_{\mathbf{l}}$ the irreducible \mathfrak{gl}_k -module with highest weight \mathbf{l} . Also, for any $m \in \mathbb{Z}_{\geq 0}$ we write V_m instead of $V_{(1^m)}$, where $(1^m) = (\underbrace{1, \dots, 1}_k, 0, \dots, 0)$.

Thus, $V_0 = \mathbb{C}$ is the trivial \mathfrak{gl}_k -module, $V_1 = \mathbb{C}^k$ with the natural action of \mathfrak{gl}_k , and V_m is the m -th antisymmetric power of V_1 .

The element $I = \sum_{a,b=1}^k e_{ab}e_{ba}$ is central in $U(\mathfrak{gl}_k)$. It acts as multiplication by $(\mathbf{l}, \mathbf{l} + 2\rho)$ in the irreducible \mathfrak{gl}_k -module $V_{\mathbf{l}}$.

Consider the algebra $\mathfrak{X}_k = \bigwedge^{\bullet} \mathbb{C}^k$, which will be identified with the ring of all polynomials in anticommuting variables x_1, \dots, x_k . In particular, $x_i^2 = 0$ for all $i = 1, \dots, k$.

We define left derivation ∂_a as follows: if $g(x) = x_{b_1} \dots x_{b_m}$ for some m , where $b_s \neq a$ for any s , then

$$\partial_a g(x) = 0, \quad \partial_a(x_a g(x)) = g(x).$$

The operators of left multiplication by x_1, \dots, x_k and left derivations $\partial_1, \dots, \partial_k$ make the space \mathfrak{X}_k into the irreducible representation of the Clifford algebra Cliff_k .

The Lie algebra \mathfrak{gl}_k acts on the space \mathfrak{X}_k by the rule: $e_{ab} \cdot p = x_a \partial_b p$ for any $p \in \mathfrak{X}_k$. Denote the obtained \mathfrak{gl}_k -module by V_{\bullet} , then

$$V_{\bullet} = \bigoplus_{l=0}^k V_l, \tag{2.1}$$

the submodule V_l being spanned by homogeneous polynomials of degree l . A highest weight vector of the submodule V_l is $x_1 x_2 \cdots x_l$.

The \mathfrak{gl}_k -action on \mathfrak{X}_k naturally extends to a $U(\mathfrak{gl}_k)^{\otimes n}$ -action on $(\mathfrak{X}_k)^{\otimes n}$. For any $g \in U(\mathfrak{gl}_k)$, set $g_{(i)} = 1 \otimes \cdots \otimes g \otimes \cdots \otimes 1 \in U(\mathfrak{gl}_k)^{\otimes n}$. We consider $U(\mathfrak{gl}_k)$ as the diagonal subalgebra of $U(\mathfrak{gl}_k)^{\otimes n}$, that is, the embedding $U(\mathfrak{gl}_k) \hookrightarrow U(\mathfrak{gl}_k)^{\otimes n}$ is given by the n -fold coproduct: $x \mapsto x_{(1)} + \cdots + x_{(n)}$ for any $x \in \mathfrak{gl}_k$. This corresponds to the standard \mathfrak{gl}_k -module structure on $(\mathfrak{X}_k)^{\otimes n}$ as the tensor product of \mathfrak{gl}_k -modules.

Let $\Omega = \sum_{a,b=1}^k e_{ab} \otimes e_{ba}$ be the Casimir tensor, and let

$$\begin{aligned}\Omega^+ &= \frac{1}{2} \sum_{a=1}^k e_{aa} \otimes e_{aa} + \sum_{1 \leq a < b \leq k} e_{ab} \otimes e_{ba}, \\ \Omega^- &= \frac{1}{2} \sum_{a=1}^k e_{aa} \otimes e_{aa} + \sum_{1 \leq a < b \leq k} e_{ba} \otimes e_{ab},\end{aligned}$$

so that $\Omega = \Omega^+ + \Omega^-$.

3 The KZ, qKZ, DD and qDD operators

Fix a nonzero complex number κ . Consider differential operators $\nabla_{z_1}, \dots, \nabla_{z_n}$, and $\widehat{\nabla}_{z_1}, \dots, \widehat{\nabla}_{z_n}$ with coefficients in $U(\mathfrak{gl}_k)^{\otimes n}$ depending on complex variables z_1, \dots, z_n , and $\lambda_1, \dots, \lambda_k$:

$$\begin{aligned}\nabla_{z_i}(z; \lambda) &= \kappa \frac{\partial}{\partial z_i} - \sum_{a=1}^k \lambda_a (e_{aa})_{(i)} - \sum_{j=1, j \neq i}^n \frac{\Omega_{(ij)}}{z_i - z_j}, \\ \widehat{\nabla}_{z_i}(z; \lambda) &= \kappa z_i \frac{\partial}{\partial z_i} - \sum_{a=1}^k \left(\lambda_a - \frac{e_{aa}}{2} \right) (e_{aa})_{(i)} - \sum_{j=1, j \neq i}^n \frac{z_i \Omega_{(ij)}^+ + z_j \Omega_{(ij)}^-}{z_i - z_j}.\end{aligned}$$

The differential operators $\nabla_{z_1}, \dots, \nabla_{z_n}$ (resp., $\widehat{\nabla}_{z_1}, \dots, \widehat{\nabla}_{z_n}$) are called the *rational* (resp., *trigonometric*) *Knizhnik–Zamolodchikov (KZ) operators*.

Introduce differential operators $D_{\lambda_1}, \dots, D_{\lambda_k}$, and $\widehat{D}_{\lambda_1}, \dots, \widehat{D}_{\lambda_k}$ with coefficients in $U(\mathfrak{gl}_k)^{\otimes n}$ depending on complex variables z_1, \dots, z_n , and $\lambda_1, \dots, \lambda_k$:

$$\begin{aligned}D_{\lambda_a}(z; \lambda) &= \kappa \frac{\partial}{\partial \lambda_a} - \sum_{i=1}^n z_i (e_{aa})_{(i)} - \sum_{b=1, b \neq a}^k \frac{e_{ab} e_{ba} - e_{aa}}{\lambda_a - \lambda_b}, \\ \widehat{D}_{\lambda_a}(z; \lambda) &= \kappa \lambda_a \frac{\partial}{\partial \lambda_a} + \frac{e_{aa}^2}{2} - \sum_{i=1}^n z_i (e_{aa})_{(i)} \\ &\quad - \sum_{b=1}^k \sum_{1 \leq i < j \leq n} (e_{ab})_{(i)} (e_{ba})_{(j)} - \sum_{b=1, b \neq a}^k \frac{\lambda_b}{\lambda_a - \lambda_b} (e_{ab} e_{ba} - e_{aa}).\end{aligned}$$

The differential operators $D_{\lambda_1}, \dots, D_{\lambda_k}$ (resp., $\widehat{D}_{\lambda_1}, \dots, \widehat{D}_{\lambda_k}$) are called the *rational* (resp., *trigonometric*) *differential dynamical (DD) operators*, see [4, 7].

For any $a, b = 1, \dots, k$, $a \neq b$, introduce the series $B_{ab}(t)$ depending on a complex variable t :

$$B_{ab} = 1 + \sum_{s=1}^{\infty} \frac{e_{ba}^s e_{ab}^s}{s!} \prod_{j=1}^s (t - e_{aa} + e_{bb} - j)^{-1}.$$

The action of this series is well defined in any finite-dimensional \mathfrak{gl}_k -module W , giving an $\text{End}(W)$ -valued rational function of t .

Denote $\lambda_{bc} = \lambda_b - \lambda_c$. Consider the products X_1, \dots, X_k depending on the complex variables z_1, \dots, z_n , and $\lambda_1, \dots, \lambda_k$:

$$X_a(z; \lambda) = (B_{ak}(\lambda_{ak}) \cdots B_{a,a+1}(\lambda_{a,a+1}))^{-1} \\ \times \prod_{i=1}^n (z_i^{-e_{aa}})_{(i)} B_{1a}(\lambda_{1a} - \kappa) \cdots B_{a-1,a}(\lambda_{a-1,a} - \kappa).$$

The products X_1, \dots, X_k act in any n -fold tensor product $W_1 \otimes \cdots \otimes W_n$ of finite-dimensional \mathfrak{gl}_k -modules.

Denote by T_u a difference operator acting on a function $f(u)$ by

$$(T_u f)(u) = f(u + \kappa).$$

Introduce difference operators $Q_{\lambda_1}, \dots, Q_{\lambda_k}$:

$$Q_{\lambda_a}(z; \lambda) = X_a(z; \lambda) T_{\lambda_a}.$$

The operators $Q_{\lambda_1}, \dots, Q_{\lambda_k}$ are called the (*rational*) *difference dynamical* (qDD) *operators* [5].

For any finite-dimensional irreducible \mathfrak{gl}_k -modules V and W , there is a distinguished rational function $R_{VW}(t)$ of t with values in $\text{End}(V \otimes W)$ called the rational R -matrix. It is uniquely determined by the \mathfrak{gl}_k -invariance

$$[R_{VW}(t), g \otimes 1 + 1 \otimes g] = 0 \quad \text{for any } g \in \mathfrak{gl}_k, \quad (3.1)$$

the commutation relations

$$R_{VW}(t) \left(te_{ab} \otimes 1 + \sum_{c=1}^k e_{ac} \otimes e_{cb} \right) = \left(te_{ab} \otimes 1 + \sum_{c=1}^k e_{cb} \otimes e_{ac} \right) R_{VW}(t), \quad (3.2)$$

and the normalization condition

$$R_{VW}(t)v \otimes w = v \otimes w, \quad (3.3)$$

where v and w are the highest weight vectors of V and W , respectively.

Denote $z_{ij} = z_i - z_j$ and $R_{ij}(t) = (R_{W_i W_j}(t))_{(ij)}$. Consider the products K_1, \dots, K_n depending on the complex variables z_1, \dots, z_n , and $\lambda_1, \dots, \lambda_k$:

$$K_i(z; \lambda) = (R_{in}(z_{in}) \cdots R_{i,i+1}(z_{i,i+1}))^{-1} \prod_{a=1}^k (\lambda_a^{-e_{aa}})_{(i)} R_{1i}(z_{1i} - \kappa) \cdots R_{i-1,i}(z_{i-1,i} - \kappa).$$

The products K_1, \dots, K_n act in any n -fold tensor product $W_1 \otimes \cdots \otimes W_n$ of \mathfrak{gl}_k -modules.

Introduce difference operators Z_{z_1}, \dots, Z_{z_n} :

$$Z_{z_i}(z; \lambda) = K_i(z; \lambda) T_{z_i}.$$

The operators Z_{z_1}, \dots, Z_{z_n} are called (*rational*) *quantized Knizhnik–Zamolodchikov* (qKZ) operators.

It is known that the introduced operators combine into three commutative families, see [4, 5, 7] for more references.

Theorem 3.1. *The operators $\nabla_{z_1}, \dots, \nabla_{z_k}, D_{\lambda_1}, \dots, D_{\lambda_k}$ pairwise commute.*

Theorem 3.2. *The operators $\widehat{\nabla}_{z_1}, \dots, \widehat{\nabla}_{z_n}, Q_{\lambda_1}, \dots, Q_{\lambda_k}$ pairwise commute.*

Theorem 3.3. *The operators $\widehat{D}_{\lambda_1}, \dots, \widehat{D}_{\lambda_k}, Z_{z_1}, \dots, Z_{z_n}$ pairwise commute.*

4 The $(\mathfrak{gl}_k, \mathfrak{gl}_n)$ -duality

Consider the ring \mathfrak{P}_{kn} of polynomials in kn anticommuting variables x_{ai} , $a = 1, \dots, k$, $i = 1, \dots, n$. As a vector space, \mathfrak{P}_{kn} is isomorphic to $(\mathfrak{X}_k)^{\otimes n}$, the isomorphism $\phi_1: (\mathfrak{X}_k)^{\otimes n} \rightarrow \mathfrak{P}_{kn}$ being given by

$$\phi_1(p_1 \otimes \cdots \otimes p_n)(x_{11}, \dots, x_{kn}) = p_1(x_{11}, \dots, x_{k1}) p_2(x_{12}, \dots, x_{k2}) \cdots p_n(x_{1n}, \dots, x_{kn}), \quad (4.1)$$

and to $(\mathfrak{X}_n)^{\otimes k}$, the isomorphism $\phi_2: (\mathfrak{X}_n)^{\otimes k} \rightarrow \mathfrak{P}_{kn}$ being given by

$$\phi_2(p_1 \otimes \cdots \otimes p_k)(x_{11}, \dots, x_{kn}) = p_1(x_{11}, \dots, x_{1n}) p_2(x_{21}, \dots, x_{2n}) \cdots p_k(x_{k1}, \dots, x_{kn}). \quad (4.2)$$

We transfer the \mathfrak{gl}_k -action on $(\mathfrak{X}_k)^{\otimes n}$ to \mathfrak{P}_{kn} using isomorphism ϕ_1 . Similarly, we transfer the \mathfrak{gl}_n -action on $(\mathfrak{X}_n)^{\otimes k}$ to \mathfrak{P}_{kn} using isomorphism ϕ_2 .

We will write superscripts $\langle k \rangle$ and $\langle n \rangle$ to distinguish objects associated with Lie algebras \mathfrak{gl}_k and \mathfrak{gl}_n , respectively. For example, $e_{ab}^{\langle k \rangle}$, $a, b = 1, \dots, k$ denote the generators of \mathfrak{gl}_k , and $e_{ij}^{\langle n \rangle}$, $i, j = 1, \dots, n$ denote the generators of \mathfrak{gl}_n . Then \mathfrak{P}_{kn} is isomorphic to $(V_{\bullet}^{\langle k \rangle})^{\otimes n}$ as a \mathfrak{gl}_k -module by (4.1), and it is isomorphic to $(V_{\bullet}^{\langle n \rangle})^{\otimes k}$ as a \mathfrak{gl}_n -module by (4.2).

It is easy to check that \mathfrak{gl}_k - and \mathfrak{gl}_n -actions on \mathfrak{P}_{kn} commute, therefore \mathfrak{P}_{kn} is a $\mathfrak{gl}_k \oplus \mathfrak{gl}_n$ -module. We have the following theorem, see for example [1]:

Theorem 4.1. *For any partition \mathbf{l} , denote its transpose by \mathbf{l}' . The $\mathfrak{gl}_k \oplus \mathfrak{gl}_n$ -module \mathfrak{P}_{kn} has the decomposition:*

$$\mathfrak{P}_{kn} = \bigoplus_{\substack{\mathbf{l}=(l_1, \dots, l_k), \\ l_1 \leq n.}} V_{\mathbf{l}}^{\langle k \rangle} \otimes V_{\mathbf{l}'}^{\langle n \rangle}.$$

Fix vectors $\mathbf{l} = (l_1, \dots, l_n) \in \mathbb{Z}_{\geq 0}^n$ and $\mathbf{m} = (m_1, \dots, m_k) \in \mathbb{Z}_{\geq 0}^k$ such that $\sum_{i=1}^n l_i = \sum_{a=1}^k m_a$.

Let

$$\mathcal{Z}_{kn}[\mathbf{l}, \mathbf{m}] = \left\{ (d_{ai})_{\substack{a=1, \dots, k \\ i=1, \dots, n}} \in \{0, 1\}^{kn} \mid \sum_{a=1}^k d_{ai} = l_i, \sum_{i=1}^n d_{ai} = m_a \right\}.$$

Denote by $\mathfrak{P}_{kn}[\mathbf{l}, \mathbf{m}] \subset \mathfrak{P}_{kn}$ the span of all monomials $x^{\mathbf{d}} = x_{11}^{d_{11}} \cdots x_{k1}^{d_{k1}} \cdots x_{1n}^{d_{1n}} \cdots x_{kn}^{d_{kn}}$ such that $\mathbf{d} = (d_{ai}) \in \mathcal{Z}_{kn}[\mathbf{l}, \mathbf{m}]$. Then by (2.1), the maps ϕ_1 and ϕ_2 induce the isomorphisms of the respective weight subspaces $(V_{l_1}^{\langle k \rangle} \otimes \cdots \otimes V_{l_n}^{\langle k \rangle})[m_1, \dots, m_k]$ and $(V_{m_1}^{\langle n \rangle} \otimes \cdots \otimes V_{m_k}^{\langle n \rangle})[l_1, \dots, l_n]$ with the space $\mathfrak{P}_{kn}[\mathbf{l}, \mathbf{m}]$.

There is another description of the isomorphisms ϕ_1 and ϕ_2 .

For any $\mathbf{a} = (a_1, \dots, a_r)$, $\mathbf{i} = (i_1, \dots, i_s)$, such that $1 \leq a_1 < \cdots < a_r \leq k$, $1 \leq i_1 < \cdots < i_s \leq n$, define: $e_{\mathbf{a}}^{\langle k \rangle} = e_{a_1 1}^{\langle k \rangle} e_{a_2 2}^{\langle k \rangle} \cdots e_{a_r r}^{\langle k \rangle}$, $e_{\mathbf{i}}^{\langle n \rangle} = e_{i_1 1}^{\langle n \rangle} e_{i_2 2}^{\langle n \rangle} \cdots e_{i_s s}^{\langle n \rangle}$.

Fix $\mathbf{d} = (d_{ai}) \in \mathcal{Z}_{kn}[\mathbf{l}, \mathbf{m}]$. Let $\mathbf{a}_j = (a_1^j, \dots, a_{l_j}^j)$ be such that $a_1^j < a_2^j < \cdots < a_{l_j}^j$ and $d_{a_p^j j} = 1$ for all $j = 1, \dots, n$, $p = 1, \dots, l_j$. Similarly, let $\mathbf{i}_s = (i_1^s, \dots, i_{m_s}^s)$ be such that $i_1^s < i_2^s < \cdots < i_{m_s}^s$ and $d_{s, i_p^s} = 1$ for all $s = 1, \dots, k$, $p = 1, \dots, m_s$. Introduce the following vectors:

$$v_{\mathbf{d}}^{\langle k \rangle} = e_{\mathbf{a}_1}^{\langle k \rangle} v_{l_1}^{\langle k \rangle} \otimes \cdots \otimes e_{\mathbf{a}_n}^{\langle k \rangle} v_{l_n}^{\langle k \rangle}, \quad v_{\mathbf{d}}^{\langle n \rangle} = e_{\mathbf{i}_1}^{\langle n \rangle} v_{m_1}^{\langle n \rangle} \otimes \cdots \otimes e_{\mathbf{i}_k}^{\langle n \rangle} v_{m_k}^{\langle n \rangle},$$

where $v_{l_j}^{\langle k \rangle} = x_1 x_2 \cdots x_{l_j}$ and $v_{m_s}^{\langle n \rangle} = x_1 x_2 \cdots x_{m_s}$ are highest weight vectors for the modules $V_{l_j}^{\langle k \rangle}$ and $V_{m_s}^{\langle n \rangle}$, respectively.

Lemma 4.2. *The vectors $v_{\mathbf{d}}^{(k)}$, $\mathbf{d} \in \mathcal{Z}_{kn}[\mathbf{l}, \mathbf{m}]$ form a basis of the weight subspace $(V_{l_1}^{(k)} \otimes \cdots \otimes V_{l_n}^{(k)})[m_1, \dots, m_k]$. Similarly, the vectors $v_{\mathbf{d}}^{(n)}$, $\mathbf{d} \in \mathcal{Z}_{kn}[\mathbf{l}, \mathbf{m}]$ form a basis of the weight subspace $(V_{m_1}^{(n)} \otimes \cdots \otimes V_{m_k}^{(n)})[l_1, \dots, l_n]$.*

Let $\varepsilon(\mathbf{d})$ be a sign function such that $x_{11}^{d_{11}} \cdots x_{1n}^{d_{1n}} \cdots x_{k1}^{d_{k1}} \cdots x_{kn}^{d_{kn}} = \varepsilon(\mathbf{d})x^{\mathbf{d}}$.

Lemma 4.3. *We have $\phi_1(v_{\mathbf{d}}^{(k)}) = x^{\mathbf{d}}$ and $\phi_2(v_{\mathbf{d}}^{(n)}) = \varepsilon(\mathbf{d})x^{\mathbf{d}}$.*

Consider the action of KZ, qKZ, DD and qDD operators for the Lie algebras \mathfrak{gl}_k and \mathfrak{gl}_n on \mathfrak{P}_{kn} -valued functions of z_1, \dots, z_n , and $\lambda_1, \dots, \lambda_k$, treating the space \mathfrak{P}_{kn} as a tensor product $(V_{\bullet}^{(k)})^{\otimes n}$ of \mathfrak{gl}_k -modules and as a tensor product $(V_{\bullet}^{(n)})^{\otimes k}$ of \mathfrak{gl}_n -modules. We will write $F \simeq G$ if the operators F and G act on the \mathfrak{P}_{kn} -valued functions in the same way.

Denote $\mathbf{1}^{(k)} = (\underbrace{1, 1, \dots, 1}_k)$ and $\mathbf{1}^{(n)} = (\underbrace{1, 1, \dots, 1}_n)$

Theorem 4.4. *For any $i = 1, \dots, n$ and $a = 1, \dots, k$ the following relations hold*

$$\nabla_{z_i}^{(k)}(z, \lambda, \kappa) \simeq D_{z_i}^{(n)}(\lambda, -z, -\kappa), \quad (4.3)$$

$$\nabla_{\lambda_a}^{(n)}(\lambda, z, \kappa) \simeq D_{\lambda_a}^{(k)}(z, -\lambda, -\kappa), \quad (4.4)$$

$$\widehat{\nabla}_{z_i}^{(k)}(z, \lambda, \kappa) \simeq -\widehat{D}_{z_i}^{(n)}(-\lambda + \mathbf{1}^{(k)}, z, -\kappa), \quad (4.5)$$

$$\widehat{\nabla}_{\lambda_a}^{(n)}(\lambda, z, \kappa) \simeq -\widehat{D}_{\lambda_a}^{(k)}(-z + \mathbf{1}^{(n)}, \lambda, -\kappa), \quad (4.6)$$

$$Z_{z_i}^{(k)}(z, \lambda, \kappa) \simeq N_i^{(n)}(z)Q_{z_i}^{(n)}(\lambda, -z, -\kappa), \quad (4.7)$$

$$Z_{\lambda_a}^{(n)}(\lambda, z, \kappa) \simeq N_a^{(k)}(\lambda)Q_{\lambda_a}^{(k)}(z, -\lambda, -\kappa), \quad (4.8)$$

where

$$N_i^{(n)}(z) = \frac{\prod_{1 \leq j < i} C_{ji}^m(z_{ji} - \kappa)}{\prod_{i < j \leq n} C_{ij}^m(z_{ij})}, \quad N_a^{(k)}(\lambda) = \frac{\prod_{1 \leq b < a} C_{ba}^k(\lambda_{ba} - \kappa)}{\prod_{a < b \leq k} C_{ab}^k(\lambda_{ab})}.$$

and

$$C_{ab}^{(k)}(t) = \frac{\Gamma(t + e_{aa}^{(k)} + 1)\Gamma(t - e_{bb}^{(k)})}{\Gamma(t + e_{aa}^{(k)} - e_{bb}^{(k)} + 1)\Gamma(t)}, \quad C_{ij}^{(n)}(t) = \frac{\Gamma(t + e_{ii}^{(n)} + 1)\Gamma(t - e_{jj}^{(n)})}{\Gamma(t + e_{ii}^{(n)} - e_{jj}^{(n)} + 1)\Gamma(t)}. \quad (4.9)$$

Proof. Verification of relations (4.3), (4.4), (4.5), (4.6) is straightforward. For (4.7) and (4.8), we have to show that

$$R_{ij}^{(k)}(t) \simeq C_{ij}^{(n)}(t)B_{ij}^{(n)}(-t), \quad (4.10)$$

$$R_{ab}^{(n)}(t) \simeq C_{ab}^{(k)}(t)B_{ab}^{(k)}(-t). \quad (4.11)$$

We will prove relation (4.11). Relation (4.10) can be proved similarly.

Note, that both action of $R_{ab}^{(n)}(t)$ on \mathfrak{P}_{kn} and action of $C_{ab}^{(k)}(t)B_{ab}^{(k)}(-t)$ on \mathfrak{P}_{kn} involve only the variables $x_{a1}, \dots, x_{an}, x_{b1}, \dots, x_{bn}$. Therefore, it is sufficient to prove (4.11) for the case of $k = 2, a = 1, b = 2$.

The \mathfrak{gl}_n -module $\mathfrak{P}_{2,n}$ is isomorphic to $V_{\bullet}^{(n)} \otimes V_{\bullet}^{(n)}$. Consider the submodule $V_{m_1}^{(n)} \otimes V_{m_2}^{(n)} \subset \mathfrak{P}_{2,n}$. We have the following decomposition of the \mathfrak{gl}_n -module:

$$V_{m_1}^{(n)} \otimes V_{m_2}^{(n)} = \bigoplus_{m=\max(0, m_1+m_2-n)}^{\min(m_1, m_2)} V_{\mathbf{l}^{(m)}}^{(n)}. \quad (4.12)$$

Here $\mathbf{l}(m) = (2, 2, \dots, 2, 1, \dots, 1, 0, \dots, 0)$, where 2 repeats m times and 1 repeats $m_1 + m_2 - 2m$ times. Denote by v_m a highest weight vector of the summand $V_{\mathbf{l}(m)}^{(n)}$ given by formula (A.1).

Define the scalar product on $\mathfrak{P}_{2,n}$ by the rule: $\langle f, f \rangle = 1$, if $f \in \mathfrak{P}_{2,n}$ is a nonzero monomial, and $\langle f, h \rangle = 0$, if $f, h \in \mathfrak{P}_{2,n}$ are two non-proportional monomials.

Lemma 4.5. *We have $\langle v_m, v_m \rangle \neq 0$ for every m .*

The proof is straightforward by formula (A.1).

Lemma 4.6. *$\langle w_1, e_{ij}^{(n)} w_2 \rangle = \langle e_{ji}^{(n)} w_1, w_2 \rangle$ for any $w_1, w_2 \in \mathfrak{P}_{2,n}$, and $i, j = 1, \dots, n$.*

The proof is straightforward.

Corollary 4.7. *If vectors w and \tilde{w} belong to distinct summands of decomposition (4.12), then $\langle w, \tilde{w} \rangle = 0$.*

Proof. The summands of decomposition (4.12) are eigenspaces of the operator $I^{(n)}$, and the corresponding eigenvalues are distinct. Lemma 4.6 implies that the operator $I^{(n)}$ is symmetric with respect to the scalar product $\langle \cdot, \cdot \rangle$, which implies the statement. ■

Denote

$$\begin{aligned} L_{ij}(t) &= t(e_{ij}^{(n)})_{(1)} + \sum_{k=1}^n (e_{ik}^{(n)})_{(1)} (e_{kj}^{(n)})_{(2)}, & M_{ij}(t) &= t(e_{ij}^{(n)})_{(1)} + \sum_{k=1}^n (e_{kj}^{(n)})_{(1)} (e_{ik}^{(n)})_{(2)}, \\ \alpha_m(t) &= \langle L_{m_1+m_2-m+1,m}(t) \cdot v_m, v_{m-1} \rangle, & \beta_m(t) &= \langle M_{m_1+m_2-m+1,m}(t) \cdot v_m, v_{m-1} \rangle. \end{aligned}$$

Lemma 4.8. *The functions $\alpha_m(t)$ and $\beta_m(t)$ are nonzero, and*

$$\frac{\alpha_m(t)}{\beta_m(t)} = \frac{t + 1 + m_1 - m}{t - 1 + m - m_2}. \quad (4.13)$$

The proof is given in Appendix A.

Due to relation (3.1), for any m , there exists a scalar function $\rho_m(t)$ such that $R_{12}^{(n)}(t)w = \rho_m(t)w$ for any $w \in V_{\mathbf{l}(m)}^{(n)}$.

Lemma 4.9. *It holds that*

$$\frac{\rho_m(t)}{\rho_{m-1}(t)} = \frac{\alpha_m(t)}{\beta_m(t)}. \quad (4.14)$$

Proof. Let us single out the term $V_{\mathbf{l}(m-1)}$ in the decomposition (4.12):

$$V_{m_1}^{(n)} \otimes V_{m_2}^{(n)} = V_{\mathbf{l}(m-1)} \oplus \tilde{V}.$$

Then we can write $L_{m_1+m_2-m+1,m}(t) \cdot v_m = w + \tilde{w}$, where $w \in V_{\mathbf{l}(m-1)}$ and $\tilde{w} \in \tilde{V}$. By the definition of $L_{m_1+m_2-m+1,m}(t)$, the vector w has weight $\mathbf{l}(m-1)$. Therefore, $w = av_{m-1}$ for some scalar a . By Corollary 4.7, we have

$$\alpha_m(t) = \langle L_{m_1+m_2-m+1,m}(t) \cdot v_m, v_{m-1} \rangle = a \langle v_{m-1}, v_{m-1} \rangle.$$

Notice that $R_{12}^{(n)}(t)\tilde{w} \in \tilde{V}$, because R -matrix $R_{12}^{(n)}(t)$ acts as a multiplication by a scalar function on each summand of the decomposition (4.12). Then, by Corollary 4.7, $\langle R_{12}^{(n)}(t)\tilde{w}, v_{m-1} \rangle = 0$, and

$$\begin{aligned} &\langle R_{12}^{(n)}(t)L_{m_1+m_2-m+1,m}(t) \cdot v_m, v_{m-1} \rangle \\ &= \langle R_{12}^{(n)}(t)w, v_{m-1} \rangle = \rho_{m-1}(t)a \langle v_{m-1}, v_{m-1} \rangle = \rho_{m-1}(t)\alpha_m(t). \end{aligned}$$

On the other hand, relation (3.2) gives

$$\begin{aligned} & \langle R_{12}^{(n)}(t) L_{m_1+m_2-m+1, m}(t) \cdot v_m, v_{m-1} \rangle \\ &= \langle M_{m_1+m_2-m+1, m}(t) R_{12}^{(n)}(t) \cdot v_m, v_{m-1} \rangle = \rho_m(t) \beta_m(t). \end{aligned}$$

Thus we get $\alpha_m(t) \rho_{m-1}(t) = \rho_m(t) \beta_m(t)$, which is relation (4.14). \blacksquare

By formulae (4.14), (4.13),

$$\rho_m(t) = \prod_{s=1}^m \frac{\rho_s(t)}{\rho_{s-1}(t)} = \prod_{s=0}^{m-1} \frac{t + m_1 - s}{t - m_2 + s}, \quad (4.15)$$

where we used that $\rho_0 = 1$ by the normalization condition (3.3).

Consider a decomposition of the \mathfrak{gl}_2 -module:

$$V_{l_1}^{(2)} \otimes \cdots \otimes V_{l_n}^{(2)} = \bigoplus_{0 \leq m \leq |l|/2} V_{(|l|-m, m)}^{(2)} \otimes W_m^{(2)},$$

where $|l| = \sum_i l_i$ and $W_m^{(2)}$ are multiplicity spaces. Let

$$(V_{l_1}^{(2)} \otimes \cdots \otimes V_{l_n}^{(2)})[m_1, m_2]_m = (V_{l_1}^{(2)} \otimes \cdots \otimes V_{l_n}^{(2)})[m_1, m_2] \cap (V_{(|l|-m, m)}^{(2)} \otimes W_m^{(2)}).$$

Lemma 4.10. *It holds that*

$$B_{12}^{(2)}(t) \Big|_{(V_{l_1}^{(2)} \otimes \cdots \otimes V_{l_n}^{(2)})[m_1, m_2]_m} = \prod_{s=m}^{m_2-1} \frac{t + m_2 - s}{t - m_1 + s}. \quad (4.16)$$

Proof. The modules $V_{l_i}^{(2)}$ can have only the following highest weights: $(1, 1)$, $(1, 0)$, $(0, 0)$. Moreover, the modules $V_{(0,0)}^{(2)}$ and $V_{(1,1)}^{(2)}$ are one-dimensional and the elements e_{12} , e_{21} , and $e_{11} - e_{22}$ act there by zero. Thus, it is enough to consider the case when $V_{l_i}^{(2)} = V_{(1,0)}^{(2)}$ for all i , so formula (4.16) follows from [6, formula (5.13)]. \blacksquare

Comparing formulae (4.15), (4.16), and (4.9), we conclude

$$\rho_m(t) = (B_{12}^{(2)}(-t) C_{12}^{(2)}(t)) \Big|_{(V_{l_1}^{(2)} \otimes \cdots \otimes V_{l_n}^{(2)})[m_1, m_2]_m}. \quad (4.17)$$

Lemma 4.11. *For the Casimir elements $I^{(2)}$ and $I^{(n)}$, we have*

$$I^{(2)} - 2 \sum_{a=1}^2 e_{aa}^{(2)} \simeq -I^{(n)} + n \sum_{i=1}^n e_{ii}^{(n)}. \quad (4.18)$$

The proof is straightforward.

Recall that in the irreducible \mathfrak{gl}_k -module V_l the element $I^{(k)}$ acts as a multiplication by $(l, l + 2\rho)$. Then it is easy to verify that

$$\left(I^{(2)} - 2 \sum_{a=1}^2 e_{aa}^{(2)} \right) \Big|_{V_{(|l|-m, m)}^{(2)} \otimes W_m^{(2)}} = \left(-I^{(n)} + n \sum_{i=1}^n e_{ii}^{(n)} \right) \Big|_{V_{l(m)}^{(n)}}. \quad (4.19)$$

Comparing formulae (4.18) and (4.19), and using the fact that the Casimir elements act on distinct irreducible modules as a multiplication by distinct scalar functions, we get that under

isomorphisms (4.1) and (4.2) the respective images of $V_{(|\mathbf{l}|-m,m)}^{(2)} \otimes W_m^{(2)}$ and $V_{\mathbf{l}(m)}^{(n)}$ in $\mathfrak{P}_{2,n}$ coincide. To indicate that, we will write $V_{(|\mathbf{l}|-m,m)}^{(2)} \otimes W_m^{(2)} \simeq V_{\mathbf{l}(m)}^{(n)}$.

We also have $(V_{l_1}^{(2)} \otimes \cdots \otimes V_{l_n}^{(2)})[m_1, m_2] \simeq (V_{m_1}^{(n)} \otimes V_{m_2}^{(n)})[l_1, \dots, l_n]$. Therefore $(V_{m_1}^{(n)} \otimes V_{m_2}^{(n)})[l_1, \dots, l_n] \cap V_{\mathbf{l}(m)}^{(n)} \simeq (V_{l_1}^{(2)} \otimes \cdots \otimes V_{l_n}^{(2)})[m_1, m_2]_m$. Now we see that (4.17) gives us relation between actions of operators $B_{12}^{(2)}(t)$, $C_{12}^{(2)}(t)$, and $R_{12}^{(n)}(t)$ on one particular submodule of $\mathfrak{P}_{2,n}$.

Theorem 4.4 is proved. \blacksquare

A Proof of Lemma 4.8

Let

$$v_m = \prod_{i=1}^m x_{1i} x_{2i} \sum_{\varepsilon} x_{\varepsilon_1, m+1} x_{\varepsilon_2, m+2} \cdots x_{\varepsilon_{m_1+m_2-2m}, m_1+m_2-m} \quad (\text{A.1})$$

with $\{\varepsilon\} = \{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m_1+m_2-2m}) : \varepsilon_i = 1 \text{ or } 2, \sum_i \varepsilon_i = m_1 - m + 2(m_2 - m)\}$. One can easily prove that v_m is a highest weight vector of weight $\mathbf{l}(m)$.

It follows from the construction of the scalar product $\langle \cdot, \cdot \rangle$ that $\alpha_m(t)$ (resp. $\beta_m(t)$) equals the sum of the coefficients of those monomials presented in

$$L_{m_1+m_2-m+1, m}(t) \cdot v_m \quad (\text{resp. } M_{m_1+m_2-m+1, m}(t) \cdot v_m)$$

that also appear in v_{m-1} . In fact, all monomials either in

$$L_{m_1+m_2-m+1, m}(t) \cdot v_m \quad \text{or in } M_{m_1+m_2-m+1, m}(t) \cdot v_m$$

appear in v_{m-1} as well.

We will start with α_m . Denote $s = m_1 + m_2 - m + 1$. Let us inspect what happens when we apply various terms of $L_{sm}(t)$ to v_m . For the sum $\sum_{k=1}^n (e_{sk})_{(1)}(e_{km})_{(2)}$, we can assume that $m \leq k < s$. If $k > m$, then the operator $(e_{sk})_{(1)}(e_{km})_{(2)}$ will send a monomial in v_m to zero if and only if this monomial does not depend on x_{1k} . That is, we look at all terms in v_m corresponding to $\varepsilon_{k-m} = 1$. There are $C_{m_1+m_2-2m-1}^{m_2-m}$ such terms with the same contribution $(-1)^{m_1+m_2+m}$. We leave the details of this calculation to a reader. Under the assumption $m < k < s$, there are $m_1 + m_2 - 2m$ different values of k , which yield the overall contribution $(-1)^{m_1+m_2+m} (m_1 + m_2 - 2m) C_{m_1+m_2-2m-1}^{m_2-m}$ to $\alpha_m(t)$.

If $k = m$, then we have $(e_{sk})_{(1)}(e_{km})_{(2)} \cdot v_m = (e_{sm})_{(1)} \cdot v_m$. Therefore all $C_{m_1+m_2-2m}^{m_1-m}$ terms in v_m equally contribute $(-1)^{m_1+m_2+m} (m_1 + m_2 - 2m)$.

Finally, the term $t(e_{ij})_{(1)}$ in $L_{sm}(t)$ generates the contribution $t(-1)^{m_1+m_2+m} C_{m_1+m_2-2m}^{m_1-m}$ to $\alpha_m(t)$, which can be seen similarly to the case $k = m$ considered above.

Thus we obtained

$$(-1)^{m_1+m_2+m} \alpha_m = (t+1) C_{m_1+m_2-2m}^{m_1-m} + (m_1 + m_2 - 2m) C_{m_1+m_2-2m-1}^{m_2-m}.$$

The similar arguments give us

$$(-1)^{m_1+m_2+m} \beta_m = (t-1) C_{m_1+m_2-2m}^{m_1-m} - (m_1 + m_2 - 2m) C_{m_1+m_2-2m-1}^{m_1-m}.$$

Since

$$(m_1 + m_2 - 2m) C_{m_1+m_2-2m-1}^{m_1-m} = (m_1 - m) C_{m_1+m_2-2m}^{m_1-m},$$

and

$$(m_1 + m_2 - 2m) C_{m_1+m_2-2m-1}^{m_2-m} = (m_2 - m) C_{m_1+m_2-2m}^{m_1-m}.$$

Lemma 4.8 is proved.

Acknowledgements

V. Tarasov was supported in part by Simons Foundation grant 430235.

References

- [1] Cheng S.-J., Wang W., Dualities and representations of Lie superalgebras, *Graduate Studies in Mathematics*, Vol. 144, [Amer. Math. Soc.](#), Providence, RI, 2012.
- [2] Etingof P., Frenkel I., Kirillov A., Lectures on representation theory and Knizhnik–Zamolodchikov equations, *Mathematical Surveys and Monographs*, Vol. 58, [Amer. Math. Soc.](#), Providence, RI, 1998.
- [3] Etingof P., Varchenko A., Dynamical Weyl groups and applications, *Adv. Math.* **167** (2002), 74–127, [arXiv:math.QA/0011001](#).
- [4] Felder G., Markov Y., Tarasov V., Varchenko A., Differential equations compatible with KZ equations, *Math. Phys. Anal. Geom.* **3** (2000), 139–177, [arXiv:math.QA/0001184](#).
- [5] Tarasov V., Varchenko A., Difference equations compatible with trigonometric KZ differential equations, *Int. Math. Res. Not.* **2000** (2000), 801–829, [arXiv:math.QA/0002132](#).
- [6] Tarasov V., Varchenko A., Duality for Knizhnik–Zamolodchikov and dynamical equations, *Acta Appl. Math.* **73** (2002), 141–154, [arXiv:math.QA/0112005](#).
- [7] Tarasov V., Varchenko A., Dynamical differential equations compatible with rational qKZ equations, *Lett. Math. Phys.* **71** (2005), 101–108, [arXiv:math.QA/0403416](#).
- [8] Toledano-Laredo V., The trigonometric Casimir connection of a simple Lie algebra, *J. Algebra* **329** (2011), 286–327, [arXiv:1003.2017](#).
- [9] Toledano-Laredo V., Yang Y., The elliptic Casimir connection of a simple Lie algebra, [arXiv:1805.12261](#).
- [10] Vicedo B., Young C., $(\mathfrak{gl}_M, \mathfrak{gl}_N)$ -dualities in Gaudin models with irregular singularities, *SIGMA* **14** (2018), 040, 28 pages, [arXiv:1710.08672](#).