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Nonstandard Quantum Complex Projective Line

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Abstract. In our attempt to explore how the quantum nonstandard complex projective spaces $\mathbb{C}P^n_{q,c}$ studied by Korogodsky, Vaksman, Dijkhuizen, and Noumi are related to those arising from the geometrically constructed Bohr–Sommerfeld groupoids by Bonechi, Ciccoli, Qiu, Staffolani, and Tarlini, we were led to establish the known identification of $C\big(\mathbb{C}P^1_{q,c}\big)$ with the pull-back of two copies of the Toeplitz C^*-algebra along the symbol map in a more direct way via an operator theoretic analysis, which also provides some interesting non-obvious details, such as a prominent generator of $C\big(\mathbb{C}P^1_{q,c}\big)$ being a concrete weighted double shift.

Key words: quantum homogeneous space; Toeplitz algebra; weighted shift

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1 Introduction

In [9], the C^*-algebra $C\big(\mathbb{C}P^n_{q,c}\big)$ of nonstandard quantum complex projective spaces studied by Korogodsky and Vaksman [5] and Dijkhuizen and Noumi [3] is embedded in a concrete groupoid C^*-algebra, and then shown to have $C\big(\mathbb{S}^{2n-1}_q\big)$ as a quotient algebra, which reflects the geometric observation [10] that the nonstandard $\mathrm{SU}(n+1)$-covariant Poisson complex projective space $\mathbb{C}P^n$ contains a copy of the standard Poisson sphere $\mathbb{S}^{2n-1}$.

Although the work in [9] involves realizing $C\big(\mathbb{C}P^n_{q,c}\big)$ as part of a concrete groupoid C^*-algebra in order to analyze the algebra structure and extract useful information, it is not clear whether one can actually realize $C\big(\mathbb{C}P^n_{q,c}\big)$ as a groupoid C^*-algebra itself. However from a purely differential geometric consideration, an elegant program of constructing some quantum homogeneous spaces as the groupoid C^*-algebras of geometrically constructed Bohr–Sommerfeld groupoids is later successfully developed by Bonechi, Ciccoli, Qiu, Staffolani, Tarlini [1, 2]. Naturally, it is of great interest to decide whether the quantum complex projective space arising from this new program is indeed the same as the known version of $C\big(\mathbb{C}P^n_{q,c}\big)$. Indeed they are the same for the case of $n=1$ because the underlying groupoids are shown to be isomorphic in Proposition 7.2 of [1]. But the higher dimensional cases are far from being settled.

It is hoped that by analyzing more carefully the embedding of $C\big(\mathbb{C}P^n_{q,c}\big)$ in a concrete groupoid C^*-algebra found in [9] via a representation theoretic approach, one can see some direct connection with the geometrically constructed Bohr–Sommerfeld groupoid and then possibly find a way to identify these two different versions of quantum complex projective spaces. While attempting this approach, we come to recognize the need of a more direct understanding of the algebra structure of $C\big(\mathbb{C}P^n_{q,c}\big)$ based on some known representations of the ambient algebra $C(\mathrm{SU}_q(n+1))$.

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In particular, for $n = 1$, we want to directly derive the algebra structure of $C\big(\mathbb{C}P^1_{q,c}\big)$ from the basic representations of $C(\mathrm{SU}_q(2))$, instead of via identifying $C\big(\mathbb{C}P^1_{q,c}\big)$ with the algebra $C\big(\mathbb{S}^2_{\mu c}\big)$ of the Podleś quantum 2-sphere [7] as indicated in [3, 5]. In this note, we show how to accomplish it. Along the way, our detailed analysis reveals some nontrivial hidden structures, for example, a distinguished generator $x_1^* x_2$ of $C\big(\mathbb{C}P^1_{q,c}\big)$ is a weighted double shift (on a core Hilbert space that determines the C^*-algebra structure of $C\big(\mathbb{C}P^1_{q,c}\big)$) with respect to an orthonormal basis, and its weights are determined by a concrete formula.

2 Nonstandard quantum $\mathbb{C}P^1_{q,c}$

We recall the description of $C\big(\mathbb{C}P^n_{q,c}\big)$ with $c \in (0,\infty)$ and $q \in (1,\infty)$ obtained by Dijkhuizen and Noumi [3] as

$$C\big(\mathbb{C}P^n_{q,c}\big) \cong C^*(\{x_i^* x_j \,|\, 1 \le i,j \le n+1\}) \subset C(\mathrm{SU}_q(n+1)),$$

where

$$x_i = \sqrt{c}\, u_{1,i} + u_{n+1,i}$$

for the standard generators $\{u_{i,j}\}_{i,j=1}^{n+1}$ of $C(\mathrm{SU}_q(n+1))$.

In this paper, we focus on the case of $n = 1$, with the goal to directly show that $C\big(\mathbb{C}P^1_{q,c}\big)$ is the pullback $\mathcal{T} \oplus_{C(\mathbb{T})} \mathcal{T}$ of two copies of the standard symbol map

$$\sigma\colon\ \mathcal{T} \to C(\mathbb{T})$$

for the Toeplitz algebra $\mathcal{T}$ that is the C^*-algebra generated by the (forward) unilateral shift $\mathcal{S}$ on $\ell^2(\mathbb{Z}_\ge)$, with $\ker(\sigma) = \mathcal{K}\big(\ell^2(\mathbb{Z}_\ge)\big)$ the ideal of all compact operators.

For $C(\mathrm{SU}_q(2))$, consider the known faithful representation π of $C(\mathrm{SU}_q(2))$ determined by

$$\pi(u) \equiv \begin{pmatrix} \pi(u_{11}) & \pi(u_{12}) \\ \pi(u_{21}) & \pi(u_{22}) \end{pmatrix} := \left\{ \begin{pmatrix} t_1\alpha & -q^{-1}t_1\gamma \\ t_2\gamma & t_2\alpha^* \end{pmatrix} \right\}_{t_2=\overline{t_1}\in\mathbb{T}}$$

as a $\mathbb{T}$-family of representations of $C(\mathrm{SU}_q(2))$ on $\ell^2(\mathbb{Z}_\ge)$ with parameter $t_1 \equiv \overline{t_2} \in \mathbb{T}$, where

$$\alpha = \begin{pmatrix} 0 & \sqrt{1-q^{-2}} & 0 & & \\ 0 & 0 & \sqrt{1-q^{-4}} & 0 & \\ 0 & 0 & 0 & \sqrt{1-q^{-6}} & \ddots \\ & 0 & 0 & 0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix} \in \mathcal{B}\big(\ell^2(\mathbb{Z}_\ge)\big)$$

and

$$\gamma = \begin{pmatrix} 1 & 0 & 0 & & \\ 0 & q^{-1} & 0 & 0 & \\ 0 & 0 & q^{-2} & 0 & \ddots \\ & 0 & 0 & q^{-3} & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix} \in \mathcal{B}\big(\ell^2(\mathbb{Z}_\ge)\big) \quad \text{self-adjoint}$$

satisfying

$$\alpha^*\alpha + \gamma\gamma^* \equiv \alpha^*\alpha + \gamma^2 = I = \alpha\alpha^* + q^{-2}\gamma^2 \equiv \alpha\alpha^* + q^{-2}\gamma^*\gamma$$

and

$$\gamma\alpha^* - q^{-1}\alpha^*\gamma = \alpha\gamma^* - q^{-1}\gamma^*\alpha = 0 \equiv \alpha\gamma - q^{-1}\gamma\alpha = \gamma^*\alpha^* - q^{-1}\alpha^*\gamma^*$$

which ensure the required condition $\pi(u)\pi(u)^* = I = \pi(u)^*\pi(u)$.

In this paper, we identify every element of $C(\mathrm{SU}_q(2)) \supset C(\mathbb{C}P^1_{q,c})$ with a $\mathbb{T}$-family of operators on $\ell^2(\mathbb{Z}_{\geq})$ via this faithful representation π, and we analyze such a $\mathbb{T}$-family of operators pointwise at each fixed $t_1 \in \mathbb{T}$.

More explicitly, the generators $x_1^*x_2, x_1^*x_1, x_2^*x_2$ of $C\big(\mathbb{C}P^1_{q,c}\big)$ are $\mathbb{T}$-families of operators with

$$x_1 := \sqrt{c}t_1\alpha + t_2\gamma = \begin{pmatrix} t_2 & \sqrt{c}t_1\sqrt{1-q^{-2}} & 0 & & \\ 0 & t_2q^{-1} & \sqrt{c}t_1\sqrt{1-q^{-4}} & 0 & \\ 0 & 0 & t_2q^{-2} & \sqrt{c}t_1\sqrt{1-q^{-6}} & \ddots \\ & 0 & 0 & t_2q^{-3} & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

and

$$x_2 := -q^{-1}\sqrt{c}t_1\gamma + t_2\alpha^* = \begin{pmatrix} -q^{-1}\sqrt{c}t_1 & 0 & 0 & & \\ t_2\sqrt{1-q^{-2}} & -q^{-2}\sqrt{c}t_1 & 0 & 0 & \\ 0 & t_2\sqrt{1-q^{-4}} & -q^{-3}\sqrt{c}t_1 & 0 & \ddots \\ & 0 & t_2\sqrt{1-q^{-6}} & -q^{-4}\sqrt{c}t_1 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

At any fixed $t_1 \in \mathbb{T}$, it is easy to see that $\ker(x_2) = 0$ since

$$\begin{aligned} 0 = x_2\left(\sum_{n=0}^{\infty} z_n e_n\right) &= -q^{-1}\sqrt{c}z_0e_0 + \big(\sqrt{1-q^{-2}}z_0 - q^{-2}\sqrt{c}z_1\big)e_1 \\ &\quad + \big(\sqrt{1-q^{-4}}z_1 - q^{-3}\sqrt{c}z_2\big)e_2 + \cdots \end{aligned}$$

implies $z_0 = z_1 = z_2 = \cdots = 0$, and $\dim(\mathrm{coker}(x_2)) = 1$ since $x_2 \equiv t_2\alpha^* \equiv t_2\mathcal{S}$ modulo $\mathcal{K}$ is a Fredholm operator of index -1. On the other hand, $\dim(\ker(x_1)) = 1$ by solving $0 = x_1\Big(\sum_{n=0}^{\infty} z_ne_n\Big)$ to get that if $z_0 = 1$, then

$$z_n = (-1)^n t_2^n q^{\frac{-n(n-1)}{2}} \sqrt{c}^{-n} t_1^{-n} \sqrt{1-q^{-2}}^{-1} \cdots \sqrt{1-q^{-2n}}^{-1}$$

for all $n \in \mathbb{N}$, and x_1 is surjective since $x_1 \equiv \sqrt{c}t_1\alpha \equiv \sqrt{c}t_1\mathcal{S}^*$ modulo $\mathcal{K}$ is a Fredholm operator of index 1. So $x_1^*x_2$ is a Fredholm operator of index -2. Actually, $\ker(x_1^*x_2) = 0$ (hence $(x_1^*x_2)^*(x_1^*x_2)$ is invertible) and $\dim(\mathrm{coker}(x_1^*x_2)) = 2$, and hence the partial isometry $(x_1^*x_2)|x_1^*x_2|^{-1}$ in the polar decomposition of $x_1^*x_2$ is $\mathcal{S} \oplus \mathcal{S}$ (up to a unitary direct summand) after a suitable choice of orthonormal basis. This observation is consistent with our goal to show that $C\big(\mathbb{C}P^1_{q,c}\big)$ is isomorphic to the pullback C^*-algebra $\mathcal{T} \oplus_{C(\mathbb{T})} \mathcal{T}$, but is far from sufficient to make such a conclusion. We need to do a much more detailed analysis which starts with the following computation.

First we compute

$$\begin{aligned} x_1^*x_1 &= c\alpha^*\alpha + \sqrt{c}\overline{t_1}^2\alpha^*\gamma + \sqrt{c}t_1^2\gamma\alpha + \gamma^2 \\ &= c + (1-c)\gamma^2 + \sqrt{c}\overline{t_1}^2\alpha^*\gamma + \sqrt{c}t_1^2\gamma\alpha \equiv c \mod \mathcal{K}, \end{aligned}$$

$$\begin{aligned}
x_2^*x_2 &= \alpha\alpha^* - q^{-1}\sqrt{c}t_1^2\alpha\gamma - q^{-1}\sqrt{c}\overline{t_1}^2\gamma\alpha^* + q^{-2}c\gamma^2\\
&= 1 + q^{-2}(c-1)\gamma^2 - q^{-1}\sqrt{c}t_1^2\alpha\gamma - q^{-1}\sqrt{c}\overline{t_1}^2\gamma\alpha^*\\
&= 1 + q^{-2}(c-1)\gamma^2 - q^{-2}\sqrt{c}t_1^2\gamma\alpha - q^{-2}\sqrt{c}\overline{t_1}^2\alpha^*\gamma \equiv 1 \mod \mathcal{K},\\
x_1^*x_2 &= \sqrt{c}\overline{t_1}^2(\alpha^*)^2 - cq^{-1}\alpha^*\gamma + \gamma\alpha^* - q^{-1}\sqrt{c}t_1^2\gamma^2 \equiv \sqrt{c}\overline{t_1}^2\mathcal{S}^2 \mod \mathcal{K},\\
x_2^*x_1 &= \sqrt{c}t_1^2\alpha^2 - cq^{-1}\gamma\alpha + \alpha\gamma - q^{-1}\sqrt{c}\overline{t_1}^2\gamma^2,
\end{aligned}$$

which imply

$$x_1^*x_1 + q^2x_2^*x_2 = q^2 + c.$$

So the C^*-algebra $C(\mathbb{C}P^1_{q,c})$ is generated by $x_1^*x_2$ and $x_1^*x_1$, i.e.,

$$C(\mathbb{C}P^1_{q,c}) = C^*(\{x_1^*x_2, x_1^*x_1\}),$$

since $x_2^*x_1 = (x_1^*x_2)^*$ and $x_2^*x_2 = 1 + cq^{-2} - q^{-2}x_1^*x_1$ are generated by $x_1^*x_2$ (with $(x_1^*x_2)^*(x_1^*x_2)$ invertible) and $x_1^*x_1$. As a remark, we note that $x_1^*x_1$ and $x_2^*x_2$ commute.

We also note that

$$x_1x_1^* + x_2x_2^* = 1 + c$$

and hence $x_1x_1^*$ and $x_2x_2^*$ commute. Indeed

$$\begin{aligned}
x_1x_1^* &= (\sqrt{c}t_1\alpha + t_2\gamma)(\sqrt{c}t_2\alpha^* + t_1\gamma) = c\alpha\alpha^* + \sqrt{c}t_1^2\alpha\gamma + \sqrt{c}t_2^2\gamma\alpha^* + \gamma^2\\
&= c - cq^{-2}\gamma^2 + \sqrt{c}t_1^2q^{-1}\gamma\alpha + \sqrt{c}t_2^2q^{-1}\alpha^*\gamma + \gamma^2,
\end{aligned}$$

while

$$\begin{aligned}
x_2x_2^* &= (t_2\alpha^* - q^{-1}\sqrt{c}t_1\gamma)(t_1\alpha - q^{-1}\sqrt{c}t_2\gamma) = \alpha^*\alpha - q^{-1}\sqrt{c}t_2^2\alpha^*\gamma - q^{-1}\sqrt{c}t_1^2\gamma\alpha + q^{-2}c\gamma^2\\
&= 1 - \gamma^2 - q^{-1}\sqrt{c}t_2^2\alpha^*\gamma - q^{-1}\sqrt{c}t_1^2\gamma\alpha + q^{-2}c\gamma^2.
\end{aligned}$$

Before proceeding further, we recall some operator-theoretic properties often used implicitly in the following analysis, including that $\overline{\mathrm{range}(T)} = \ker(T^*)^\perp$ and $\ker(T^*) = \ker(T^*T)$ for general bounded linear operators T on a Hilbert space $\mathcal{H}$ easily derived from $\langle T^*(v), w\rangle = \langle v, T(w)\rangle$ and $\langle (T^*T)(v), v\rangle = \langle T(v), T(v)\rangle$ for all $v, w \in \mathcal{H}$ respectively. In C^*-algebra theory, a projection refers to a self-adjoint idempotent. For an operator T in the C^*-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$, we recall that T is a projection if and only if T is geometrically the orthogonal projection from $\mathcal{H}$ onto a closed subspace of $\mathcal{H}$.

We will need some basic knowledge of Fredholm operators, i.e., operators $T \in \mathcal{B}(\mathcal{H})$ with its quotient class $[T]$ an invertible element of the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, which can be found in [4, 6]. Any such operator has closed finite-codimensional range and finite-dimensional kernel, and the intersection of $\mathbb{R}\backslash\{0\}$ and the spectrum $\mathrm{Sp}(T)$ of any positive Fredholm operator T is a compact subset of $(0, \infty)$. Also we note that the set of all Fredholm operators is closed under taking adjoint and composition of operators. For any positive Fredholm operator T we will denote by $\underline{T^{-1/2}}$ the positive operator $f(T)$ defined by functional calculus, where f is the nonnegative continuous function on $\{0\} \sqcup K$ such that $f(\bullet) = \bullet^{-1/2}$ on $K := \mathrm{Sp}(T)\backslash\{0\}$ and $f(0) = 0$.

Below we recall a folklore result with a proof.

Lemma 1. *For any Fredholm operator T on a Hilbert space $\mathcal{H}$,*

$$\tilde{T} := T\underline{(T^*T)^{-1/2}}$$

is a partial isometry sending the closed subspace $(\ker(T))^\perp \equiv \mathrm{range}(T^*)$ isometrically onto the closed subspace $\mathrm{range}(T)$ while annihilating $\ker(T)$.

Proof. Note that since T^*T is a positive Fredholm operator, the set K is a compact subset of $(0,\infty)$.

By the spectral theory of self-adjoint operators,

$$\tilde{T}^*\tilde{T} \equiv \underline{(T^*T)^{-1/2}}T^*T\underline{(T^*T)^{-1/2}} \equiv f(T^*T)(T^*T)f(T^*T) = \chi_K(T^*T)$$

for the characteristic function χ_K on $\mathrm{Sp}(T^*T)$, and hence $\tilde{T}^*\tilde{T}$ is the orthogonal projection from $\mathcal{H}$ onto range(T^*T). Thus $\tilde{T}$ annihilates

$$\ker\left(\tilde{T}\right) \equiv \ker\left(\tilde{T}^*\tilde{T}\right) \equiv \left(\text{range}\left(\tilde{T}^*\tilde{T}\right)\right)^{\perp} = (\text{range}(T^*T))^{\perp} \equiv \ker(T^*T) \equiv \ker(T)$$

and is metric preserving on

$$\text{range}(T^*T) \equiv (\ker(T^*T))^{\perp} \equiv (\ker(T))^{\perp} \equiv \text{range}(T^*)$$

due to

$$\left\langle \tilde{T}(v), \tilde{T}(w)\right\rangle = \left\langle \left(\tilde{T}^*\tilde{T}\right)(v), w\right\rangle = \langle v, w\rangle \qquad \text{for all } v, w \in \text{range}(T^*T),$$

i.e., $\tilde{T}$ is a partial isometry sending range(T^*T) isometrically onto range$(\tilde{T})$ while annihilating $\ker(T)$.

It remains to show that range$(\tilde{T})$ = range(T). In fact, since $\frac{1}{f|_K}$ is a well-defined continuous function on K, the restriction of $(T^*T)^{-1/2} \equiv f(T^*T)$ to range(T^*T) is an invertible linear operator on range(T^*T) and hence

$$\text{range}\left((T^*T)^{-1/2}\right) = \text{range}(T^*T) \equiv (\ker(T))^{\perp}.$$

Thus we get

$$\text{range}(\tilde{T}) \equiv \text{range}\left(T(T^*T)^{-1/2}\right) = T((\ker(T))^{\perp}) = \text{range}(T). \qquad \blacksquare$$

At each fixed $t_1 \in \mathbb{T}$, by applying Lemma 1 to the Fredholm operator values of the norm continuous $\mathbb{T}$-families x_1 and x_2, we get two partial isometries

$$\tilde{x}_1 := x_1\underline{(x_1^*x_1)^{-1/2}}$$

and

$$\tilde{x}_2 := x_2\underline{(x_2^*x_2)^{-1/2}} = x_2(x_2^*x_2)^{-1/2}$$

where

$$\underline{(x_2^*x_2)^{-1/2}} = (x_2^*x_2)^{-1/2} \equiv \left((x_2^*x_2)^{-1}\right)^{1/2} \equiv \sqrt{(x_2^*x_2)^{-1}}$$

is a well-defined invertible operator on $\ell^2(\mathbb{Z}_{\geq})$ since $\ker(x_2) = 0$ and hence the spectrum of the positive Fredholm operator $x_2^*x_2$ is a compact subset of $(0,\infty)$, implying the invertibility of $x_2^*x_2$ and making the functional calculus $(x_2^*x_2)^{-1/2} \equiv \left((x_2^*x_2)^{-1}\right)^{1/2}$ meaningful.

Theorem 1. *The C^*-algebra $C\left(\mathbb{C}P^1_{q,c}\right)$ coincides with $C^*(\{\tilde{x}_1^*\tilde{x}_2, x_1^*x_1\})$, i.e. the C^*-algebra generated by two $\mathbb{T}$-families $x_1^*x_1 \geq 0$ and $\tilde{x}_1^*\tilde{x}_2$ of operators on $\ell^2(\mathbb{Z}_{\geq})$ such that at each fixed $t_1 \in \mathbb{T}$, $\tilde{x}_1^*\tilde{x}_2$ is an isometry of index -2 (with a zero kernel and a range of codimension 2), where $\tilde{x}_1^*$ and $\tilde{x}_2 = x_2(x_2^*x_2)^{-1/2}$ are isometries of index -1 while $\tilde{x}_1 = x_1\underline{(x_1^*x_1)^{-1/2}}$ and $\tilde{x}_2^*$ are surjective partial isometries of index 1.*

Proof. By Lemma 1, the surjective Fredholm operator x_1 with kernel of dimension 1 yields a surjective partial isometry $\tilde{x}_1 = x_1\underline{(x_1^*x_1)^{-1/2}}$ of index 1 and the injective Fredholm x_2 with cokernel of dimension 1 yields an isometry $\tilde{x}_2 = x_2(x_2^*x_2)^{-1/2}$ of index -1.

Now both $\tilde{x}_2$ and the adjoint $\tilde{x}_1^*$ of the surjective partial isometry $\tilde{x}_1$ are isometries of index -1, and hence $\tilde{x}_1^*\tilde{x}_2$ is an isometry of index -2 with $(\tilde{x}_1^*\tilde{x}_2)^*(\tilde{x}_1^*\tilde{x}_2) = 1$.

From the definition of $\tilde{x}_i$, we get

$$\tilde{x}_1^*\tilde{x}_2 = \underline{(x_1^*x_1)^{-1/2}}x_1^*x_2(x_2^*x_2)^{-1/2} \in C^*(\{x_1^*x_2, x_1^*x_1, x_2^*x_2\}) = C\big(\mathbb{C}P^1_{q,c}\big).$$

Since $(x_1^*x_1)^{1/2}\underline{(x_1^*x_1)^{-1/2}}$ is the orthogonal projection onto $\mathrm{range}(x_1^*x_1) \equiv \mathrm{range}(x_1^*)$, by spectral theory:

$$x_1^*x_2 = (x_1^*x_1)^{1/2}\underline{(x_1^*x_1)^{-1/2}}x_1^*x_2 = (x_1^*x_1)^{1/2}\tilde{x}_1^*\tilde{x}_2(x_2^*x_2)^{1/2} \in C^*(\{\tilde{x}_1^*\tilde{x}_2, x_1^*x_1, x_2^*x_2\}).$$

So we get

$$C\big(\mathbb{C}P^1_{q,c}\big) \equiv C^*(\{x_1^*x_2, x_1^*x_1, x_2^*x_2\}) = C^*(\{\tilde{x}_1^*\tilde{x}_2, x_1^*x_1, x_2^*x_2\}).$$

Furthermore the generator $x_2^*x_2$ is redundant since

$$x_2^*x_2 = 1 + cq^{-2} - q^{-2}x_1^*x_1 = \big(1 + cq^{-2}\big)(\tilde{x}_1^*\tilde{x}_2)^*(\tilde{x}_1^*\tilde{x}_2) - q^{-2}x_1^*x_1$$

can be generated by $\tilde{x}_1^*\tilde{x}_2$ and $x_1^*x_1$. Thus

$$C\big(\mathbb{C}P^1_{q,c}\big) \equiv C^*(\{\tilde{x}_1^*\tilde{x}_2, x_1^*x_1, x_2^*x_2\}) = C^*(\{\tilde{x}_1^*\tilde{x}_2, x_1^*x_1\}). \qquad \blacksquare$$

We remark that for each $i \in \{1, 2\}$,

$$\tilde{x}_i^*\tilde{x}_i = \underline{(x_i^*x_i)^{-1/2}}x_i^*x_i\underline{(x_i^*x_i)^{-1/2}} = \chi_{\mathrm{Sp}(x_i^*x_i)\setminus\{0\}}(x_i^*x_i) \in C^*(\{x_i^*x_i\})$$

and hence belongs to $C\big(\mathbb{C}P^1_{q,c}\big) \equiv C^*(\{\tilde{x}_1^*\tilde{x}_2, x_1^*x_1, x_2^*x_2\})$, since any C^*-algebra is closed under functional calculus by continuous functions vanishing at 0.

3 Invariant subspace decomposition

In this section, fixing an arbitrary value of the parameter $t_1 \equiv \overline{t_2} \in \mathbb{T}$, we study and treat any $\mathbb{T}$-family of operators on $\ell^2(\mathbb{Z}_{\geq})$, including any element of $C\big(\mathbb{C}P^1_{q,c}\big)$, as an operator in $\mathcal{B}\big(\ell^2(\mathbb{Z}_{\geq})\big)$.

As such the index-1 surjective partial isometry $\tilde{x}_1$ has $\ker(\tilde{x}_1) = \ker(x_1) = \mathbb{C}v_1$ for some unit vector v_1, and

$$p_1 := 1 - \tilde{x}_1^*\tilde{x}_1 \in C^*(\{\tilde{x}_1^*\tilde{x}_2, \tilde{x}_1^*\tilde{x}_1\}) \subset C\big(\mathbb{C}P^1_{q,c}\big)$$

is the rank-1 orthogonal projection onto $\mathbb{C}v_1$. Note that $(\tilde{x}_1^*\tilde{x}_1)(v_1) = 0$ is equivalent to $\tilde{x}_1(v_1) = 0$ (or equivalently $v_1 \perp \mathrm{range}(\tilde{x}_1^*) = \mathrm{range}(\tilde{x}_1^*\tilde{x}_1)$).

Note that

$$p_2 := \tilde{x}_1^*\tilde{x}_1 - (\tilde{x}_2^*\tilde{x}_1)^*(\tilde{x}_2^*\tilde{x}_1) = \tilde{x}_1^*\tilde{x}_1 - (\tilde{x}_1^*\tilde{x}_2)(\tilde{x}_2^*\tilde{x}_1) = \tilde{x}_1^*(1 - \tilde{x}_2\tilde{x}_2^*)\tilde{x}_1$$

is also a rank-1 projection onto $\mathbb{C}v_2$ for some unit vector v_2. In fact p_2 clearly annihilates $\ker(\tilde{x}_1)$ and can be viewed as the conjugation of the rank-1 projection $1 - \tilde{x}_2\tilde{x}_2^*$ (onto the kernel of $\tilde{x}_2^*$) by the unitary operator $\tilde{x}_1|_{(\ker(\tilde{x}_1))^\perp}$, from $(\ker(\tilde{x}_1))^\perp$ onto $\ell^2(\mathbb{Z}_{\geq})$. In an explicit description,

v_2 can be taken as the inverse image under $\tilde{x}_1|_{(\ker(\tilde{x}_1))^\perp}$, of any unit vector in the 1-dimensional range of $1-\tilde{x}_2\tilde{x}_2^*$, and, in particular, $v_2 \in (\ker(\tilde{x}_1))^\perp \equiv \mathrm{range}(\tilde{x}_1^*\tilde{x}_1)$. The inequalities

$$0 \leq p_2 = \tilde{x}_1^*\tilde{x}_1 - (\tilde{x}_1^*\tilde{x}_2)(\tilde{x}_2^*\tilde{x}_1) \leq \tilde{x}_1^*\tilde{x}_1$$

relate the three projections p_2, $(\tilde{x}_1^*\tilde{x}_2)(\tilde{x}_2^*\tilde{x}_1)$, and $\tilde{x}_1^*\tilde{x}_1$ in $C^*(\{\tilde{x}_1^*\tilde{x}_2, \tilde{x}_1^*\tilde{x}_1\})$, and clarify their geometric relation: p_2 and $(\tilde{x}_1^*\tilde{x}_2)(\tilde{x}_2^*\tilde{x}_1)$ are projections onto two mutually orthogonal subspaces which add up to the range of the projection $\tilde{x}_1^*\tilde{x}_1$, i.e.,

$$\mathrm{range}(p_2) \oplus^\perp \mathrm{range}((\tilde{x}_1^*\tilde{x}_2)(\tilde{x}_2^*\tilde{x}_1)) = \mathrm{range}(\tilde{x}_1^*\tilde{x}_1),$$

indicating, in particular, $v_2 \in \mathrm{range}(p_2) \subset \mathrm{range}(\tilde{x}_1^*\tilde{x}_1)$.

Now $v_2 \perp v_1$ since v_2 is in the range of the self-adjoint operator $\tilde{x}_1^*\tilde{x}_1$ and hence is perpendicular to $\ker(\tilde{x}_1^*\tilde{x}_1) = \mathbb{C}v_1$. Furthermore,

$$v_{i+2n} := (\tilde{x}_1^*\tilde{x}_2)^n(v_i)$$

with $n \geq 0$ and $i \in \{1,2\}$ are orthonormal vectors, since

$$v_1 \perp \mathrm{range}(\tilde{x}_1^*) \supset \mathrm{range}(\tilde{x}_1^*\tilde{x}_2) \ni (\tilde{x}_1^*\tilde{x}_2)(v_1)$$

and

$$v_2 \perp \mathrm{range}((\tilde{x}_1^*\tilde{x}_2)(\tilde{x}_2^*\tilde{x}_1)) = \mathrm{range}(\tilde{x}_1^*\tilde{x}_2) \qquad \text{with } \ v_1 \perp v_2\,.$$

Thus $\mathcal{V} := \mathrm{Span}\{v_1, v_2\} \perp \mathrm{range}(\tilde{x}_1^*\tilde{x}_2)$ or more precisely, by combining with the fact that the index-(-2) isometry $\tilde{x}_1^*\tilde{x}_2$ has $\mathrm{range}(\tilde{x}_1^*\tilde{x}_2)$ of codimension 2,

$$\mathcal{H} = \mathcal{V} \ \oplus^\perp \mathrm{range}(\tilde{x}_1^*\tilde{x}_2) \qquad \text{as Hilbert space orthogonal direct sum.}$$

Hence, since $\tilde{x}_1^*\tilde{x}_2$ is an isometry, we inductively get:

$$\mathrm{range}(\tilde{x}_1^*\tilde{x}_2)^{k-1} = (\tilde{x}_1^*\tilde{x}_2)^{k-1}(\mathcal{V}) \oplus^\perp \mathrm{range}(\tilde{x}_1^*\tilde{x}_2)^k$$

for all $k \geq 1$.

Clearly, for each $i \in \{1,2\}$, the operator $\tilde{x}_1^*\tilde{x}_2$ restricted to the closed linear span $\mathcal{H}_i \subset \ell^2(\mathbb{Z}_\geq)$ of $\{v_{i+2n} \colon n \geq 0\}$ is a unilateral shift $\mathcal{S}$, while the orthogonal projection onto $\mathbb{C}v_i$ is $p_i \in C^*(\{\tilde{x}_1^*\tilde{x}_2, \tilde{x}_1^*\tilde{x}_1\})$.

Since $\tilde{x}_1^*\tilde{x}_2$ is a unilateral shift simultaneously on both $\mathcal{H}_1$ and $\mathcal{H}_2$, it generates a C^*-algebra $C^*(\{\tilde{x}_1^*\tilde{x}_2\}|_{\mathcal{H}_1\oplus\mathcal{H}_2})$ containing two "synchronized" copies of the ideal of compact operators, i.e.,

$$C^*(\{\tilde{x}_1^*\tilde{x}_2\}|_{\mathcal{H}_1\oplus\mathcal{H}_2}) \supset \big\{T \oplus T \colon T \in \mathcal{K}\big(\ell^2(\mathbb{Z}_\geq)\big)\big\} \cong \mathcal{K}\big(\ell^2(\mathbb{Z}_\geq)\big),$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are identified with the same Hilbert space $\ell^2(\mathbb{Z}_\geq)$ in a canonical way, i.e., identifying v_{i+2n} for $i \in \{1,2\}$ with the canonical orthonormal basis vector $e_n \in \ell^2(\mathbb{Z}_\geq)$.

However our goal is to show that $C\big(\mathbb{C}P^1_{q,c}\big)$ or for now $C^*(\{\tilde{x}_1^*\tilde{x}_2, \tilde{x}_1^*\tilde{x}_1\}|_{\mathcal{H}_1\oplus\mathcal{H}_2})$ contains the direct sum $\mathcal{K}(\mathcal{H}_1) \oplus \mathcal{K}(\mathcal{H}_2)$ of all "non-synchronized" pairs of compact operators. This can be achieved by noticing that for any $k, m \in \mathbb{Z}_\geq$,

$$\varepsilon^{(1)}_{k,m} := (\tilde{x}_1^*\tilde{x}_2)^k p_1((\tilde{x}_1^*\tilde{x}_2)^*)^m|_{\mathcal{H}_1\oplus\mathcal{H}_2} \in C^*(\{\tilde{x}_1^*\tilde{x}_2, \tilde{x}_1^*\tilde{x}_1\}|_{\mathcal{H}_1\oplus\mathcal{H}_2})$$

is a typical matrix unit in $\mathcal{K}(\mathcal{H}_1) \oplus 0$ sending v_{1+2m} to v_{1+2k} while eliminating all other v_{i+2n} with $i+2n \neq 1+2m$, and we get $\mathcal{K}(\mathcal{H}_1) \oplus 0$ as the closure of the linear span of $\varepsilon^{(1)}_{k,m}$ with $k, m \in \mathbb{Z}_\geq$. Similarly the elements

$$\varepsilon^{(2)}_{k,m} := (\tilde{x}_1^*\tilde{x}_2)^k p_2((\tilde{x}_1^*\tilde{x}_2)^*)^m|_{\mathcal{H}_1\oplus\mathcal{H}_2} \in C^*(\{\tilde{x}_1^*\tilde{x}_2, \tilde{x}_1^*\tilde{x}_1\}|_{\mathcal{H}_1\oplus\mathcal{H}_2})$$

with $k, m \in \mathbb{Z}_{\geq}$ linearly span a dense subspace of $0 \oplus \mathcal{K}(\mathcal{H}_2)$. Thus

$$\mathcal{K}(\mathcal{H}_1) \oplus \mathcal{K}(\mathcal{H}_2) = (\mathcal{K}(\mathcal{H}_1) \oplus 0) + (0 \oplus \mathcal{K}(\mathcal{H}_2)) \subset C^*(\{\tilde{x}_1^* \tilde{x}_2, \tilde{x}_1^* \tilde{x}_1\}|_{\mathcal{H}_1 \oplus \mathcal{H}_2}).$$

Next we want to show that each v_k with $k \geq 1$ is an eigenvector of $x_1^* x_1$ or equivalently of $x_2^* x_2 = 1 + cq^{-2} - q^{-2} x_1^* x_1$, and hence each $\mathcal{H}_i$ is invariant under $x_1^* x_1$ and $x_2^* x_2$.

Proposition 1. *The isometry $\tilde{x}_1^* \tilde{x}_2$ intertwines the positive operators $x_1^* x_1$ and $(1+c) - x_2^* x_2$, i.e.,*

$$(x_1^* x_1)(\tilde{x}_1^* \tilde{x}_2) = (\tilde{x}_1^* \tilde{x}_2)(1 + c - x_2^* x_2).$$

Proof. A direct computation shows

$$\begin{aligned} (x_1^* x_1)(\tilde{x}_1^* \tilde{x}_2) &= x_1^* x_1 \underline{(x_1^* x_1)^{-1/2}} x_1^* \tilde{x}_2 = \underline{(x_1^* x_1)^{-1/2}} x_1^* x_1 x_1^* \tilde{x}_2 \\ &= \underline{(x_1^* x_1)^{-1/2}} x_1^* (1 + c - x_2 x_2^*) \tilde{x}_2 = \tilde{x}_1^* (1 + c - x_2 x_2^*) \tilde{x}_2 \\ &= (1+c) \tilde{x}_1^* \tilde{x}_2 - \tilde{x}_1^* x_2 x_2^* \tilde{x}_2 = (1+c) \tilde{x}_1^* \tilde{x}_2 - \tilde{x}_1^* x_2 x_2^* x_2 (x_2^* x_2)^{-1/2} \\ &= (1+c) \tilde{x}_1^* \tilde{x}_2 - \tilde{x}_1^* x_2 (x_2^* x_2)^{-1/2} x_2^* x_2 = (1+c) \tilde{x}_1^* \tilde{x}_2 - (\tilde{x}_1^* \tilde{x}_2)(x_2^* x_2) \\ &= (\tilde{x}_1^* \tilde{x}_2)(1 + c - x_2^* x_2). \end{aligned}$$ ■

Proposition 2. *The isometry $\tilde{x}_1^* \tilde{x}_2$ intertwines the (possibly degenerate) eigenspaces $E_\lambda(x_2^* x_2)$ and $E_{1+c-\lambda}(x_1^* x_1)$, where $E_\lambda(T) := \ker(\lambda - T)$ for linear operators T and $\lambda \in \mathbb{C}$. More precisely,*

$$(\tilde{x}_1^* \tilde{x}_2)(E_\lambda(x_2^* x_2)) \subset E_{1+c-\lambda}(x_1^* x_1),$$

and

$$(\tilde{x}_1^* \tilde{x}_2)^{-1}(E_{1+c-\lambda}(x_1^* x_1)) \subset (E_\lambda(x_2^* x_2)),$$

where $(\tilde{x}_1^* \tilde{x}_2)^{-1}(E_{1+c-\lambda}(x_1^* x_1))$ is the inverse image of $E_{1+c-\lambda}(x_1^* x_1)$ under (the non-surjective) $\tilde{x}_1^* \tilde{x}_2$.

Proof. The commutation relation

$$(x_1^* x_1)(\tilde{x}_1^* \tilde{x}_2) = (\tilde{x}_1^* \tilde{x}_2)((1+c) - x_2^* x_2)$$

implies that if $v \in E_\lambda(x_2^* x_2)$ then $(\tilde{x}_1^* \tilde{x}_2)(v) \in E_{1+c-\lambda}(x_1^* x_1)$, because

$$\begin{aligned} (x_1^* x_1)((\tilde{x}_1^* \tilde{x}_2)(v)) &= (\tilde{x}_1^* \tilde{x}_2)((1+c) - x_2^* x_2)(v) \\ &= (\tilde{x}_1^* \tilde{x}_2)((1+c) - \lambda) v = ((1+c) - \lambda)((\tilde{x}_1^* \tilde{x}_2)(v)). \end{aligned}$$

On the other hand, if $(\tilde{x}_1^* \tilde{x}_2)(v) \in E_{1+c-\lambda}(x_1^* x_1)$, then

$$\begin{aligned} ((1+c) - \lambda)((\tilde{x}_1^* \tilde{x}_2)(v)) &= (x_1^* x_1)((\tilde{x}_1^* \tilde{x}_2)(v)) = (\tilde{x}_1^* \tilde{x}_2)((1+c) - x_2^* x_2)(v) \\ &= (1+c)(\tilde{x}_1^* \tilde{x}_2)(v) - (\tilde{x}_1^* \tilde{x}_2)((x_2^* x_2)(v)), \end{aligned}$$

and hence $(\tilde{x}_1^* \tilde{x}_2)(\lambda v) = (\tilde{x}_1^* \tilde{x}_2)((x_2^* x_2)(v))$. Since $\tilde{x}_1^* \tilde{x}_2$ is injective, we get $\lambda v = (x_2^* x_2)(v)$, i.e., $v \in E_\lambda(x_2^* x_2)$. ■

Corollary 1. *If λ is an eigenvalue of $x_2^* x_2$, then $1 + c - \lambda$ is an eigenvalue of $x_1^* x_1$.*

Proof. If $E_\lambda(x_2^* x_2) \neq 0$ then $E_{1+c-\lambda}(x_1^* x_1) \supset (\tilde{x}_1^* \tilde{x}_2)(E_\lambda(x_2^* x_2)) \neq 0$ since $\tilde{x}_1^* \tilde{x}_2$ is injective. ■

The equality $x_1^*x_1 + q^2x_2^*x_2 = q^2 + c$ implies

$$E_\lambda(x_2^*x_2) = E_{q^2+c-q^2\lambda}(x_1^*x_1)$$

for any $\lambda \in \mathbb{R}$, or equivalently

$$E_\lambda(x_1^*x_1) = E_{q^{-2}(q^2+c-\lambda)}(x_2^*x_2).$$

Proposition 3. *The orthonormal vectors v_k, $k \in \mathbb{N}$, are eigenvectors of $x_1^*x_1$ (and of $x_2^*x_2 \equiv 1 + q^{-2}c - q^{-2}x_1^*x_1$), and hence each of $\mathcal{H}_1$ and $\mathcal{H}_2$ is invariant under all of the generators $\tilde{x}_1^*\tilde{x}_2$, $x_1^*x_1$, and $x_2^*x_2$ of $C(\mathbb{C}P^1_{q,c})$. More explicitly, $(x_1^*x_1)(v_k) = c_kv_k$ for all $k \in \mathbb{N}$, where c_k is defined recursively by*

$$c_{k+2} = c - q^{-2}c + q^{-2}c_k \qquad \text{for } k \in \mathbb{N}, \text{ with } c_2 = 1 + c \text{ and } c_1 = 0,$$

which can be rewritten as

$$c_{2n} = q^{-2(n-1)} + c \qquad \text{and} \qquad c_{2n+1} = \big(1 - q^{-2n}\big)c$$

for all $n \in \mathbb{Z}_\geq$.

Proof. We prove $(x_1^*x_1)(v_k) = c_kv_k$ and the formula $c_{k+2} = c - q^{-2}c + q^{-2}c_k$ inductively on k.

First $v_1 \in (\operatorname{range}(\tilde{x}_1^*))^\perp = \ker(\tilde{x}_1)$, so

$$(x_1^*x_1)(v_1) = \big((x_1^*x_1)^{1/2}x_1^*x_1\underline{(x_1^*x_1)^{-1/2}}\big)(v_1) = \big((x_1^*x_1)^{1/2}x_1^*\tilde{x}_1\big)(v_1) = 0.$$

Next since $v_2 \in \operatorname{range}(\tilde{x}_1^*(1 - \tilde{x}_2\tilde{x}_2^*)\tilde{x}_1)$, so $v_2 = \tilde{x}_1^*(w)$ for some unit vector

$$w \in \operatorname{range}(1 - \tilde{x}_2\tilde{x}_2^*) = \ker(\tilde{x}_2\tilde{x}_2^*) = \ker(\tilde{x}_2^*) = \ker(x_2^*)$$

and hence

$$\begin{aligned}(x_1^*x_1)(v_2) &= (x_1^*x_1)(\tilde{x}_1^*(w)) = (x_1^*x_1)\underline{(x_1^*x_1)^{-1/2}}x_1^*(w)\\ &= \underline{(x_1^*x_1)^{-1/2}}(x_1^*x_1)x_1^*(w) = \underline{(x_1^*x_1)^{-1/2}}x_1^*(1 + c - x_2x_2^*)(w)\\ &= \tilde{x}_1^*((1 + c)w - 0) = (1 + c)\tilde{x}_1^*(w) = (1 + c)v_2.\end{aligned}$$

Now assume that $(x_1^*x_1)(v_k) = c_kv_k$, i.e., $v_k \in E_{c_k}(x_1^*x_1)$, for $k \in \mathbb{N}$. Then

$$\begin{aligned}v_{k+2} &\equiv (\tilde{x}_1^*\tilde{x}_2)(v_k) \in (\tilde{x}_1^*\tilde{x}_2)(E_{c_k}(x_1^*x_1)) \equiv (\tilde{x}_1^*\tilde{x}_2)(E_{q^{-2}(q^2+c-c_k)}(x_2^*x_2))\\ &\subset E_{1+c-q^{-2}(q^2+c-c_k)}(x_1^*x_1) = E_{c-q^{-2}c+q^{-2}c_k}(x_1^*x_1),\end{aligned}$$

and hence $(x_1^*x_1)(v_{k+2}) = c_{k+2}v_{k+2}$ for $c_{k+2} := c - q^{-2}c + q^{-2}c_k$.

The recursive formula $c_{k+2} = c - q^{-2}c + q^{-2}c_k$ rewritten as $c_{k+2} - c = q^{-2}(c_k - c)$ immediately leads to $c_{i+2n} - c = q^{-2n}(c_i - c)$ and hence

$$c_{i+2n} = q^{-2n}(c_i - c) + c$$

for any $i \in \{1, 2\}$ and $n \in \mathbb{N}$. More explicitly, we have $c_{2n} = q^{-2(n-1)} + c$ and $c_{2n+1} = \big(1 - q^{-2n}\big)c$ for all $n \in \mathbb{N}$. ■

Corollary 2. *The element $x_1^*x_2 = (x_1^*x_1)^{1/2}\tilde{x}_1^*\tilde{x}_2(x_2^*x_2)^{1/2}$ is a weighted shift on $\mathcal{H}_1$ and $\mathcal{H}_2$ with respect to the orthonormal bases $\{v_{2n-1}\}_{n\geq1}$ and $\{v_{2n}\}_{n\geq1}$ respectively. More precisely,*

$$(x_1^*x_2)(v_k) = \sqrt{c_{k+2}}\sqrt{1 + q^{-2}c - q^{-2}c_k}\,v_{k+2}$$

for the constants c_k specified in the above proposition.

Proof. This is a simple consequence of $(x_1^* x_1)(v_k) = c_k v_k$ and

$$(x_2^* x_2)(v_k) \equiv \big(1 + q^{-2}c - q^{-2}x_1^* x_1\big)(v_k) = \big(1 + q^{-2}c - q^{-2}c_k\big)v_k. \qquad \blacksquare$$

With each of $\mathcal{H}_1$ and $\mathcal{H}_2$ invariant under the self-adjoint operators $x_i^* x_i$, it is clear that the orthogonal complement $\mathcal{H}_0 := (\mathcal{H}_1 \oplus \mathcal{H}_2)^\perp$ in $\ell^2(\mathbb{Z}_\geq)$ is also invariant under each $x_i^* x_i$. On the other hand, since we know the orthonormal vectors $v_1, v_2 \in (\mathrm{range}(\tilde{x}_1^* \tilde{x}_2))^\perp$ for the index-(-2) isometry $\tilde{x}_1^* \tilde{x}_2$, we get a Wold-von Neumann decomposition (Theorem 3.5.17 of [6]) for the isometry $\tilde{x}_1^* \tilde{x}_2$ as

$$\tilde{x}_1^* \tilde{x}_2 = \tilde{x}_1^* \tilde{x}_2|_{\mathcal{H}_0} \oplus \tilde{x}_1^* \tilde{x}_2|_{\mathcal{H}_1} \oplus \tilde{x}_1^* \tilde{x}_2|_{\mathcal{H}_2}$$

with

$$\mathcal{H}_0 \equiv \big(\mathrm{Span}\,\big(\big\{(\tilde{x}_1^* \tilde{x}_2)^k (v_i) \colon i \in \{1,2\} \text{ and } k \in \mathbb{Z}_\geq\big\}\big)\big)^\perp.$$

Here $(\tilde{x}_1^* \tilde{x}_2)|_{\mathcal{H}_0}$ is a unitary operator on $\mathcal{H}_0$ (if $\mathcal{H}_0 \neq 0$) since $\tilde{x}_1^* \tilde{x}_2|_{\mathcal{H}_0}$ is an index-0 isometry in view of $\tilde{x}_1^* \tilde{x}_2|_{\mathcal{H}_i}$ being an index-(-1) isometry for each $i \in \{1,2\}$.

It is not clear whether $\mathcal{H}_0$ is actually trivial or not, so we remark that any discussion involving $\mathcal{H}_0$ below is only needed and valid when $\mathcal{H}_0 \neq 0$.

We already know that $(\tilde{x}_1^* \tilde{x}_2)|_{\mathcal{H}_i}$ is a unilateral shift for each $i \in \{1,2\}$. So with respect to the decomposition

$$\ell^2(\mathbb{Z}_\geq) = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$$

into orthogonal subspaces, the generators $\tilde{x}_1^* \tilde{x}_2$, $x_1^* x_1$, $x_2^* x_2$ and hence all elements of $C\big(\mathbb{C}P^1_{q,c}\big)$ can be viewed as block diagonal operators. Then it is easy to see that Propositions 1 and 2 hold for the restrictions of $\tilde{x}_1^* \tilde{x}_2$, $x_1^* x_1$, $x_2^* x_2$ to each $\mathcal{H}_i$.

Lemma 2. *The spectrum* $\mathrm{Sp}(x_1^* x_1|_{\mathcal{H}_0})$ *of* $x_1^* x_1|_{\mathcal{H}_0}$ *is invariant under the function*

$$f_1\colon\ s \mapsto c - q^{-2}c + q^{-2}s \equiv c + q^{-2}(s - c)$$

and its inverse function. Similarly, the spectrum $\mathrm{Sp}(x_2^* x_2|_{\mathcal{H}_0})$ *of* $x_2^* x_2|_{\mathcal{H}_0}$ *is invariant under the function*

$$f_2\colon\ s \mapsto 1 - q^{-2} + q^{-2}s \equiv 1 + q^{-2}(s - 1)$$

and its inverse function.

Proof. By Proposition 1,

$$(x_1^* x_1|_{\mathcal{H}_0})(\tilde{x}_1^* \tilde{x}_2|_{\mathcal{H}_0}) = (\tilde{x}_1^* \tilde{x}_2|_{\mathcal{H}_0})(1 + c - x_2^* x_2|_{\mathcal{H}_0})$$

with $\tilde{x}_1^* \tilde{x}_2|_{\mathcal{H}_0}$ unitary, we get $x_1^* x_1|_{\mathcal{H}_0}$ and $1 + c - x_2^* x_2|_{\mathcal{H}_0}$ unitarily equivalent and hence

$$\mathrm{Sp}(x_1^* x_1|_{\mathcal{H}_0}) = \mathrm{Sp}(1 + c - x_2^* x_2|_{\mathcal{H}_0}) = 1 + c - \mathrm{Sp}(x_2^* x_2|_{\mathcal{H}_0}).$$

On the other hand, from $x_1^* x_1 + q^2 x_2^* x_2 = q^2 + c$, we have

$$\mathrm{Sp}(x_2^* x_2|_{\mathcal{H}_0}) = q^{-2}\big(q^2 + c - \mathrm{Sp}(x_1^* x_1|_{\mathcal{H}_0})\big) = 1 + q^{-2}c - q^{-2}\,\mathrm{Sp}(x_1^* x_1|_{\mathcal{H}_0}).$$

Hence

$$\mathrm{Sp}(x_1^* x_1|_{\mathcal{H}_0}) = 1 + c - \big(1 + q^{-2}c - q^{-2}\,\mathrm{Sp}(x_1^* x_1|_{\mathcal{H}_0})\big) = c - q^{-2}c + q^{-2}\,\mathrm{Sp}(x_1^* x_1|_{\mathcal{H}_0}),$$

which shows that under the invertible function $f_1\colon s\in\mathbb{R}\mapsto c-q^{-2}c+q^{-2}s\in\mathbb{R}$, the set $\mathrm{Sp}(x_1^*x_1|_{\mathcal{H}_0})\subset\mathbb{R}$ equals itself and hence the inverse function $(f_1)^{-1}$ maps $\mathrm{Sp}(x_1^*x_1|_{\mathcal{H}_0})$ onto itself too.

Since the invertible function $g\colon s\in\mathbb{R}\mapsto 1+c-s\in\mathbb{R}$ maps $\mathrm{Sp}(x_2^*x_2|_{\mathcal{H}_0})$ onto $\mathrm{Sp}(x_1^*x_1|_{\mathcal{H}_0})$, the conjugate $g^{-1}\circ f_1\circ g$ and its inverse function map $\mathrm{Sp}(x_2^*x_2|_{\mathcal{H}_0})$ onto itself, where

$$\begin{aligned}\big(g^{-1}\circ f_1\circ g\big)(s)&=1+c-f_1(1+c-s)\\&=1+c-\big(c-q^{-2}c+q^{-2}(1+c-s)\big)=1-q^{-2}+q^{-2}s.\end{aligned}\qquad\blacksquare$$

Note that $f_1(s)-c=q^{-2}(s-c)$ and $f_2(s)-1=q^{-2}(s-1)$ for all $s\in\mathbb{R}$ with $q>1$. So the only bounded backward f_1-orbit is the constant f_1-orbit $\{c\}$, and similarly the only bounded backward f_2-orbit is the constant f_2-orbit $\{1\}$, where by a backward f_i-orbit, we mean the set $\{(f_i)^{-n}(s)\colon n\in\mathbb{N}\}$ for a point $s\in\mathbb{R}$. On the other hand, any forward f_i-orbit converges to the constant f_i-orbit, i.e., $\lim_{n\to\infty}(f_1)^n(s)=c$ and $\lim_{n\to\infty}(f_2)^n(s)=1$ for any s. Since each spectrum $\mathrm{Sp}(x_i^*x_i|_{\mathcal{H}_0})$ is a compact and hence bounded set that is invariant under backward iterations of f_i, it can contain only the constant f_i-orbit. We have, therefore

Corollary 3. $\mathrm{Sp}(x_1^*x_1|_{\mathcal{H}_0})=\{c\}$ *and* $\mathrm{Sp}(x_2^*x_2|_{\mathcal{H}_0})=\{1\}$*, i.e.,* $x_1^*x_1|_{\mathcal{H}_0}=c\,\mathrm{id}$ *and* $x_2^*x_2|_{\mathcal{H}_0}=\mathrm{id}$.

Proposition 4. *There is a unital C^*-algebra isomorphism from $C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}$ to the pullback $\mathcal{T}\oplus_{C(\mathbb{T})}\mathcal{T}$ of two copies of $\mathcal{T}\xrightarrow{\sigma}C(\mathbb{T})$, sending $\tilde{x}_1^*\tilde{x}_2|_{\mathcal{H}_1\oplus\mathcal{H}_2}$ to $\mathcal{S}\oplus\mathcal{S}$. This isomorphism provides an exact sequence*

$$0\to\mathcal{K}(\mathcal{H}_1)\oplus\mathcal{K}(\mathcal{H}_2)\to C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}\cong\mathcal{T}\oplus_{C(\mathbb{T})}\mathcal{T}\xrightarrow{\sigma}C(\mathbb{T})\to 0$$

of C^*-algebras with $\sigma(\tilde{x}_1^*\tilde{x}_2|_{\mathcal{H}_1\oplus\mathcal{H}_2})=\mathrm{id}_{\mathbb{T}}$, $\sigma(x_1^*x_1|_{\mathcal{H}_1\oplus\mathcal{H}_2})=c$, and $\sigma(x_2^*x_2|_{\mathcal{H}_1\oplus\mathcal{H}_2})=1$.

Proof. We note that the eigenvalues c_k of $x_1^*x_1|_{\mathcal{H}_1\oplus\mathcal{H}_2}$ satisfying $c_{k+2}=c-q^{-2}c+q^{-2}c_k=f_1(c_k)$ form two forward f_1-orbits and hence $\lim_{k\to\infty}c_k=c$. This limit is also clear from the explicit formulae of c_{2n} and c_{2n+1} given in Proposition 3. Similarly, one can verify that the eigenvalues c'_k of $x_2^*x_2|_{\mathcal{H}_1\oplus\mathcal{H}_2}$ form two forward f_2-orbit and hence $\lim_{k\to\infty}c'_k=1$.

So $x_1^*x_1|_{\mathcal{H}_1\oplus\mathcal{H}_2}\equiv c\oplus c \bmod \mathcal{K}(\mathcal{H}_1)\oplus\mathcal{K}(\mathcal{H}_2)$ and $x_2^*x_2|_{\mathcal{H}_1\oplus\mathcal{H}_2}\equiv 1\oplus 1 \bmod \mathcal{K}(\mathcal{H}_1)\oplus\mathcal{K}(\mathcal{H}_2)$, while $\tilde{x}_1^*\tilde{x}_2|_{\mathcal{H}_1\oplus\mathcal{H}_2}=\mathcal{S}_{\mathcal{H}_1}\oplus\mathcal{S}_{\mathcal{H}_2}$ for copies $\mathcal{S}_{\mathcal{H}_i}$ of the unilateral shift.

It has been shown earlier that $\mathcal{K}(\mathcal{H}_1)\oplus\mathcal{K}(\mathcal{H}_2)\subset C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}$, so it is not hard to see that $C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}$ is the pullback of two copies of $\mathcal{T}\xrightarrow{\sigma}C(\mathbb{T})$. In fact, $\mathcal{S}_{\mathcal{H}_1}\oplus\mathcal{S}_{\mathcal{H}_2}\equiv\tilde{x}_1^*\tilde{x}_2|_{\mathcal{H}_1\oplus\mathcal{H}_2}$ generates $\{T\oplus T\colon T\in\mathcal{T}\}$ as a C^*-subalgebra of $C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}$ and hence

$$\tilde{x}_1^*\tilde{x}_2|_{\mathcal{H}_1\oplus\mathcal{H}_2}\in\{T\oplus T\colon T\in\mathcal{T}\}+(\mathcal{K}(\mathcal{H}_1)\oplus\mathcal{K}(\mathcal{H}_2))\subset C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}.$$

On the other hand,

$$x_1^*x_1|_{\mathcal{H}_1\oplus\mathcal{H}_2}\in(c\oplus c)+(\mathcal{K}(\mathcal{H}_1)\oplus\mathcal{K}(\mathcal{H}_2))\subset\{T\oplus T\colon T\in\mathcal{T}\}+(\mathcal{K}(\mathcal{H}_1)\oplus\mathcal{K}(\mathcal{H}_2))$$

and hence

$$C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}\subset\{T\oplus T\colon T\in\mathcal{T}\}+(\mathcal{K}(\mathcal{H}_1)\oplus\mathcal{K}(\mathcal{H}_2)).$$

So we get

$$C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}=\{T\oplus T\colon T\in\mathcal{T}\}+(\mathcal{K}(\mathcal{H}_1)\oplus\mathcal{K}(\mathcal{H}_2))=\mathcal{T}\oplus_{C(\mathbb{T})}\mathcal{T},$$

where the second equality is due to that any $S\oplus T\in\mathcal{T}\oplus\mathcal{T}$ with $\sigma(S)=\sigma(T)$ can be written as

$$S\oplus T=(T\oplus T)+((S-T)\oplus 0)\in T\oplus T+(\mathcal{K}(\mathcal{H}_1)\oplus\mathcal{K}(\mathcal{H}_2)).$$

Replacing $\mathcal{T}\oplus_{C(\mathbb{T})}\mathcal{T}$ in the canonical exact sequence

$$0\to\mathcal{K}(\mathcal{H}_1)\oplus\mathcal{K}(\mathcal{H}_2)\to\mathcal{T}\oplus_{C(\mathbb{T})}\mathcal{T}\xrightarrow{\sigma}C(\mathbb{T})\to 0$$

by the isomorphic C^*-algebra $C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}$, we get the stated exact sequence with $\sigma(\tilde{x}_1^*\tilde{x}_2|_{\mathcal{H}_1\oplus\mathcal{H}_2})=\mathrm{id}_{\mathbb{T}}$, $\sigma(x_1^*x_1|_{\mathcal{H}_1\oplus\mathcal{H}_2})=c$, and $\sigma(x_2^*x_2|_{\mathcal{H}_1\oplus\mathcal{H}_2})=1$. ■

Theorem 2. *The restriction map*

$$T\in C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})\mapsto T|_{\mathcal{H}_1\oplus\mathcal{H}_2}\in C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}$$

is a C^*-algebra isomorphism, and hence $C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})$ is isomorphic to the pullback $\mathcal{T}\oplus_{C(\mathbb{T})}\mathcal{T}$ of two copies of $\mathcal{T}\xrightarrow{\sigma}C(\mathbb{T})$ with $\tilde{x}_1^*\tilde{x}_2$ corresponding to $\mathcal{S}\oplus\mathcal{S}$.

Proof. Clearly we only need to consider the case with $\mathcal{H}_0\neq 0$.

Since $\tilde{x}_1^*\tilde{x}_2|_{\mathcal{H}_0}$ is unitary, as shown in the above discussion of Wold–von Neumann decomposition, and $C(\mathbb{T})$ is the universal C^*-algebra generated by a single unitary generator, there is a unique C^*-algebra homomorphism

$$h\colon\ C(\mathbb{T})\to C^*(\{\tilde{x}_1^*\tilde{x}_2|_{\mathcal{H}_0}\})$$

sending $\mathrm{id}_{\mathbb{T}}$ to $\tilde{x}_1^*\tilde{x}_2|_{\mathcal{H}_0}$ while fixing all scalars in $\mathbb{C}\subset C(\mathbb{T})$.

Clearly with $x_1^*x_1|_{\mathcal{H}_0}=1$ and $x_2^*x_2|_{\mathcal{H}_0}=c$,

$$h\circ\sigma\colon\ C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}\to C^*(\{\tilde{x}_1^*\tilde{x}_2|_{\mathcal{H}_0}\})=C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_0}$$

is a well-defined C^*-algebra homomorphism sending $\tilde{x}_1^*\tilde{x}_2|_{\mathcal{H}_1\oplus\mathcal{H}_2}$ to $\tilde{x}_1^*\tilde{x}_2|_{\mathcal{H}_0}$ and $x_i^*x_i|_{\mathcal{H}_1\oplus\mathcal{H}_2}$ to $x_i^*x_i|_{\mathcal{H}_0}$ for $i\in\{1,2\}$. Hence the restriction map

$$T\in C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})\mapsto T|_{\mathcal{H}_1\oplus\mathcal{H}_2}\in C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}$$

gives a well-defined isomorphism. ■

In Theorem 2, we treat elements of $C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})$ as operators instead of families of operators by fixing implicitly the value of $\mathbb{T}$-parameter at any $t_1\in\mathbb{T}$, i.e., the statement of Theorem 2 is a pointwise result at any $t_1\in\mathbb{T}$. It is clear that collectively the restriction map

$$C\big(\mathbb{C}P^1_{q,c}\big)\to C\big(\mathbb{C}P^1_{q,c}\big)|_{\widetilde{\mathcal{H}_1}\oplus\widetilde{\mathcal{H}_2}}$$

is still a C^*-algebra isomorphism where elements of $C\big(\mathbb{C}P^1_{q,c}\big)$ are $\mathbb{T}$-families of operators on $\ell^2(\mathbb{Z}_{\geq})$ and $\widetilde{\mathcal{H}_1}\oplus\widetilde{\mathcal{H}_2}$ represents a $\mathbb{T}$-family of Hilbert subspaces $\mathcal{H}_1\oplus\mathcal{H}_2$ of $\ell^2(\mathbb{Z}_{\geq})$ constructed pointwise for each $t_1\in\mathbb{T}$ as described above.

4 Superfluous circle parameter

In this section, we show that the $\mathbb{T}$-parameter is superfluous for the C^*-algebra $C\big(\mathbb{C}P^1_{q,c}\big)|_{\widetilde{\mathcal{H}_1}\oplus\widetilde{\mathcal{H}_2}}$ consisting of $\mathbb{T}$-families of operators on $\mathcal{H}_1\oplus\mathcal{H}_2$, and hence $C\big(\mathbb{C}P^1_{q,c}\big)|_{\widetilde{\mathcal{H}_1}\oplus\widetilde{\mathcal{H}_2}}\cong C\big(\mathbb{C}P^1_{q,c}\big)$ is isomorphic to $C^*(\{\tilde{x}_1^*\tilde{x}_2,x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}\cong\mathcal{T}\oplus_{C(\mathbb{T})}\mathcal{T}$ (for any $t_1\in\mathbb{T}$ fixed) as obtained in Proposition 4.

Recall that by a simple change of orthonormal basis $e_k\rightsquigarrow t^ke_k$ of $\ell^2(\mathbb{Z}_{\geq})$ for any fixed $t\in\mathbb{T}$, the weighted shift operator α becomes $\tilde{\alpha}=t\alpha$ with respect to the new orthonormal basis, while the self-adjoint operator γ remains the same operator $\tilde{\gamma}=\gamma$.

Note that the earlier concrete description of $x_1^*x_2$, $x_1^*x_1$, and $x_2^*x_2$ as families of operators parametrized by $t_1 \in \mathbb{T}$ (with $t_2 = \overline{t_1}$) viewed as a representation of $C(\mathbb{C}P^1_{q,c}) \equiv C^*(\{x_1^*x_2, x_1^*x_1, x_2^*x_2\})$ can be first "consolidated" by a change of orthonormal basis converting α to $\tilde{\alpha} := t_1^2\alpha$ and γ to $\tilde{\gamma} = \gamma$, so that we can rewrite the description as

$$\begin{aligned}
x_1^*x_1 &= c + (1-c)\gamma^2 + \sqrt{c}\overline{t_1}^2\alpha^*\gamma + \sqrt{c}t_1^2\gamma\alpha \\
&= c + (1-c)\tilde{\gamma}^2 + \sqrt{c}\tilde{\alpha}^*\tilde{\gamma} + \sqrt{c}\tilde{\gamma}\tilde{\alpha}, \\
x_2^*x_2 &= 1 + q^{-2}(c-1)\gamma^2 - q^{-2}\sqrt{c}t_1^2\gamma\alpha - q^{-2}\sqrt{c}\overline{t_1}^2\alpha^*\gamma \\
&= 1 + q^{-2}(c-1)\tilde{\gamma}^2 - q^{-2}\sqrt{c}\tilde{\gamma}\tilde{\alpha} - q^{-2}\sqrt{c}\tilde{\alpha}^*\tilde{\gamma}, \\
x_1^*x_2 &= \sqrt{c}\overline{t_1}^2(\alpha^*)^2 - cq^{-1}\alpha^*\gamma + \gamma\alpha^* - q^{-1}\sqrt{c}t_1^2\gamma^2 \\
&= t_1^2\big(\sqrt{c}\overline{t_1}^4(\alpha^*)^2 - cq^{-1}\overline{t_1}^2\alpha^*\gamma + \overline{t_1}^2\gamma\alpha^* - q^{-1}\sqrt{c}\gamma^2\big) \\
&= t_1^2\big(\sqrt{c}(\tilde{\alpha}^*)^2 - cq^{-1}\tilde{\alpha}^*\tilde{\gamma} + \tilde{\gamma}\tilde{\alpha}^* - q^{-1}\sqrt{c}\tilde{\gamma}^2\big),
\end{aligned}$$

where it is understood that $\tilde{\alpha}, \tilde{\gamma}$ with respect to suitable orthonormal basis of $\ell^2(\mathbb{Z}_\geq)$ are the same familiar matrix operators α, γ, and hence we can simply replace $\tilde{\alpha}, \tilde{\gamma}$ by α, γ in the above formulas for $x_1^*x_2$, $x_1^*x_1$, and $x_2^*x_2$.

So we have

$$\begin{aligned}
x_1^*x_1 &= c + (1-c)\gamma^2 + \sqrt{c}\alpha^*\gamma + \sqrt{c}\gamma\alpha, \\
x_2^*x_2 &= 1 + q^{-2}(c-1)\gamma^2 - q^{-2}\sqrt{c}\gamma\alpha - q^{-2}\sqrt{c}\alpha^*\gamma, \\
x_1^*x_2 &= t_1^2\big(\sqrt{c}(\alpha^*)^2 - cq^{-1}\alpha^*\gamma + \gamma\alpha^* - q^{-1}\sqrt{c}\gamma^2\big),
\end{aligned}$$

where only $x_1^*x_2$ still involves $t_1 = \overline{t_2}$ as a factor. From Proposition 3 and Corollary 2, there is an orthonormal basis $\{v_{2k}, v_{2k-1} : k \in \mathbb{N}\}$ of $\mathcal{H}_1 \oplus \mathcal{H}_2$ consisting of eigenvectors of $x_1^*x_1$ and $x_2^*x_2$ and with respect to which $x_1^*x_2$ is a double weighted shift. So after the change of orthonormal basis $v_{2k} \rightsquigarrow (t_1^2)^k v_{2k}$ and $v_{2k-1} \rightsquigarrow (t_1^2)^k v_{2k-1}$, the factor t_1^2 in the formula of $x_1^*x_2$ can be dropped while the formulas of $x_1^*x_1$ and $x_2^*x_2$ remain the same, i.e., we have $C^*(\{x_1^*x_2, x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}$ for any fixed $t_1 \in \mathbb{T}$ unitarily equivalent to $C^*(\{x_1^*x_2, x_1^*x_1\})|_{\mathcal{H}_1\oplus\mathcal{H}_2}$ for $t_1 := 1$. So we conclude that the parameter $t_1 \in \mathbb{T}$ is "edundant" in the sense that representations of the generators $x_1^*x_2$, $x_1^*x_1$, and $x_2^*x_2$ of $C(\mathbb{C}P^1_{q,c})$ as operators on $\ell^2(\mathbb{Z}_\geq)$ by the above formulas for different t_1's in $\mathbb{T}$ are unitarily equivalent representations.

So we can now say that $C(\mathbb{C}P^1_{q,c}) \cong C^*(\{x_1^*x_2, x_1^*x_1\})$ where $C^*(\{x_1^*x_2, x_1^*x_1\})$ is considered as in the previous section for the operators $x_1^*x_2, x_1^*x_1$ without specifying any value of the t_1-parameter. Thus we conclude that $C(\mathbb{C}P^1_{q,c})$ is isomorphic to the pullback $\mathcal{T} \oplus_{C(\mathbb{T})} \mathcal{T}$ of two copies of $\mathcal{T} \xrightarrow{\sigma} C(\mathbb{T})$ by Proposition 4, and hence is isomorphic to the algebra $C(\mathbb{S}^2_{\mu c})$ of Podleś quantum 2-sphere by the result of [8].

We now summarize our conclusion in the following theorem, where the operators $X_1 := \sqrt{c}\alpha + \gamma$ and $X_2 := -q^{-1}\sqrt{c}\gamma + \alpha^*$ on $\ell^2(\mathbb{Z}_\geq)$ are respectively the values of the $\mathbb{T}$-families x_1 and x_2 at $t_1 = 1 = t_2$.

Theorem 3. *The C^*-algebra $C(\mathbb{C}P^1_{q,c}) \cong C^*(\{X_1^*X_2, X_1^*X_1\})$ for the linear operators $X_1 := \sqrt{c}\alpha + \gamma$ and $X_2 := -q^{-1}\sqrt{c}\gamma + \alpha^*$ on $\ell^2(\mathbb{Z}_\geq)$, and is isomorphic to the pullback*

$$\mathcal{T} \oplus_{C(\mathbb{T})} \mathcal{T} \equiv \big\{(T,S) \in \mathcal{T} \oplus \mathcal{T} : \sigma(T) = \sigma(S)\big\}$$

of two copies of the standard Toeplitz C^*-algebra $\mathcal{T}$ along the symbol map $\mathcal{T} \xrightarrow{\sigma} C(\mathbb{T})$.

We remark that the above change of orthonormal basis $v_{2k} \rightsquigarrow (t_1^2)^k v_{2k}$ and $v_{2k-1} \rightsquigarrow (t_1^2)^k v_{2k-1}$ is "compatible" and hence works well with the elements $x_1^*x_2$, $x_1^*x_1$, and $x_2^*x_2$ of $C(\mathbb{C}P^1_{q,c})$, but is not suitable for manipulating more fundamental elements like x_1 and x_2 in $C(\mathbb{S}^3_q) \equiv C(\mathrm{SU}_q(2))$.

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