The Farey Graph

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This is joint work with David Singerman and Keith Wicks, subsequently published in [2]. The modular group

$$\Gamma = PSL(2, \mathbf{Z}) = SL(2, \mathbf{Z})/\{\pm I\}$$

acts on the upper half-plane $\mathcal{U} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$ and on the rational projective line $\hat{\mathbf{Q}} = \mathbf{Q} \cup \{\infty\}$ as a group of Möbius transformations

$$z \mapsto \frac{az+b}{cz+d} \qquad (a, b, c, d \in \mathbf{Z},\ ad - bc = 1)\,. \tag{$*$}$$

Its action on $\hat{\mathbf{Q}}$ is transitive but imprimitive: for each positive integer $N \neq 2, 5$ there is a Γ-invariant equivalence relation on $\hat{\mathbf{Q}}$ with N equivalence classes. We study the action of Γ on $\hat{\mathbf{Q}}$ by using suborbital graphs (introduced in 1967 by Sims [3] for finite permutation groups). These are Γ-invariant directed graphs with vertex-set $\hat{\mathbf{Q}}$, their edge-sets being the orbits of Γ on the cartesian square $\hat{\mathbf{Q}}^2$. Apart from the trivial case, corrresponding to the diagonal orbit, there is one suborbital graph $\mathcal{G}_{u,n}$ for each integer $n \geq 1$ and for each of the $\phi(n)$ units u mod (n): its edge-set is the orbit containing the pair $(\infty, u/n)$. Reversing edges induces a pairing of suborbital graphs, in which $\mathcal{G}_{u,n}$ is paired with $\mathcal{G}_{v,n}$ where $uv \equiv -1$ mod (n); thus $\mathcal{G}_{u,n}$ is self-paired (and can be represented as an undirected graph) if and only if $u^2 \equiv -1$ mod (n).

The simplest example is the *Farey graph* $\mathcal{F} = \mathcal{G}_{1,1}$: the vertex ∞ is joined to the integers, while two rational numbers r/s and x/y (in reduced form) are adjacent in $\mathcal{F}$ if and only if $ry - sx = \pm 1$, or equivalently if they are consecutive terms in some Farey sequence F_m (consisting of the rationals x/y with $|y| \leq m$, arranged in increasing order). If we draw the edges of $\mathcal{F}$ as hyperbolic geodesics in $\mathcal{U}$ (euclidean semicircles and half-lines), they do not cross, so we have an embedding of $\mathcal{F}$; the faces are hyperbolic triangles, giving a triangulation $\mathcal{T}$ of $\mathcal{U}$ with 'ideal vertices' on the boundary. Both $\mathcal{F}$ and $\mathcal{T}$ have automorphism group $PGL(2, \mathbf{Z})$, which contains Γ as its orientation-preserving subgroup of index 2. The triangulation $\mathcal{T}$ acts as a universal object for triangular maps, each of which is isomorphic to a quotient $\mathcal{T}/M$ for some subgroup M of $PGL(2, \mathbf{Z})$. It follows from Belyĭ's Theorem [1] that the Riemann surfaces defined as algebraic curves over the field $\overline{\mathbf{Q}}$ of algebraic numbers are those obtained in this way from compact orientable triangular maps, that is, they are the compactifications of the surfaces $\mathcal{U}/M$ where M has finite index in Γ.

The Farey graph $\mathcal{G}_{1,1}$ is connected, but if $n > 1$ then $\mathcal{G}_{u,n}$ is a disjoint union of

$$\psi(n) = n \prod_{p|n} \Big(1 + \frac{1}{p}\Big)$$

subgraphs (where p ranges over the distinct primes dividing n): their vertex-sets are the equivalence classes in $\hat{\mathbf{Q}}$ where we define $r/s \equiv_n x/y$ if and only if $ry - sx \equiv 0 \bmod (n)$. For a given n these subgraphs are permuted transitively by Γ, so they are all isomorphic to the subgraph $\mathcal{F}_{u,n}$ containing ∞, consisting of the vertices x/y with $y \equiv 0 \bmod(n)$. This subgraph is connected if and only if $n \leq 4$. Each $\mathcal{F}_{u,n}$ is embedded in $\mathcal{U}$ to give a tessellation $\mathcal{T}_{u,n}$: for instance $\mathcal{T}_{1,2}$ is the universal map [4], in the sense that every map is isomorphic to a quotient of $\mathcal{T}_{1,2}$ by some group of automorphisms.

$\mathcal{G}_{u,n}$ contains directed triangles if and only if $u^2 \pm u + 1 \equiv 0 \bmod(n)$, a typical example being $\infty \to u/n \to (u \pm 1)/n \to \infty$; however, only $\mathcal{G}_{1,1} = \mathcal{F}$ contains anti-directed triangles, such as $\infty \to 1 \leftarrow 2 \to \infty$. For $n > 1$, $\mathcal{G}_{u,n}$ is a forest if it is self-paired or if n is even. We conjecture that $\mathcal{G}_{u,n}$ is a forest if and only if it contains no triangles, that is, if and only if $u^2 \pm u + 1 \not\equiv 0 \bmod(n)$. (This conjecture has subsequently been proved by Mehmet Akbas.)

[1] G. V. Belyĭ, 'On Galois extensions of a maximal cyclotomic field', *Izv. Akad. Nauk SSSR* 43 (1979), 269–276 (Russian); *Math. USSR Izvestiya* 14 (1980), 247–256 (English transl.).

[2] G. A. Jones, D. Singerman and K. Wicks, 'The modular group and generalized Farey graphs', in *Groups, St. Andrews 1989, Vol. 2* (eds. C. M. Campbell and E. F. Robertson), London Math. Soc. Lecture Note Ser. 160 (1991), 316–338.

[3] C. C. Sims, 'Graphs and finite permutation groups' *Math. Z.* 95 (1967), 76–86.

[4] D. Singerman, 'Universal tessellations', *Revista Mat. Univ. Complutense Madrid* 1 (1988), 111–123.