Publ. I.R.M.A. Strasbourg, 1984, 229/S–08
Actes 8^{e} Séminaire Lotharingien, p. 87–89

A COMMON GENERALIZATION OF BINOMIAL COEFFICIENTS, STIRLING NUMBERS AND GAUSSIAN COEFFICIENTS

BY

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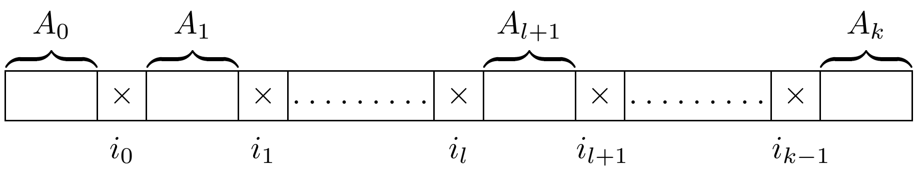
Let A_0, A_1, A_2, ... be finite sets. For nonnegative integers n and k denote by $S_k^n(a_0, a_1, a_2, \dots)$, where $a_i = |A_i|$, the number of words $w = (w_0, \dots, w_{n-1})$ such that

(1) w contains k labels, say at positions $i_0, \dots, i_{k-1}$,

(2) all entries in w before position i_0 belong to A_0, all entries in w between positions i_l and i_{l+1}, where $l = 0, \dots, k-2$, belong to A_{l+1}, all entries after position i_{k-1} belong to A_k.



As $S_n^k(\vec{a}) = \sum_{0 \le i_0 < i_1 < \cdots < i_{k-1} < n} a_0^{i_0} \cdot a_1^{i_1 - i_0 - 1} \cdots a_k^{n - i_{k-1} - 1}$, the numbers S_k^n can be obviously defined for sequences of complex numbers.

Examples.

(1) $S_n^k(1, 1, \dots) = \binom{n}{k}$ (Binomial coefficients)

(2) $S_k^n(0, 1, 2, \dots) = S_k^n$ (Stirling numbers of the second kind)

(3) $S_k^n(1, q, q^2, \dots) = \binom{n}{k}_q$ (Gaussian binomial coefficients)

(4) $S_k^n(q, q^2, q^3, \dots) =$ number of affine k-dimensional subspaces in the n-dimensional affine space over $GF(q)$.

(5) $S_k^n(2,3,4,\dots) =$ number of Boolean sublattices $P(k)$ in $P(n)$ ($P(n) \equiv$ lattice of subsets of an m-element set).

Pascal-identity : $S_k^{n+1}(\vec{a}) = S_{k-1}^n + a_k \cdot S_k^n$.

explicitly : $S_k^n(\vec{a}) = \sum_{i=0}^{k} a_i^n \cdot \prod_{j=0\ j\neq i} (a_i - a_j)^{-1}$, provided $a_0, \dots, a_k$ are mutually distinct.

Let $P_0^{\vec{a}}(x) = 1$ and $P_{k+1}^{\vec{a}}(x) = (x - a_k) \cdot P_k^{\vec{a}}(x)$, i.e.

$$P_k^{\vec{a}}(x) = (x - a_{k-1}) \cdots (x - a_0).$$

Inversion, resp. interpolation formulae :

$$x^n = \sum_{k\geq 0} S_k^n(\vec{a}) \cdot P_k^{\vec{a}}(x).$$

Now one can introduce the inverse numbers $s_k^n(\vec{a})$ by $\sum_j s_j^n(\vec{a}) \cdot S_k^j(\vec{a}) = \delta_k^n$ (Kronecker delta).

Recursion : $s_k^{n+1}(\vec{a}) = s_{k-1}^n - a_n \cdot s_k^n$.

These numbers can be used in order to describe inversion for arbitrary ascending sequences of normalized polynomials : let

$$S_k^n(\vec{a}, \vec{b}) = \sum_j s_j^n(\vec{a}) \cdot S_k^j(\vec{b}).$$

then

$$P_n^{\vec{a}}(\vec{x}) = \sum_k S_k^n(\vec{a}, \vec{v}) \cdot P_k^{\vec{b}}(x).$$

Recursion : $S_k^{n+1}(\vec{a}, \vec{b}) = S_{k-1}^n(\vec{a}, \vec{v}) + (b_k - a_k) \cdot S_k^n(\vec{a}, \vec{b})$.

THEOREM. — *Let $\vec{a}$ as before, l a complex number, then*

(i) $S_k^n(\vec{a}) = \sum_j \binom{n}{j} \cdot l^{n-j} \cdot S_k^j(\vec{a} - l)$,

(ii) $S_k^n(\vec{a}) = \sum_j \binom{j}{k} \cdot l^{j-k} \cdot S_j^n(\vec{a} - l)$,

(iii) $s_k^n(\vec{a}) = \sum_j \binom{n}{j} \cdot l^{n-j} \cdot s_k^j(\vec{a} + l)$,

(iv) $s_k^n(\vec{a}) = \sum_j \binom{j}{k} \cdot l^{j-k} \cdot s_j^n(\vec{a} + l)$.

Remark. — The case $\vec{a} = (1, q, q^2, \dots)$ and $l = 1$ is a theorem of Carlitz [1].

Corollary. — *The number of Boolean sublattices of $P(n)$ is equal to the $(n+2)$-nd Bell number B_{n+2}.*

REFERENCES

[1] Carlitz (L.). — On abelian fields, *Trans. Amer. Math. Soc.*, t. **35**, 1933, p. 122–126.

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