INTRODUCTION TO ASSOCIATION SCHEMES

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Abstract
The present paper gives an introduction to the theory of association schemes, following Bose-Mesner (1959), Biggs (1974), Delsarte (1973), Bannai-Ito (1984) and Brouwer-Cohen-Neumaier (1989). Apart from definitions and many examples, also several proofs and some problems are included. The paragraphs have the following titles:

1. Introduction
2. Distance regular graphs
3. Minimal idempotents
4. $\mathcal{A}$-modules
5. Representations
6. Root lattices
7. Generalizations
8. References

§1. Introduction
An ordinary graph on n vertices (symmetric relation Γ on an n-set Ω) is described by its symmetric $n \times n$ adjacency matrix A. We paint the edges of the complete graph on n vertices in s colours:

$$J - I = A_1 + A_2 + ... + A_s \ ,$$

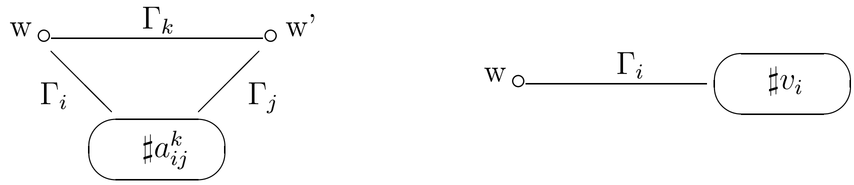
and require that the vector space

$$\mathcal{A} = \langle A_0 = I, A_1, A_2, ..., A_s \rangle_{\mathbb{R}}$$

is a *symmetric algebra* w.r.t. matrix multiplication, that is,

$$A_i A_j = A_j A_i = \sum_{k=0}^{s} a_{ij}^k A_k \ ; \qquad i,j = 0,1,...,s \ .$$

We call this algebra the *Bose-Mesner algebra* of the *s-association scheme* (Ω,{id,Γ_1, Γ_2, ..., Γ_s}), where colour i corresponds to relation (graph) Γ_i and adjacency matrix A_i. The *intersection numbers* a_{ij}^k and the *valencies* $v_i = a_{ii}^0$ have the following interpretation:

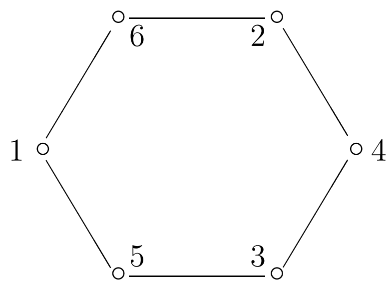


These notions go back to Bose and Mesner (1959).

Example 1.
A strongly regular graph is a 2-association scheme, where A_1 and A_2 denote the adjacency matrices of the graph and its complement.

In the next example we use the *distance* $\partial(u,v)$ of the vertices u and v of a *graph*, and the *relations* Γ_i, defined by $\{u,v\} \in \Gamma_i$ iff $\partial(u,v) = i$, for $i = 0,1,...,d =$ diameter.

Example 2.
The hexagon



gives rise to a 3-association scheme, since the distance i matrices A_i read:

$$A_1 = \begin{bmatrix} 0 & J-I \\ J-I & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} J-I & 0 \\ 0 & J-I \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} .$$

Problem.
Prove that the distance relations in the cube graph form a 3-association scheme. Determine the valencies and the intersection numbers.

Example 3 *Hamming scheme* $H(v, \mathbb{F}_2)$.
Consider $\Omega := (\mathbb{F}_2)^v$ with Hamming distance $\partial_H(x,y)$, that is, the number of coordinates in which x and $y \in \Omega$ differ. Denote by Γ_i the relation

$$\{x,y\} \in \Gamma_i \text{ iff } \partial_H(x,y) = i \ .$$

Then we have a v-association scheme with

$$n = 2^v\ , \quad v_i = \binom{v}{i}, \quad a_{ij}^k = \binom{k}{\frac{1}{2}(i-j+k)}\binom{v-k}{\frac{1}{2}(i+j-k)} \ .$$

Example 4 *Johnson scheme* $J(v,k)$.
Take Ω the set of all k-subsets of a v-set, and $\{w,w'\} \in \Gamma_i$ iff $|w \cap w'| = k - i$. Then

$$n = \binom{v}{k}, \quad v_i = \binom{k}{i}\binom{v-k}{i} \ .$$

In an association scheme $(\Omega, \{\Gamma_i\})$ we will be interested in *special subsets* $X \subset \Omega$, for instance:

– blue cliques X : only blue edges in X,

– blue cocliques X : no blue edges in X,

– code X at min. distance δ : no $\Gamma_1, \Gamma_2, ..., \Gamma_{\delta-1}$ in X,

– few-distance sets X in $\mathbb{R}^d$, etc., etc.

The problem then will be to find *bounds* for the cardinality $|X|$ of the special subsets $X \subset \Omega$, and to investigate the case of equality.

§2. Distance-regular graphs

In a graph $\Gamma = (\Omega, E)$ of diameter d we define:

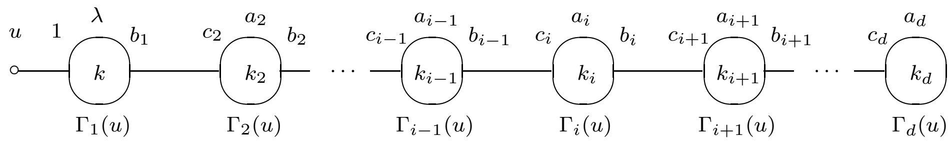
$$\text{distance } \partial(u,v) = \text{length of shortest path between } u, v \in \Omega\ ,$$

$$\Gamma_i(u) := \{x \in \Omega \ : \ \partial(x,u) = i\}\ , \quad |\Gamma_i(u)| =: k_i \ .$$

Definition.

A graph Γ is *distance regular* if for all $u \in \Omega$, for $i = 0, 1, 2, ..., d$,

$$\begin{aligned} \text{each } v \in \Gamma_i(u) \text{ has } c_i \text{ neighbours in } \Gamma_{i-1}(u) \ , \\ \text{has } b_i \text{ neighbours in } \Gamma_{i+1}(u) \ , \\ \text{has } a_i \text{ neighbours in } \Gamma_i(u) \ . \end{aligned}$$



Then

$$a_i + b_i + c_i = k \ , \quad k_{i+1}c_{i+1} = k_i b_i \ , \quad b_0 = k \ , \quad c_1 = 1 \ , \quad a_1 = \lambda \ .$$

So the independent parameters are

$$\{k = b_0, b_1, b_2, ..., b_{d-1} \, ; \, 1 = c_1, c_2, ..., c_d\} \ .$$

It is convenient to arrange the parameters into the $(d+1) \times (d+1)$ *tridiagonal* matrix T:

$$T \ := \begin{bmatrix} 0 & k & & & & & \\ c_1 & a_1 & b_1 & & & & \\ & c_2 & a_2 & b_2 & & & \\ & & c_3 & a_3 & \ddots & & \\ & & & c_4 & \ddots & & \\ & & & & \ddots & & \\ & & & & & \ddots & b_{d-1} \\ & & & & & c_d & a_d \end{bmatrix} .$$

The definition of distance-regularity translates in terms of the $n \times n$ *distance i matrices* A_i, which are defined by

$$A_i(x, y) = 1 \ \text{ if } \ \partial(x, y) = i \ , \quad = \ 0 \ \text{ otherwise.} \quad (\text{So } A_1 = A, \ A_0 = I.)$$

Theorem.

Γ is distance regular iff, for $1 \leq i \leq d-1$,

$$AA_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1} \ .$$

Proof.

$$(AA_i)(x,y) = \#\{z \in \Omega \ : \ \partial(x,z) = 1, \partial(y,z) = i\} \ .$$

There are such z only if $\partial(x,y) = i-1, i, i+1$, and their numbers are b_{i-1}, a_i, c_{i+1}, respectively. □

Corollary.
In a distance regular graph the distance i matrices A_i are polynomials p_i of degree i in the adjacency matrix A, for $i = 0, 1, ..., d$.

Proof. By recursive application of the theorem. □

Corollary.
For a distance regular graph of diameter d, the distance i relations constitute a d-association scheme.

Proof. Conversely to $A_i = p_i(A)$, $\deg p_i = i$, the powers $I, A, A^2, ..., A^d$ are linear combinations of $A_0, A_1, ..., A_d$. This implies that $\langle A_0 = I, A_1 = A, A_2, ..., A_d\rangle_{\mathbb{R}}$ is a Bose-Mesner algebra. □

Example.
The distance 1 relation in the Hamming scheme $H(d, \mathbb{F}_2)$ defines a distance regular graph. The vertices are the vectors of $\mathbb{F}_2^d$, two vertices being adjacent whenever they differ in one coordinate. Hence

$$k = d \, , \quad c_i = i \, , \quad b_i = d - i \, , \quad k_i = \left(\begin{array}{c} d \\ i \end{array} \right) \, .$$

Problem.
Find the parameters b_i and c_i for the distance regular graph formed by the d-subsets of an n-set, $n \geq 2d$, adjacency whenever two d-subsets differ in one element.

The tridiagonal matrix T, of size $d+1$, is useful for eigenvalues.

Lemma.
The eigenvalues of A are those of T (not counting multiplicities).

Proof. Let λ be an eigenvalue of A. Then $A_i = p_i(A)$ has the eigenvalue $p_i(\lambda)$.

The theorem implies

$$\lambda p_i(\lambda) = b_{i-1}p_{i-1}(\lambda) + a_i p_i(\lambda) + c_{i+1}p_{i+1}(\lambda) \ .$$

But this reads

$$T^t \underline{p}(\lambda) = \lambda \underline{p}(\lambda) \,, \quad \text{for } \underline{p}(\lambda) \ := \ \Big(p_0(\lambda), p_1(\lambda), ..., p_d(\lambda)\Big) \ .$$

and λ is an eigenvalue of T^t, hence of T. There are $d+1$ distinct eigenvalues of A, hence of T. □

Although T and T^t have the same eigenvalues, they do not have the same eigenvectors. We shall denote by $\underline{u}(\vartheta)$ the eigenvector of T corresponding to the eigenvalue ϑ:

$$T^t \underline{p}(\lambda) = \lambda \underline{p}(\lambda) \,; \quad T\underline{u}(\vartheta) = \vartheta \underline{u}(\vartheta) \,; \quad u_0 = p_0 = 1 \ ,$$

hence

$$c_i u_{i-1} + a_i u_i + b_i u_{i+1} = \vartheta u_i \ ; \qquad i = 1, ..., d-1 \ .$$

<u>Lemma</u>.

$$\Big(\underline{u}(\vartheta), \underline{p}(\lambda)\Big) = 0 \ , \qquad \text{for } \vartheta \neq \lambda \ .$$

<u>Proof</u>.

$$\vartheta\Big(\underline{u}(\vartheta), \underline{p}(\lambda)\Big) = \Big(T\underline{u}(\vartheta), \underline{p}(\lambda)\Big) = \Big(\underline{u}(\vartheta), T^t\underline{p}(\lambda)\Big) = \lambda\Big(\underline{u}(\vartheta), \underline{p}(\lambda)\Big) \ . \quad \square$$

<u>Theorem</u>.
Let the adjacency matrix A of a distance regular graph have the eigenvalue ϑ of multiplicity f. Let the tridiagonal T have eigenvector $\underline{u}(\vartheta)$. Then

$$L \ := \ \frac{f}{n}\,(I + u_1 A_1 + u_2 A_2 + ... + u_d A_d)$$

is an idempotent matrix of rank f.

<u>Proof</u>. If λ is any other eigenvalue of A, then the corresponding eigenvalue of L equals

$$\frac{f}{n} \sum_{i=0}^{d} u_i(\vartheta) p_i(\lambda) = \frac{f}{n} \Big(\underline{u}(\vartheta), \underline{p}(\lambda)\Big) = \delta_{\vartheta,\lambda} \ .$$

Indeed, the lemma gives 0 for $\lambda \neq \vartheta$. For $\lambda = \vartheta$ the corresponding eigenvalue of L, which also has multiplicity f, equals 1, since trace $L = f$. □

Remark.
The theory in this section goes back to Biggs (1974). By the present theorem a distance regular graph may be viewed as a set of vectors at equal length in $\mathbb{R}^f$, at cosines u_i. For certain classes of DRG this paves the way to characterization, by use of root lattices, cf. BCN (1989) and §6.

§3. Minimal idempotents

We return to the general case of an association scheme with Bose-Mesner algebra

$$\mathcal{A} = \langle A_0 = I, A_1, A_2, ..., A_s \rangle_{\mathbb{R}} \ .$$

The commuting A_i are simultaneously diagonalizable, hence there exists a basis of minimal orthogonal idempotents:

$$\mathcal{A} = \left\langle E_0 = \frac{1}{n}\, J, E_1, ..., E_s \right\rangle_{\mathbb{R}} \ .$$

Example.
$s = 2$, $\operatorname{spec} A = (k^1, r^f, s^g)$.

$$E_1 = \frac{1}{r-s}\left(A - sI - \frac{k-s}{n}\, J\right) \ , \quad \text{of rank } f \ ,$$

$$E_2 = \frac{1}{r-s}\left(rI - A + \frac{k-r}{n}\, J\right) \ , \quad \text{of rank } g \ .$$

The algebra $\mathcal{A}$ is closed with respect to matrix multiplication. It is also closed with respect to Schur (= entry-wise) multiplication with idempotents $A_0, A_1, ..., A_s$. We have:

$$
\begin{array}{ccc}
\textit{Matrix multiplication } \cdot & , & \textit{Schur multiplication } \circ \\
E_i E_j = \delta_{ij} E_i & , & A_i \circ A_j = \delta_{ij} A_i \\
A_i A_j = \sum_{k=0}^{s} a_{ij}^k A_k & , & E_i \circ E_j = \sum_{k=0}^{s} b_{ij}^k E_k \\
\textit{intersection numbers } a_{ij}^k \in \mathbb{N} & , & \textit{Krein parameters } b_{ij}^k \geq 0
\end{array}
$$

Transition between the two bases of $\mathcal{A}$:

$$
\begin{array}{ccc}
A_k = \sum_{i=1}^{s} p_{ik} E_i & , & E_i = \frac{1}{n} \sum_{k=0}^{s} q_{ki} A_k \\
A_k E_i = p_{ik} E_i & , & E_i \circ A_k = \frac{1}{n} q_{ki} A_k \\
\textit{valency } v_k = p_{ok} & , & \textit{multiplicity } f_i = q_{oi} \\
\Delta_v \ := \ \mathrm{diag}(v_k) & , & \Delta_f \ := \ \mathrm{diag}(f_i) \\
P = [p_{ik}], \text{ the character table} & , & Q \text{ from } PQ = nI = QP \, .
\end{array}
$$

Theorem.

$$\Delta_f P = Q^t \Delta_v \ .$$

Proof.

$$f_i p_{ik} = p_{ik} \operatorname{tr} E_i = \operatorname{tr} A_k E_i = \sum E_i \circ A_k = \frac{1}{n} q_{ki} \sum A_k = q_{ki} v_k \ ,$$

with trace $MN^t = \sum_{\text{elts}} M \circ N$. □

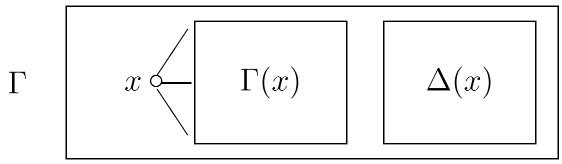
Problem.

Prove the *Krein inequalities* $b_{ij}^k \geq 0$, by considering $E_i \circ E_j$ and $E_i \otimes E_j$, and by using that, for fixed i, j, the matrix $E_i \circ E_j$ has the eigenvalues b_{ij}^k.

Remark.

For strongly regular graphs the vanishing of the Krein parameter b_{11}^1 allows the following combinatorial interpretation.

> Let Γ be a strongly regular graph having $b_{11}^1 = 0$. Then, for every vertex x, the subconstituents $\Gamma(x)$ and $\Delta(x)$ are both strongly regular.



Essentially, also the converse holds (under the assumption that Γ, $\Gamma(x)$, $\Delta(x)$ are strongly regular for some vertex x). Such graphs are called *Smith graphs.* For $r = 1, 2$ they are the following unique graphs, with order and eigenvalues (n, k, r, s):

$$\begin{array}{lllll} (16,5,1,-3) & , & (27,10,1,-5) & , & (100,22,2,-8) \quad , \\ (112,30,2,-10) & , & (162,56,2,-16) & , & (275,112,2,-28) \quad . \end{array}$$

The automorphism groups of these graphs are well-known groups, such as the 27 lines-group, the Higman-Sims group on 100, the McLaughlin group on 275 vertices, cf. BCN (1989).

Example.
Elimination of Q from $\Delta_f P = Q^t \Delta_v$, $PQ = QP = nI$ yields

$$P^t \Delta_f P = n\Delta_v \ , \quad \sum_{z=0}^{s} f_z p_{zk} p_{zl} = n v_k \delta_{k,l} \ .$$

In the case of distance regular graphs, the

$$p_{zi} \text{ are (degree } i\text{)-polynomials in } p_{z1} \qquad (0 \le i \le s) \ .$$

From the equations above it follows that the p_{zi} form a family of orthogonal polynomials with weights f_z. For the Hamming scheme $H(v, \mathbb{F}_2)$ these are the Krawchouk polynomials, for the Johnson scheme $J(v, l)$ the dual Hahn polynomials, cf. Delsarte (1973).

Remark.
Similarly, elimination of P leads to Q-polynomial association schemes, cf. the classification theorems in Bannai-Ito (1984).

§4. The $\mathcal{A}$-module V

Let $\mathcal{A}$ be the Bose-Mesner algebra of an association scheme on Ω. Consider the vector space

$$V = \mathbb{R}\Omega = \{x = \sum_{w \in \Omega} x(w) w\} = \{f \ : \ \Omega \to \mathbb{R}\} \ ,$$

provided with the inner product $(x,y) = \sum_{w \in \Omega} x(w)y(w)$.
$\mathcal{A}$ acts on V, with simultaneous eigenspaces

$$V = V_0 \perp V_1 \perp ... \perp V_s \ ; \qquad \pi_i \ : \ V \to V_i \ ;$$

$$A_k V_i = p_{ik} V_i \ , \qquad E_i = \text{Gram}\left\{ \pi_i w \ : \ w \in \Omega \right\} \ .$$

A subset $X = \{w_1, ..., w_m\} \subset \{w_1, ..., w_n\} = \Omega$ is represented by its characteristic vector

$$x = (111..100..0) \in \mathbb{R}\Omega \ .$$

Then $|X| = (x,x)$, $|X \cap Y| = (x,y)$, and the *average valency* of A_k over S is

$$a_k \ := \ \frac{(x, A_k x)}{(x,x)} \ , \qquad k = 0, 1, ..., s \ .$$

<u>Example</u>.
For a code X in the Hamming scheme: $a_1 = a_2 = ... = a_{\delta-1} = 0$.

<u>Theorem</u>.

$$\sum_{k=0}^{s} \frac{(x, A_k x)}{v_k} A_k = \sum_{i=0}^{s} \frac{(x, E_i x)}{f_i} nE_i \ .$$

<u>Proof</u>. Apply §3, then

$$\text{left} = \sum_{k,i,j} (x, E_i x) E_j p_{ik} p_{jk} / v_k = \sum_{k,i,j} (x, E_i x) E_j p_{ik} q_{kj} / f_j = \text{right} \ . \qquad \square$$

<u>Corollary</u>.

$$Q^t \underline{a} \geq 0 \ , \qquad \text{for } \underline{a} = (1, a_1, a_2, ..., a_s) \ .$$

<u>Proof</u>. Multiply the theorem by E_i, then

$$(x,x) \sum_{k=0}^{s} a_k q_{ki} = n(x, E_i x) \geq 0 \ .$$

<u>Remark</u>.
The constraints $Q^t \underline{a} \geq 0$, $\underline{a} \geq \underline{0}$, and $|X| = 1 + a_1 + a_2 + ... + a_s$, provide a setting for the application of *linear programming*, cf. Delsarte (1973).

A further application is the following *Code-Clique theorem.*

Let $T = \{1, 2, ..., t\} \subset S = \{1, 2, ..., s\}$.
$X \subset \Omega$ is called a T-*clique* if only T-relations in X,
$Y \subset \Omega$ is called a T-*code* if no T-relations in Y:
$(x, A_k x) = 0$ for $t < k \le s$; $(y, A_k y) = 0$ for $1 \le k \le t$.

Theorem.
$|X| \cdot |Y| \le |\Omega|$ and equality iff $|X \cap Y| = 1$.

Proof.

$$
\begin{aligned}
n(x,x)(y,y) &= n \sum_{k=0}^{s} (x, A_k x)(y, A_k y)/v_k = \\
&= n^2 \sum_{i=0}^{s} (x, E_i x)(y, E_i y)/f_i \ge n^2 (x, E_0 x)(y, E_0 y) = \\
&= |X|^2 |Y|^2 . \qquad \square
\end{aligned}
$$

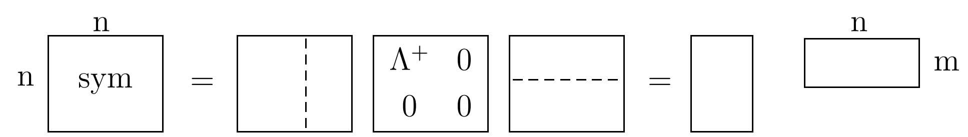
Problem.
Handle the case of equality.

§5. Representations

Combinatorial objects are represented as sets X of vectors in Euclidean space $\mathbb{R}^d$. The set X can be investigated by means of its Gram matrix. Another way is to confront $L(X)$ and $L(\mathbb{R}^d)$, where L denotes a linear space of certain test functions.

Theorem.
Any real symmetric semidefinite marix of rank m is the Gram matrix of n vectors in Euclidean space $\mathbb{R}^m$.

Proof. Use diagonalization of symmetric matrices:



As an example we consider a graph Γ on n vertices, say regular of valency k, whose adjacency matrix A has smallest eigenvalue s of multiplicity $n - d - 1$. From A the following matrix G is constructed:

$$AJ = kJ\,, \quad G \;:=\; c\Big(A - sI - \frac{k-s}{n}\,J\Big) = \begin{bmatrix} 1 & & \alpha/\beta \\ & \ddots & \\ \alpha/\beta & & 1 \end{bmatrix}\,.$$

Then G is symmetric, positive semidefinite of rank d, has constant diagonal (say 1) and two off-diagonal entries. By the theorem, G is the Gram matrix of a two-distance set X on the unit sphere S in Euclidean space $\mathbb{R}^d$. The following general geometric theorem has consequences for graph theory.

Theorem.
Any 2-distance set X on the unit sphere S in Euclidean $\mathbb{R}^d$ has cardinality at most $\frac{1}{2}\,d(d+3)$.

Proof. For any $y \in X$ we define the polynomial

$$F_y(\xi) \;:=\; \frac{\Big((y,\xi)-\alpha\Big)\Big((y,\xi)-\beta\Big)}{(1-\alpha)(1-\beta)}\;, \qquad \xi \in S\,.$$

The n polynomials in $\xi \in S$, thus obtained, have degree ≤ 2 and are independent, as a consequence of

$$F_y(x) = \delta_{y,x}\;; \qquad x, y \in X\,.$$

Therefore, their number n is at most the dimension of the space of all polynomials of degree ≤ 2 in d variables, restricted to S. This dimension equals $\frac{1}{2}\,d(d+1) + d + 0 =$
$\frac{1}{2}\,d(d+3)$. $\square$

Only three examples are known for the case of equality, viz.

$$(n,d) = (5,2)\,, \quad (27,6)\,, \quad (275,22)\,.$$

These 2-distance sets correspond to the pentagon graph, and the graphs of Schäfli, and McLaughlin, respectively. We illustrate the second case.

Example.
The 28 vectors $(3^2, (-1)^6)$ in 7-space span 28 lines which are equiangular at $\cos\varphi = 1/3$. Select a unit vector z along any line, then the 27 unit vectors along the other lines at $\cos\varphi = -1/3$ with z determine a 2-distance set in 6-space at

$$\cos\alpha = 1/4\,, \quad \cos\beta = -1/2\,.$$

Problem.
From the Johnson scheme $J(8,2)$ find the Schäfli graph on 27 vertices (which corresponds to the 2-distance set just constructed). Find the parameters of the Schäfli graph.

We now turn to representation in eigenspaces. Let the real symmetric $n \times n$ matrix A have an eigenvalue ϑ of multiplicity m, and a corresponding eigenmatrix U of size $n \times m$:

$$AU = \vartheta U \ , \quad U^t U = I_m \ , \quad UU^t = E \ .$$

Then the $n \times n$ matrix E is idempotent of rank m. The n row vectors $u_i \in \mathbb{R}^m$ of the matrix U have E as their Gram matrix. Now let A be the adjacency matrix of a graph $\Gamma = (V, A)$ on n vertices. Then U defines a representation of the graph in $\mathbb{R}^m$:

$$u \ : \ \Gamma \to \mathbb{R}^m \ : \ V \to U \ : \ i \mapsto u_i \ .$$

For distance regular graphs the inner products (u_i, u_j) are determined by the distance $\partial(i,j) \ =: \ r$, hence

$$(u_i, u_i) = \text{constant} \ , \quad w_r \ := \ \frac{(u_i, u_j)}{(u_i, u_i)} = \cos \varphi_{ij} \ .$$

The adjacencies imply

$$\vartheta u_i = \sum_{j \sim i} u_j \ , \quad \vartheta = \sum_{j \sim i} \frac{(u_i, u_j)}{(u_i, u_i)} = k w_1$$

and the first cosines are

$$w_0 = 1 \ , \quad w_1(\vartheta) = \vartheta / k \ , \quad w_2(\vartheta) = (\vartheta^2 - a_1 \vartheta - k)/k b_1 \ .$$

Theorem.
Let $m > 2$ denote the multiplicity of an eigenvalue of a distance regular graph. Then the valency k and the diameter d satisfy Godsil's bound

$$k \leq (m-1)(m+2)/2 \ , \qquad (d \leq 3m - 4) \ .$$

Proof. For any vector p of a distance regular graph let K denote the set of the neighbours of p. For any $i, j \in K$ their distance $\partial(i,j)$ equals 1 or 2, hence $u(K)$

is a 2-distance set of k vectors in $\mathbb{R}^{m-1}$. Now apply the bound above to obtain the inequality for k. $\square$

Problem.
Prove Godsil's diameter bound.

§6. Euclidean root lattices

A *lattice* is a free Abelian subgroup of rank d in Euclidean $\mathbb{R}^d$. The lattice is *integral* if the inner products of its vectors are integral, and *even* if its vectors have even norm (x,x). A *root* is a vector of norm 2. A *root lattice* is a lattice generated by roots. A root lattice is invariant under the *reflection* in the hyperplane perpendicular to any root r:

$$x \mapsto x - 2\frac{(x,r)}{(r,r)}\, r = x - (x,r)r \ .$$

The *Weyl group* of the root lattice is the group generated by the reflections on the roots.

Theorem (Witt).
The only irreducible root lattices in $\mathbb{R}^d$ are those of type A_d, D_d, E_6, E_7, E_8.

To explain the root systems of type D_d and E_8 (which contain the others: $A_d \subset D_{d+1}$; $E_6, E_7 \subset E_8$), we select an orthonormal basis $e_1, e_2, ..., e_d$ in $\mathbb{R}^d$.

$$D_d \ := \ \left\{ x \in \mathbb{R}^d \ : \ x_i \in \mathbb{Z}, \sum_1^d \ x_i \in 2\mathbb{Z} \right\} ;$$

the root system consists of the $2d(d-1)$ vectors $\pm\, e_i \ \pm\ e_j$ $(i \neq j)$, and is situated on $d(d-1)$ lines at 60° and 90° in $\mathbb{R}^d$.

$$E_8 \ := \ \left\langle D_8, \frac{1}{2}\,(e_1 + e_2 + ... + e_8) \right\rangle_{\mathbb{Z}} ;$$

the root system consists of the $240 = 112{+}128$ vectors $\pm e_i \pm e_j$ and $\frac{1}{2}\,(\pm\, e_1 \ \pm\ e_2 \ \pm\ ... \pm\, e_8)$, even number of minusses, on 120 lines at 60°, 90° in $\mathbb{R}^8$.

Witt's theorem plays a role in the proof of the following theorems, cf. CGSS (1976), Terw (1986), Neu (1985), BCN (1989).

Theorem.
All graphs having smallest eigenvalue -2 are represented in the root systems of types D_d and E_8.

Theorem.
The Hamming graphs $H(d,q)$ for $q \neq 4$, and the Johnson graphs $J(d,k)$ for $(d,k) \neq (8,2)$ are characterized by their parameters.

In order to illustrate this, we mention an ingredient used by Terwilliger:

$$E_1 = \frac{1}{n} \sum_{i=0}^{d} q_{i1} A_i = \sum_{i=0}^{d} (a - bi) A_i = \mathrm{Gram}(x, y, z, \ldots \in \mathbb{R}^f) \ .$$

Then

$$\left\langle \frac{1}{\sqrt{b}} (x - y) \ : \ (x,y) \in A_1 \right\rangle_{\mathbb{Z}} \quad \text{is a root lattice, etc.}$$

An ingredient used by Neumaier:

$$G = I + u_1 A_1 + \ldots + u_d A_d$$

is an idempotent matrix; for

$$\vartheta = k - \lambda - 2 \, , \quad u_i = \frac{k}{\lambda + 2} - i \ ,$$

this leads to root lattices, etc.

§7. Generalizations

We briefly indicate three recent developments which generalize the theory exposed in the present survey.

a. Coherent algebras, cf. Higman (1987).
These are subalgebras of the matrix algebra $M_n(\mathbb{C})$ which are closed under Schur multiplication, and contain J. No symmetry, commutativity, containment of I is presupposed. This leads to the earlier coherent configurations by the same author.

b. Association schemes on triples, cf. Mesner, Bhattacharya (1990).
The paper deals with partitions of $\Omega \times \Omega \times \Omega$ into $m+1$ relations R_i, and with 3-dimensional matrices satisfying

$$A_i A_j A_k = \sum_{l=0}^{m} p_{ijk}^{l} A_l \ .$$

Here the triple product $D = ABC$ is the $v \times v \times v$ matrix having the entries

$$D_{xyz} = \sum_{w \in \Omega} A_{wyz} B_{xwz} C_{xyw} \ .$$

c. Polynomial spaces, cf. Godsil (1988).

Ω :	$J(n,k)$	$S \subset \mathbb{R}^n$	$\mathrm{Sym}(n)$	$O(n)$
$\rho(x,y)$:	$\lvert x \cap y \rvert$	(x,y)	$\lvert \mathrm{fix}\, x^{-1} y \rvert$	$\mathrm{tr}(x^t y)$

The paper deals with a general set-up involving linear inner-product spaces of polynomials defined on a set Ω provided with a distance function $\rho \ : \ \Omega \times \Omega \to \mathbb{R}$. The axioms are:

$$\rho(x,y) = \rho(y,x) \,, \quad \dim \mathrm{Pol}(\Omega, 1) < \infty \ ,$$

and for the inner products:

$$\langle f, g \rangle = \langle 1, fg \rangle$$

and

$$\langle 1, f \rangle \geq 0 \ \text{ for } \ f \geq 0 \,, \quad = \ 0 \ \text{ iff } \ f = 0 \ .$$

The polynomials are defined in terms of *zonals* $\zeta_a(f)$, defined by

$$\Big(\zeta_a(f)\Big)(x) \ := \ f\Big(\rho(a,x)\Big) \ , \qquad x \in \Omega \ .$$

We refer to the original papers for further details.

§8. References

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