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SCHUR FUNCTIONS : THEME AND VARIATIONS

BY

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Introduction and theme

In this article we shall survey various generalizations, analogues and deformations of Schur functions — some old, some new — that have been proposed at various times. We shall present these as a sequence of variations on a theme and (unlike e.g. Bourbaki) we shall proceed from the particular to the general. Thus Variations 1 and 2 are included in Variation 3; Variations 4 and 5 are particular cases of Variation 6; and in their turn Variations 6, 7 and 8 (in part) are included in Variation 9.

To introduce our theme, we recall [M_1, Ch. I, § 3] that the *Schur function* $s_\lambda(x_1,\dots,x_n)$ (where $x_1,\dots,x_n$ are independent indeterminates and $\lambda=(\lambda_1,\dots,\lambda_n)$ is a partition of length $\le n$) may be defined as the quotient of two alternants :

$$s_\lambda(x_1,\dots,x_n)=\frac{\det\big(x_i^{\lambda_j+n-j}\big)_{1\le i,j\le n}}{\det\big(x_i^{n-j}\big)_{1\le i,j\le n}}. \tag{0.1}$$

The denominator on the right-hand side is the Vandermonde determinant, equal to the product $\prod_{i<j}(x_i-x_j)$.

When $\lambda=(r)$, s_λ is the *complete symmetric function* h_r, and when $\lambda=(1^r)$, s_λ is the *elementary symmetric function* e_r. In terms of the h's, the Schur function s_λ (in any number of variables) is given by the Jacobi-Trudi formula

$$s_\lambda=\det\big(h_{\lambda_i-i+j}\big)_{1\le i,j\le n}. \tag{0.2}$$

Dually, in terms of the elementary symmetric functions, s_λ is given by the Nägelsbach-Kostka formula

$$s_\lambda=\det\big(e_{\lambda'_i-i+j}\big)_{1\le i,j\le m} \tag{0.3}$$

in which $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ is the conjugate [M_1, Ch. I, § 1] of the partition λ.

There are (at least) two other determinantal formulas for s_λ : one in terms of "hooks" due to Giambelli, and the other in terms of "ribbons" discovered quite recently by Lascoux and Pragacz [LP_2]. If $\lambda = (\alpha_1, \ldots, \alpha_p \mid \beta_1, \ldots, \beta_p)$ in Frobenius notation [M_1, Ch. I, § 1], Giambelli's formula is

$$s_\lambda = \det\big(s_{(\alpha_i|\beta_j)}\big)_{1\le i,j\le p}. \tag{0.4}$$

To state the formula of Lascoux and Pragacz, let

$$\lambda^{(i,j)} = (\alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \alpha_p \mid \beta_1, \ldots, \widehat{\beta}_j, \ldots, \beta_p)$$

for $1 \le i, j \le p$, where the circumflexes indicate deletion of the symbols they cover; and let

$$[\alpha_i \mid \beta_j] = [\alpha_i \mid \beta_j]_\lambda = \lambda - \lambda^{(i,j)}.$$

In particular, $[\alpha_1 \mid \beta_1]$ is the *rim* or *border* of λ, and $[\alpha_i \mid \beta_j]$ is that part of the border consisting of the squares (h, k) such that $h \ge i$ and $k \ge j$. With this notation explained, the "ribbon formula" is

$$s_\lambda = \det\big(s_{[\alpha_i|\beta_j]}\big)_{1\le i,j\le p}. \tag{0.5}$$

Finally, we recall [M_1, Ch. I, § 5] the expression of a Schur function as a sum of monomials : namely

$$s_\lambda = \sum_T x^T \tag{0.6}$$

summed over all column-strict tableaux T of shape λ, where $x^T = \prod_{s\in\lambda} x_{T(s)}$. (Throughout this article, we shall find it convenient to think of a tableau T as a mapping from (the shape of) λ into the positive integers, so that $T(s)$ is the integer occupying the square $s \in \lambda$.)

All these formulas, with the exception of the original definition (0.1), have their extensions to skew Schur functions $s_{\lambda/\mu}$. In place of (0.2) we have

$$s_{\lambda/\mu} = \det\big(h_{\lambda_i-\mu_j-i+j}\big), \tag{0.7}$$

and in place of (0.3) we have

$$s_{\lambda/\mu} = \det\big(e_{\lambda'_i-\mu'_j-i+j}\big) \tag{0.8}$$

where λ', μ' are the partitions conjugate to λ, μ respectively. For the skew versions of (0.4) and (0.5) we refer to [LP$_1$], [LP$_2$]. Finally, in place of (0.6) we have

$$s_{\lambda/\mu} = \sum_T x^T \tag{0.9}$$

where now T runs over column-strict tableaux of shape $\lambda - \mu$ [M$_1$, Ch. I, § 5].

To complete this introduction we should mention the *Cauchy identity*

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y) \tag{0.10}$$

and its dual version

$$\prod_{i,j}(1 + x_i y_j) = \sum_\lambda s_\lambda(x) s_{\lambda'}(y) \tag{0.11}$$

where λ' is the conjugate of λ.

If we replace each y_j by y_j^{-1} and then multiply by a suitable power of $y_1 y_2 \ldots$, (0.11) takes the equivalent form (when the number of variables x_i, y_j is finite)

$$\prod_{\substack{1\le i\le n\\ 1\le j\le m}} (x_i + y_j) = \sum_\lambda s_\lambda(x) s_{\widehat{\lambda}'}(y) \tag{0.11$'$}$$

summed over partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \le m$, where $\widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n)$ is the complementary partition defined by $\widehat{\lambda}_i = m - \lambda_{n+1-i}$, and $\widehat{\lambda}'$ is the conjugate of $\widehat{\lambda}$.

The left-hand side of (0.10) may be regarded as defining a scalar product $\langle f, g\rangle$ on the ring of symmetric functions, as follows. For each $r \ge 1$ let p_r denote the r th power sum $\sum x_i^r$, and for each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ let p_λ denote the product $p_{\lambda_1} p_{\lambda_2} \ldots$ The p_λ form a $\mathbb{Q}$-basis of the ring of symmetric functions (in infinitely many variables, *cf.* [M$_1$, Ch. I]) with rational coefficients, and the scalar product may be defined by

$$\langle p_\lambda, p_\mu\rangle = \delta_{\lambda\mu} z_\lambda \tag{0.12}$$

where $\delta_{\lambda\mu}$ is the Kronecker delta, and

$$z_\lambda = \prod_{i\ge1} i^{m_i}.m_i!,$$

$m_i = m_i(\lambda)$ being the number of parts λ_j of λ equal to i, for each $i \geq 1$.

The Cauchy formula (0.10) is now equivalent to the statement that the Schur functions s_λ form an orthonormal basis of the ring of symmetric functions, i.e.,

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}. \tag{0.13}$$

Also, from this point of view, the skew Schur function $s_{\lambda/\mu}$ may be defined to be $s_\mu^\perp(s_\lambda)$, where $s_\mu^\perp$ is the adjoint of multiplication by s_μ, so that $\langle s_\mu^\perp f, g\rangle = \langle f, s_\mu g\rangle$ for any symmetric functions f, g.

1st Variation : Hall-Littlewood symmetric functions

Let $x_1, \ldots, x_n$, t be independent variables and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition of length $\leq n$. The *Hall-Littlewood symmetric function* indexed by λ [M_1, Ch. III] is defined by

$$P_\lambda(x_1, \ldots, x_n; t) = \frac{1}{v_\lambda(t)} \sum_{w \in \mathfrak{S}_n} w\Big(x_1^{\lambda_1} \ldots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\Big) \tag{1.1}$$

in which $v_\lambda(t) \in \mathbb{Z}[t]$ is a polynomial (with constant term equal to 1) chosen so that the leading monomial in P_λ is $x^\lambda = x_1^{\lambda_1} \ldots x_n^{\lambda_n}$. When $t = 0$, the right-hand side of (1.1) is just the expansion of the determinant $\det(x_i^{\lambda_j+n-j})$, divided by the Vandermonde determinant, so that when $t = 0$ the formula (1.1) reduces to the definition (0.1) of the Schur function.

None of determinantal formulas (0.2) – (0.5) have counterparts for the Hall-Littlewood functions (so far as I am aware). In place of (0.6) we have

$$P_\lambda(x; t) = \sum_T \psi_T(t) x^T \tag{1.2}$$

summed over column-strict tableaux T of shape λ, where $\psi_T(t) \in \mathbb{Z}[t]$ is a polynomial given explicitly in [M_1, Ch. III, § 5].

Finally, in place of the Cauchy identity (0.10) we have

$$\prod_{i,j} \frac{1 - tx_iy_j}{1 - x_iy_j} = \sum_\lambda b_\lambda(t) P_\lambda(x; t) P_\lambda(y; t). \tag{1.3}$$

As in the case of the Schur functions, this identity may be interpreted as saying that the symmetric functions $P_\lambda(x; t)$ are pairwise orthogonal with respect to the scalar product defined in terms of the power-sum products by

$$\langle p_\lambda, p_\mu \rangle_t = \delta_{\lambda\mu} z_\lambda \prod_{i \geq 1} (1 - t^{\lambda_i})^{-1}. \tag{1.4}$$

For more details, and in particular for the definition of the polynomials $b_\lambda(t)$ featuring in the right-hand side of (1.3), we refer to [M_1, Ch. III].

2nd Variation : Jack symmetric functions

These are symmetric functions $P_\lambda^{(\alpha)}(x)$ depending on a parameter α, but unlike the Hall-Littlewood functions (Variation 1) there is no closed formula such as (1.1) that can serve as definition. The simplest (and original) definition is the following : analogously to (0.12) and (1.4), we define a scalar product by

$$\langle p_\lambda, p_\mu \rangle^{(\alpha)} = \delta_{\lambda\mu} z_\lambda \alpha^{l(\lambda)} \tag{2.1}$$

where $l(\lambda)$ is the length of the partition λ, that is to say the number of non zero parts λ_i. For each positive integer n, arrange the partitions of n in lexicographical order (so that (1^n) comes first and (n) comes last). Then the $P_\lambda^{(\alpha)}(x)$ are uniquely determined by the two requirements

$$P_\lambda^{(\alpha)}(x) = x^\lambda + \text{lower terms} \tag{2.2}$$

where x^λ denotes the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \ldots$, and by "lower terms" is meant a sum of monomials x^β corresponding to sequences $\beta = (\beta_1, \beta_2, \ldots)$ that precede λ in the lexicographical order ; and

$$\langle P_\lambda^{(\alpha)}, P_\mu^{(\alpha)} \rangle^{(\alpha)} = 0 \qquad \text{if } \lambda \neq \mu. \tag{2.3}$$

The two conditions mean that the $P_\lambda^{(\alpha)}$ may be constructed from the monomial symmetric functions by the Gram-Schmidt process, starting (for partitions of n) with $P_{(1^n)} = e_n$, the n th elementary symmetric function.

Since the scalar product (2.1) reduces to (0.12) when $\alpha = 1$, it follows that $P_\lambda^{(\alpha)} = s_\lambda$ when $\alpha = 1$.

In view of the definition (2.1) of the scalar product, the orthogonality property (2.3) is equivalent to the following generalization of the Cauchy identity (0.10) :

$$\prod_{i,j} (1 - x_i y_j)^{-1/\alpha} = \sum_\lambda c_\lambda(\alpha) P_\lambda^{(\alpha)}(x) P_\lambda^{(\alpha)}(y) \tag{2.4}$$

where the $c_\lambda(\alpha)$ are rational functions of the parameter α which have been calculated explicitly by Stanley [S] — note, however, that his normalization of the Jack symmetric functions is different from ours.

As in the case of the Hall-Littlewood symmetric functions, none of the determinantal formulas (0.2) – (0.5) generalize, so far as is known, to the present situation. In place of (0.6) there is an explicit expression for $P_\lambda^{(\alpha)}(x)$ as a weighted sum of monomials, namely

$$P_\lambda^{(\alpha)}(x) = \sum_T f_T(\alpha) x^T \tag{2.5}$$

summed over column-strict tableaux T of shape λ, where $f_T(\alpha)$ is a rational function of α, computed explicitly by Stanley [S], to whom we refer for more details.

Finally, the dual Cauchy formula (0.11) generalizes as follows :

$$\prod_{i,j}(1+x_iy_j) = \sum_\lambda P_\lambda^{(\alpha)}(x)P_{\lambda'}^{(1/\alpha)}(y) \tag{2.6}$$

where as before λ' is the conjugate of λ.

3rd Variation

Our third variation is a family of symmetric functions $P_\lambda(x;q,t)$, indexed as usual by partitions λ, and depending on two parameters q and t. They include the two previous variations (the Hall-Littlewood symmetric functions and the Jack symmetric functions) as particular cases (see below). Since I have given an extended account of these symmetric functions at a previous Séminaire Lotharingien [M_2], I shall be brief here and refer to *loc. cit.* for all details. The functions may be most simply defined along the same lines as in Variation 2 : we define a new scalar product on the ring of symmetric functions by

$$\langle p_\lambda, p_\mu\rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i\geq 1} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \tag{3.1}$$

and then the symmetric functions $P_\lambda(x;q,t)$ are uniquely determined by the two requirements

$$P_\lambda(x;q,t) = x^\lambda + \text{lower terms}, \tag{3.2}$$

$$\langle P_\lambda, P_\mu\rangle_{q,t} = 0 \qquad \text{if } \lambda \neq \mu. \tag{3.3}$$

If we set $q = t^\alpha$ and then let $t \to 1$, in the limit the scalar product (3.1) becomes that defined in (2.1), from which it follows that the Jack symmetric function $P_\lambda^{(\alpha)}(x)$ is the limit of $P_\lambda(x;t^\alpha,t)$ as $t \to 1$. Again, if we set $q = 0$ the scalar product (3.1) reduces to (1.4), and it follows that $P_\lambda(x;0,t)$ is the Hall-Littlewood symmetric function $P_\lambda(x;t)$. Finally, if $q = t$ then (3.1) reduces to the original scalar product (0.12), and correspondingly $P_\lambda(x;q,q)$ is the Schur function $s_\lambda(x)$.

In view of the definition (3.1) of the scalar product, the orthogonality condition (3.3) is equivalent to the following extension of the Cauchy identity (0.10) :

$$\prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} = \sum_\lambda b_\lambda(q,t) P_\lambda(x;q,t) P_\lambda(y;q,t). \tag{3.4}$$

On the left-hand side of (3.4) we have used the standard notation

$$(x;q)_\infty = \prod_{i\geq 0}(1-xq^i).$$

On the right-hand side, $b_\lambda(q,t)$ is a rational function of q and t, given explicitly in [M$_2$, § 5].

As in the previous two variations, none of the determinantal formulas for Schur functions quoted in the introduction appear to generalize to the present situation. However, the formula (0.6) for s_λ as a sum of monomials does generalize : namely we have

$$P_\lambda(x;q,t) = \sum_T \varphi_T(q,t)x^T \tag{3.5}$$

where $\varphi_T(q,t)$ is a rational function of q and t, again given explicit expression in [M$_2$, § 5].

Finally, the dual Cauchy formula (0.11) generalizes as follows [M$_2$, § 5] :

$$\prod_{i,j}(1+x_iy_j) = \sum_\lambda P_\lambda(x;q,t)P_{\lambda'}(y;t,q). \tag{3.6}$$

4 th Variation : factorial Schur functions

Let $z = (z_1,\ldots,z_n)$ be a sequence of independent variables. For each pair of partitions λ, μ Biedenharn and Louck have defined a *skew factorial Schur function* $t_{\lambda/\mu}(z)$ in [BL$_1$]. Their original definition (*loc. cit.*) was couched in terms of Gelfand patterns, and in the equivalent language of tableaux it reads as follows. If $T:\lambda-\mu\to[1,n]$ is a column-strict tableau of shape $\lambda-\mu$, containing only the integers 1, 2, … , n, let

$$z^{(T)} = \prod_{s\in\lambda-\mu}\big(z_{T(s)} - T^*(s)+1\big), \tag{4.1}$$

where $T^*(i,j) = T(i,j)+j-i$ (so that T^* is a *row-strict* tableau of shape $\lambda-\mu$). Then $t_{\lambda/\mu}(z)$ is defined by

$$t_{\lambda/\mu}(z) = \sum_T z^{(T)} \tag{4.2}$$

summed over all column-strict tableaux $T:\lambda-\mu\to[1,n]$.

When $\mu=0$ they write t_λ in place of $t_{\lambda/0}$.

It is not particularly obvious from this definition that $t_{\lambda/\mu}(z)$ is in fact a (non-homogeneous) *symmetric* polynomial in $z_1, \dots, z_n$, and Biedenharn and Louck had some trouble (see [BL$_1$] pp. 407–412) in establishing this fact directly from their definition (4.2).

Some time ago I noticed that it followed rather simply from one of their results (Th. 5 of [BL$_2$]) that an alternative definition of $t_\lambda(z)$ could be given which brought out its analogy with the Schur function s_λ : namely (for $\lambda = (\lambda_1, \dots, \lambda_n)$ a partition of length $\leq n$)

$$t_\lambda(z) = \det\bigl(z_i^{(\lambda_j+n-j)}\bigr) \,/\, \det\bigl(z_i^{(n-j)}\bigr), \tag{4.3}$$

where $z^{(r)}$ is the "falling factorial"

$$z^{(r)} = z(z-1)\dots(z-r+1) \qquad (r \geq 0). \tag{4.4}$$

Note that since $z^{(r)}$ is a monic polynomial in z of degree r, the denominator in (4.3) is just the Vandermonde determinant :

$$\det\bigl(z_i^{(n-j)}\bigr) = \det\bigl(z_i^{n-j}\bigr) = \prod_{i<j}(z_i - z_j).$$

Hence t_λ as defined by (4.3) is the quotient of a skew-symmetric polynomial in $z_1, \dots, z_n$ by the Vandermonde determinant, and is therefore a (non-homogeneous) symmetric polynomial in the z_i. Moreover, it is clear from (4.3) that $t_\lambda(z)$ is of the form

$$t_\lambda(z) = s_\lambda(z) + \text{terms of lower degree},$$

and hence that the $t_\lambda(z)$, as λ runs through the partitions of length $\leq n$, form a $\mathbb{Z}$-basis of the ring Λ_n of symmetric polynomials in $z_1, \dots, z_n$.

In [CL], Chen & Louck show that t_λ (and more generally $t_{\lambda/\mu}$) satisfies a determinantal identity analogous to (0.2) and (0.7). Namely if

$$w_r(z) = t_{(r)}(z)$$

for all $r \geq 0$ (and $w_r(z) = 0$ when $r < 0$) then we have (*loc. cit.*, Th. 5.1)

$$t_{\lambda/\mu}(z) = \det\bigl(w_{\lambda_i-\mu_j-i+j}(z - \mu_j + j - 1)\bigr) \tag{4.5}$$

where in general $z + r$ denotes the sequence $(z_1 + r, \dots, z_n + r)$.

The other determinantal formulas quoted in the introduction all have their analogues for factorial Schur functions. If we define

$$f_r(z) = t_{(1^r)}(z) \qquad (0 \leq r \leq n)$$

(and $f_r(z) = 0$ for $r < 0$ and $r > n$), so that the f_r are the analogues of the elementary symmetric functions, then we have

$$t_{\lambda/\mu}(z) = \det\big(f_{\lambda'_i-\mu'_j-i+j}(z+\mu'_j-j+1)\big). \tag{4.6}$$

We shall not stop to prove (4.6) here, nor the hook and ribbon formulas

$$\begin{aligned} t_\lambda(z) &= \det\big(t_{(\alpha_i\,|\,\beta_j)}(z)\big)_{1\le i,j\le r} \\ &= \det\big(t_{[\alpha_i\,|\,\beta_j]}(z)\big)_{1\le i,j\le r} \end{aligned} \tag{4.7}$$

(where $\lambda = (\alpha_1,\dots,\alpha_r\,|\,\beta_1,\dots,\beta_r)$ in Frobenius notation, and for the explanation of the notation $[\alpha_i\,|\,\beta_j]$ we refer to (0.5)), since they are special cases of the corresponding results in Variation 6, which in their turn are contained in Variation 9. In this development we take (4.3) and (4.5) as definitions of t_λ and $t_{\lambda/\mu}$ respectively, and deduce (4.2) from them (see (6.16) below), very much in the spirit of [M_1], Chapter I, § 5.

5 th Variation : α-paired factorial Schur functions

Let $z = (z_1,\dots,z_n)$ again be a sequence of independent variables, and let α be another variable (or parameter). In parallel with the factorial Schur functions (Variation 4) Biedenharn and Louck [BL_1] have defined *α-paired factorial Schur functions* $T_{\lambda/\mu}(\alpha;z)$. As in the previous case, their definition was couched in terms of Gelfand patterns, and in the equivalent language of tableaux it reads as follows. Let

$$\overline{z}_i = -\alpha - z_i \qquad (1 \le i \le n)$$

and for each column-strict tableau $T : \lambda - \mu \to [1,n]$ let

$$(\alpha : z)^{(T)} = \prod_{s\in\lambda-\mu} \big(z_{T(s)} - T^*(s) + 1\big)\big(\overline{z}_{T(s)} - T^*(s) + 1\big) \tag{5.1}$$

where (as in § 4) T^* is the row-strict tableau associated with T (i.e., $T^*(i,j) = T(i,j) + j - i$). Then

$$T_{\lambda/\mu}(\alpha;z) = \sum_T (\alpha : z)^{(T)} \tag{5.2}$$

summed over all column-strict tableaux $T : \lambda - \mu \to [1,n]$. (When $\mu = 0$, they write T_λ in place of $T_{\lambda/0}$.)

Chen and Louck remark ([CL], p. 18) that "it is quite surprising that the α-paired factorial Schur function enjoys all the properties of the ordinary factorial Schur function." The reason for this, we believe, lies in the fact that both these classes of symmetric functions are special cases of those to be defined in our 6 th Variation. In the present situation the falling factorial $z^{(r)}$ is replaced by

$$z^{(r)}\,\overline{z}^{\,(r)} = \prod_{i=0}^{r-1}(z-i)(\overline{z}-i)$$

where $\overline{z} = -\alpha - z$; and since

$$(z-i)(\overline{z}-i) = z\overline{z} + \alpha i + i^2$$

it follows that we may write

$$z^{(r)}\,\overline{z}^{\,(r)} = \prod_{i=1}^{r}(x+a_i)$$

where $x = z\overline{z}$ and $a_i = \alpha(i-1) + (i-1)^2$. In Variation 6 below the building blocks are the products $(x\,|\,a)^r = \prod_{i=1}^{r}(x+a_i)$ defined by an *arbitrary* sequence $a_1, a_2, \ldots$

We may then take as an alternative definition of $T_\lambda(\alpha; z)$, where λ is a partition of length $\leq n$,

$$T_\lambda(\alpha; z) = \frac{\det\big(z_i^{(\lambda_j+n-j)}\,\overline{z}_i^{\,(\lambda_j+n-j)}\big)}{\det\big(z_i^{(n-j)}\,\overline{z}_i^{\,(n-j)}\big)} \tag{5.3}$$

([CL], Th. 6.2) ; all the determinantal formulas (Jacobi-Trudi etc.) together with the tableau definition (5.2) are consequences of (5.3), as we shall show in a more general context in the next section.

6 th Variation

Let R be any commutative ring and let $a = (a_n)_{n\in\mathbb{Z}}$ be any (doubly infinite) sequence of elements of R. For each $r \in \mathbb{Z}$ we define $\tau^r a$ to be the sequence whose nth term is a_{n+r} :

$$(\tau^r a)_n = a_{n+r}.$$

Let

$$(x\,|\,a)^r = (x+a_1)\ldots(x+a_r)$$

for each $r \geq 0$. Clearly we have

$$(x\,|\,a)^{r+s} = (x\,|\,a)^r\,(x\,|\,\tau^r a)^s \tag{6.1}$$

for all r, $s \geq 0$.

Now let $x = (x_1, \ldots, x_n)$ be a sequence of independent indeterminates over R, and for each $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^n$ define

$$(6.2) \qquad A_\alpha(x\,|\,a) = \det\big((x_i\,|\,a)^{\alpha_j}\big)_{1\le i,j\le n}.$$

In particular, when $\alpha = \delta = (n-1, n-2, \ldots, 1, 0)$, since $(x_i\,|\,a)^{n-j}$ is a monic polynomial in x_i of degree $(n-j)$, it follows that

$$(6.3) \qquad A_\delta(x\,|\,a) = \det\big(x_i^{n-j}\big) = \prod_{i<j}(x_i - x_j)$$

is the Vandermonde determinant $\Delta(x)$, independent of the sequence a. Since $A_\alpha(x\,|\,a)$ is a skew symmetric polynomial in $x_1, \ldots, x_n$, it is therefore divisible by $A_\delta(x\,|\,a)$ in $R[x_1, \ldots, x_n]$. Moreover, the determinant $A_\alpha(x\,|\,a)$ clearly vanishes if any two of the α_i are equal, and hence (up to sign) we may assume that $\alpha_1 > \cdots > \alpha_n \ge 0$, i.e., that $\alpha = \lambda + \delta$ where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of length $\le n$. It follows therefore that

$$(6.4) \qquad s_\lambda(x\,|\,a) = A_{\lambda+\delta}(x\,|\,a) \,\big/\, A_\delta(x\,|\,a)$$

is a symmetric (but not homogeneous) polynomial in $x_1, \ldots, x_n$ with coefficients in R. Moreover it is clear from the definitions that

$$A_{\lambda+\delta}(x\,|\,a) = a_{\lambda+\delta}(x) + \text{lower terms},$$

in the notation of [M$_1$], ch. I, and hence that

$$s_\lambda(x\,|\,a) = s_\lambda(x) + \text{terms of lower degree}.$$

Hence the $s_\lambda(x\,|\,a)$ form an R-basis of the ring $\Lambda_{n,R} = R[x_1, \ldots, x_n]^{\mathfrak{S}_n}$.

These polynomials $s_\lambda(x\,|\,a)$, and their skew analogues $s_{\lambda/\mu}(x\,|\,a)$ to be defined later, form our 6th Variation. They include Variations 4 and 5 as special cases : for Variation 4 we take $R = \mathbb{Z}$, $x_i = z_i$ and $a_n = 1 - n$ for all $n \in \mathbb{Z}$; for Variation 5 we take $R = \mathbb{Z}[\alpha]$, $x_i = z_i\overline{z}_i$ and $a_n = (n-1)\alpha + (n-1)^2$. The Schur functions themselves are given by the zero sequence : $a_n = 0$ for all $n \in \mathbb{Z}$. When $\lambda = (r)$ we shall write

$$h_r(x\,|\,a) = s_{(r)}(x\,|\,a) \qquad (r \ge 0)$$

with the usual convention that $h_r(x\,|\,a) = 0$ if $r < 0$; and when $\lambda = (1^r)$ $(0 \le r \le n)$ we shall write

$$e_r(x\,|\,a) = s_{(1^r)}(x\,|\,a) \qquad (0 \le r \le n)$$

with the convention that $e_r(x\,|\,a) = 0$ if $r < 0$ or $r > n$.

Let t be another indeterminate and let

$$f(t) = \prod_{i=1}^{n}(t - x_i).$$

From (6.3) it follows that

$$f(t) = A_{\delta_{n+1}}(t, x_1, \ldots, x_n \,|\, a) \,/\, A_{\delta_n}(x_1, \ldots, x_n \,|\, a).$$

By expanding the determinant $A_{\delta_{n+1}}$ along the top row we shall obtain

$$(6.5) \qquad f(t) = \sum_{r=0}^{n}(-1)^r e_r(x \,|\, a)(t \,|\, a)^{n-r}.$$

Let $\mathbf{E}(x \,|\, a)$, $\mathbf{H}(x \,|\, a)$ be the (infinite) matrices

$$\mathbf{H}(x \,|\, a) = \big(h_{j-i}(x \,|\, \tau^{i+1}a)\big)_{i,j \in \mathbb{Z}},$$
$$\mathbf{E}(x \,|\, a) = \big((-1)^{j-i} e_{j-i}(x \,|\, \tau^{j}a)\big)_{i,j \in \mathbb{Z}}.$$

Both are upper unitriangular, and they are related by

$$(6.6) \qquad \mathbf{E}(x \,|\, a) = \mathbf{H}(x \,|\, a)^{-1}.$$

Proof. — We have to show that

$$\sum_j (-1)^{k-j} e_{k-j}(x \,|\, \tau^k a)\, h_{j-i}(x \,|\, \tau^{i+1}a) = \delta_{ik}$$

for all i, k. This is clear if $i \geq k$, so we may assume $i < k$. Since $f(x_i) = 0$ it follows from (6.5) that

$$\sum_{r=0}^{n}(-1)^r e_r(x \,|\, a)\,(x_i \,|\, a)^{n-r} = 0$$

and hence, replacing a by $\tau^{s-1}a$ and multiplying by $(x_i \,|\, a)^{s-1}$, that

$$(1) \qquad \sum_{r=0}^{n}(-1)^r e_r(x \,|\, \tau^{s-1}a)\,(x_i \,|\, a)^{n-r+s-1} = 0$$

for all $s > 0$ and $1 \leq i \leq n$. Now it is clear, from expanding the determinant $A_{(m)+\delta}(x \,|\, a)$ down the first column, that $h_m(x \,|\, a)$ is of the form

$$(2) \qquad h_m(x \,|\, a) = \sum_{i=1}^{n}(x_i \,|\, a)^{m+n-1} u_i(x)$$

with coefficients $u_i(x)$ rational functions of $x_1, \ldots, x_n$ independent of m. (In fact, it is easily seen that $u_i(x) = 1/f'(x_i)$.)

From (1) and (2) it follows that

$$\sum_{r=0}^{n} (-1)^r e_r(x\,|\,\tau^{s-1}a)\, h_{s-r}(x\,|\,a) = 0$$

for each $s > 0$. Putting $s = k - i$ and replacing a by $\tau^{i+1}a$ we obtain

$$\sum_{i \le j \le k} (-1)^{k-j} e_{k-j}(x\,|\,\tau^k a)\, h_{j-i}(x\,|\,\tau^{i+1}a) = 0,$$

as required. ∎

Next, we have analogues of the Jacobi-Trudi and Nägelsbach-Kostka formulas (0.2), (0.3) :

(6.7) *If λ is a partition of length $\le n$, then*

$$\begin{aligned} s_\lambda(x\,|\,a) &= \det\big(h_{\lambda_i - i + j}(x \mid \tau^{1-j}a)\big) \\ &= \det\big(e_{\lambda'_i - i + j}(x\,|\,\tau^{j-1}a)\big). \end{aligned}$$

Proof. — Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. From equation (2) above we have

$$\begin{aligned} h_{\alpha_i - n + j}(x\,|\,\tau^{1-j}a) &= \sum_{k=1}^{n} (x_k\,|\,\tau^{1-j}a)^{\alpha_i + j - 1} u_k(x) \\ &= \sum_{k=1}^{n} (x_k\,|\,a)^{\alpha_i}\, (x_k\,|\,\tau^{1-j}a)^{j-1} u_k(x) \end{aligned}$$

by (6.1). This shows that the matrix $H_\alpha = \big(h_{\alpha_i - n + j}(x\,|\,\tau^{1-j}a)\big)_{i,j}$ is the product of the matrices $\big((x_k\,|\,a)^{\alpha_i}\big)_{i,k}$ and $B = \big((x_k\,|\,\tau^{1-j}a)^{j-1} u_k(x)\big)_{k,j}$. On taking determinants it follows that

$$\det(H_\alpha) = A_\alpha \det(B).$$

In particular, when $\alpha = \delta$, the matrix $H_\delta = \big(h_{j-i}(x\,|\,\tau^{1-j}a)\big)$ is unitriangular and hence has determinant equal to 1. It follows that $A_\delta \det(B) = 1$ and hence that

$$\det(H_\alpha) = A_\alpha(x\,|\,a) \,/\, A_\delta(x\,|\,a),$$

for all $\alpha \in \mathbb{N}^n$. Taking $\alpha = \lambda + \delta$, we obtain the first of the formulas (6.7). The second formula, involving the e's, is then deduced from it and (6.6), exactly as in the case of Schur functions ([M_1], ch. I, (2.9)). ∎

Remark. — A consequence of (6.7) is that the determinant

$$\det\big(h_{\lambda_i-i+j}(x\,|\,\tau^{1-j}a)\big),$$

which appears to involve not only $a_1, a_2, \ldots$ but also $a_0, a_{-1}, \ldots, a_{2-l(\lambda)}$, is in fact independent of the latter.

More generally, if λ and μ are partitions we define

$$s_{\lambda/\mu}(x\,|\,a) = \det\big(h_{\lambda_i-\mu_j-i+j}(x\,|\,\tau^{\mu_j-j+1}a)\big) \tag{6.8}$$

and then it follows as above from (6.6) that

$$s_{\lambda/\mu}(x\,|\,a) = \det\big(e_{\lambda'_i-\mu'_j-i+j}(x\,|\,\tau^{-\mu'_j+j-1}a)\big). \tag{6.9}$$

Moreover,

(6.10) $s_{\lambda/\mu}(x\,|\,a) = 0$ *unless* $0 \le \lambda'_i - \mu'_i \le n$ *for all* i.

The proof is the same as for Schur functions : [M$_1$] ch. I, § 5.

The hook and ribbon formulas (0.4), (0.5) remain valid in the present context : if $\lambda = (\alpha_1, \ldots, \alpha_p\,|\,\beta_1, \ldots, \beta_p)$ in Frobenius notation, then

$$\begin{aligned} s_\lambda(x\,|\,a) &= \det\big(s_{(\alpha_i\,|\,\beta_j)}(x\,|\,a)\big)_{1\le i,j\le p} \\ &= \det\big(s_{[\alpha_i\,|\,\beta_j]}(x\,|\,a)\big)_{1\le i,j\le p}. \end{aligned} \tag{6.11}$$

This will be considered in a more general context in § 9.

Let $y = (y_1, \ldots, y_m)$ be another set of indeterminates, and let (x, y) denote $(x_1, \ldots, x_n, y_1, \ldots, y_m)$. Then we have

(6.12) (i) $$\mathbf{E}(x, y\,|\,a) = \mathbf{E}(y\,|\,\tau^n a)\,\mathbf{E}(x\,|\,a),$$

(ii) $$\mathbf{H}(x, y\,|\,a) = \mathbf{H}(x\,|\,a)\,\mathbf{H}(y\,|\,\tau^n a).$$

Proof. — It is enough to prove (i), since (ii) then follows by taking inverses and invoking (6.6). From (6.5) we have

$$\begin{aligned} \sum_{i=0}^{m+n} (-1)^i e_i(x, y\,|\,a)(t\,|\,a)^{m+n-i} &= \prod_{i=1}^{n}(t - x_i)\prod_{j=1}^{m}(t - y_j) \\ &= \sum_{j=0}^{n}(-1)^j e_j(x\,|\,a)(t\,|\,a)^{n-j}\sum_{k=0}^{m}(-1)^k e_k(y\,|\,\tau^{n-j}a)(t\,|\,\tau^{n-j}a)^{m-k} \\ &= \sum_{j,k}(-1)^{j+k} e_j(x\,|\,a)\,e_k(y\,|\,\tau^{n-j}a)\,(t\,|\,a)^{m+n-j-k} \end{aligned}$$

by use of (6.1). Since the polynomials $(t\,|\,a)^r$, $r \geq 0$ are linearly independent, we may equate coefficients to obtain

$$e_i(x, y\,|\,a) = \sum_{j+k=i} e_j(x\,|\,a)\, e_k(y\,|\,\tau^{n-j}a).$$

With a change of notation this relation takes the form

$$(-1)^{k-i} e_{k-i}(x, y\,|\,\tau^k a) = \sum_j (-1)^{k-j} e_{k-j}(x\,|\,\tau^k a)\,(-1)^{j-i} e_{j-k}(y\,|\,\tau^{n+j} a)$$

which establishes (i). ∎

(6.13) *Let λ, μ be partitions. Then*

$$s_{\lambda/\mu}(x, y\,|\,a) = \sum_\nu s_{\nu/\mu}(x\,|\,a)\, s_{\lambda/\nu}(y\,|\,\tau^n a).$$

Proof. — Let $r \geq \max(l(\lambda), l(\mu))$. By definition (6.8), $s_{\lambda/\mu}(x, y\,|\,a)$ is the $r \times r$ minor of $\mathbf{H}(x, y\,|\,a)$ corresponding to the row indices $\mu_1 - 1, \ldots, \mu_r - r$ and the column indices $\lambda_1 - 1, \ldots, \lambda_r - r$, that is to say, it is the element of $\bigwedge^r \mathbf{H}(x, y\,|\,a)$ indexed by these sets of indices. The formula (6.13) now follows from (6.12) (ii) and the functoriality of exterior powers,* which together imply that $\bigwedge^r \mathbf{H}(x, y\,|\,a) = \bigwedge^r \mathbf{H}(x\,|\,a).\bigwedge^r \mathbf{H}(y\,|\,\tau^n a)$. ∎

By iterating (6.13) we obtain the following result. Let $x^{(i)}, \ldots, x^{(n)}$ be n sets of variables, where $x^{(i)} = (x_1^{(1)}, \ldots, x_{r_i}^{(i)})$, and let λ, μ be partitions. Then

$$(6.14) \qquad s_{\lambda/\mu}(x^{(i)}, \ldots, x^{(n)}\,|\,a) = \sum_{(\nu)} \prod_{i=1}^{n} s_{\nu^{(i)}/\nu^{(i-1)}}(x^{(i)}\,|\,\tau^{r_1+\cdots+r_{i-1}} a)$$

summed over all sequences $(\nu) = (\nu^{(0)}, \ldots, \nu^{(n)})$ of partitions, such that $\mu = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(n)} = \lambda$. ∎

We shall apply (6.14) in the case that each $x^{(i)}$ consists of a single variable x_i (so that $r_i = 1$ for $1 \leq i \leq n$). For a single x we have $s_{\lambda/\mu}(x\,|\,a) = 0$ unless $\lambda - \mu$ is a horizontal strip, by (6.10); and if $\lambda - \mu$ is a horizontal strip it follows from (6.8) that

$$\begin{aligned} s_{\lambda/\mu}(x\,|\,a) &= \prod_{i \geq 1} h_{\lambda_i - \mu_i}(x\,|\,\tau^{\mu_i - i + 1} a) \\ &= \prod_{i \geq 1} (x\,|\,\tau^{\mu_i - i + 1} a)^{\lambda_i - \mu_i}. \end{aligned}$$

* also known as the Cauchy-Binet identity.

since $h_r(x\,|\,a) = s_{(r)}(x\,|\,a) = (x\,|\,a)^r$ in the case of a single x, from the definition (6.4). Hence

(6.15) *For a single x we have*

$$s_{\lambda/\mu}(x\,|\,a) = \prod_{s\in\lambda-\mu} (x + a_{c(s)+1})$$

if $\lambda-\mu$ is a horizontal strip, and $s_{\lambda/\mu}(x\,|\,a) = 0$ otherwise.
(Here $c(s)$ is the *content* of s, i.e., $c(s) = j - i$ if $s = (i,j)$.)

From (6.14) and (6.15) it now follows that if $x = (x_1, \dots, x_n)$

$$s_{\lambda/\mu}(x\,|\,a) = \sum_T (x\,|\,a)^T \tag{6.16}$$

summed over column strict tableaux $T : \lambda - \mu \to [1, n]$, where

$$(x\,|\,a)^T = \prod_{s\in\lambda-\mu} \left(x_{T(s)} + a_{T^*(s)}\right)$$

and $T^*(i,j) = T(i,j) + j - i$ (so that T^* is row-strict).

When $a_i = 1 - i$ for all $i \in \mathbb{Z}$ (Variation 4), (6.16) reduces to the definition (4.2) of the factorial Schur functions.

Finally, there is an analogue of the dual Cauchy formula : namely (with the notation of (0.11′))

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(x_i + y_j) = \sum_\lambda s_\lambda(x\,|\,a)\, s_{\widehat{\lambda'}}(y\,|-a) \tag{6.17}$$

where $-a$ is the sequence $(-a_n)_{n\in\mathbb{Z}}$.

Proof. — Consider the quotient

$$A_{\delta_{m+n}}(x,y) \,/\, A_{\delta_n}(x)A_{\delta_m}(y)$$

which by (6.3) is equal to $\prod_{i,j}(x_i - y_j)$. On the other hand, Laplace expansion of the determinant $A_{\delta_{m+n}}(x,y)$ gives

$$A_{\delta_{m+n}}(x,y) = \sum_{\lambda\subset(m^n)} (-1)^{|\widehat{\lambda}|} A_{\lambda+\delta_n}(x)\, A_{\widehat{\lambda'}+\delta_m}(y).$$

Hence we have

$$\prod_{i,j}(x_i - y_j) = \sum_{\lambda\subset(m^n)} (-1)^{|\widehat{\lambda}|} s_\lambda(x\,|\,a)\, s_{\widehat{\lambda}'}(y\,|\,a)$$

and by replacing each y_j by $-y_j$ we obtain (6.17). ∎

Remark. — From the definition (6.1) it follows that

$$(x\,|\,a)^r = \sum_{k\geq 0} x^k e_{r-k}\big(a^{(r)}\big),$$

where $a^{(r)} = (a_1, a_2, \dots, a_r)$. Hence, with $x = (x_1, \dots, x_n)$,

$$\begin{aligned} A_\alpha(x\,|\,a) &= \det\Big(\sum_{\beta_k\geq 0} x_i^{\beta_k} e_{\beta_k-\alpha_j}\big(a^{(\alpha_j)}\big)\Big) \\ &= \sum_\beta \det\big(x_i^{\beta_k}\big) \det\Big(e_{\beta_k-\alpha_j}\big(a^{(\alpha_j)}\big)\Big) \end{aligned}$$

summed over $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ such that $\beta_1 > \beta_2 > \cdots > \beta_n$.

On dividing both sides by the Vandermonde determinant $\Delta(x)$ and replacing α, β by $\lambda + \delta$, $\mu + \delta$ respectively, we obtain

$$(6.18) \qquad s_\lambda(x\,|\,a) = \sum_{\mu\subset\lambda} s_\mu(x) \det\Big(e_{\lambda_i-\mu_j-i+j}\big(a^{(\lambda_j+n-j)}\big)\Big),$$

symmetric in the x's but not in the a's.

Now assume that the a's are independent variables; then we can let $n \to \infty$ (which would not have been possible in the contexts of Variations 4 and 5). In the limit the right-hand side of (6.18) becomes, by virtue of (0.8),

$$\sum_{\mu\subset\lambda} s_\mu(x) s_{\lambda'/\mu'}(a)$$

where $x = (x_1, x_2, \dots)$ and $a = (a_1, a_2, \dots)$. It follows that

$$(6.19) \qquad \lim_{n\to\infty} s_\lambda(x_1, \dots, x_n\,|\,a) = s_\lambda(x\,\|\,a),$$

where $s_\lambda(x\,\|\,a)$ is the "supersymmetric Schur function" defined by

$$s_\lambda(x\,\|\,a) = \det\big(h_{\lambda_i-i+j}(x\,\|\,a)\big)$$

in which $h_r(x\,\|\,a)$ is the coefficient of t^r in the power series expansion of $\prod_{i\geq 1}(1-tx_i)^{-1}\prod_{j\geq 1}(1+ta_j)$. Thus the limit as $n\to\infty$ of $s_\lambda(x_1,\dots,x_n\,|\,a)$ is symmetric in the a's as well as in the x's. From (6.19) and (6.16) we conclude that, with the notation of (6.16),

$$s_\lambda(x\,\|\,a)=\sum_T (x\,|\,a)^T \tag{6.20}$$

summed over all column-strict tableaux T of shape λ with positive integer entries.

For the skew functions the corresponding result reads as follows. Let $x=(x_n)_{n\in\mathbb{Z}}$, $a=(a_n)_{n\in\mathbb{Z}}$ now be two doubly infinite sequences of independent variables, and let λ, μ be partitions such that $\lambda\supset\mu$. The "skew supersymmetric Schur function" $s_{\lambda/\mu}(x\,\|\,a)$ is defined by

$$s_{\lambda/\mu}(x\,\|\,a)=\det\bigl(h_{\lambda_i-\mu_j-i+j}(x\,\|\,a)\bigr),$$

where $h_r(x\,\|\,a)$ is now the coefficient of t^r in the power series expansion of $\prod_{i\in\mathbb{Z}}(1-tx_i)^{-1}\prod_{j\in\mathbb{Z}}(1+ta_j)$. Then we have

$$s_{\lambda/\mu}(x\,\|\,a)=\sum_T (x\,|\,a)^T \tag{6.21}$$

summed over all column-strict tableaux $T:\lambda-\mu\to\mathbb{Z}$. (6.20) and (6.21) were found independently by Ian Goulden and Curtis Greene.

7 th Variation

Here we shall work over a finite field $F=\mathbb{F}_q$ of cardinality q (so that q is a prime power). Let $x_1,\dots,x_n$ be independent indeterminates over F, and let $V\subset F[x_1,\dots,x_n]$ denote the F-vector space spanned by the x_i, so that $F[x_1,\dots,x_n]$ is the symmetric algebra $\mathbf{S}(V)$ of V over F.

For each $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$ we define

$$A_\alpha=\det\bigl(x_i^{q^{\alpha_j}}\bigr)_{1\leq i,j\leq n}. \tag{7.1}$$

If $v\in V$, $v\neq 0$, so that

$$v=a_1x_1+\cdots+a_nx_n \tag{7.2}$$

with coefficients $a_i\in F$, not all zero, then we have

$$v^{q^r}=a_1x_1^{q^r}+\cdots+a_nx_n^{q^r}$$

for all integers $r\geq 0$, from which it follows that the determinant (7.1) is divisible by v in $\mathbf{S}(V)$. Hence if V_0 is the subset of V consisting of all the vectors (7.2) for which the first non zero coefficient a_i is equal to 1, we see that A_α is divisible in $\mathbf{S}(V)$ by the product

(7.3) $$P = P(x_1, \ldots, x_n) = \prod_{v \in V_0} v,$$

which is homogeneous of degree

$$\mathrm{Card}(V_0) = q^{n-1} + q^{n-2} + \cdots + 1.$$

In particular, when $\alpha = \delta_n = \delta = (n-1, n-2, \ldots, 1, 0)$, A_δ is divisible by P, and is a homogeneous polynomial of the same degree $q^{n-1} + q^{n-2} + \cdots + 1$; moreover the leading term in each of P and A_δ is the monomial $x_1^{q^{n-1}} x_2^{q^{n-2}} \ldots x_n$, and therefore

(7.4) $$P = A_\delta.$$

The determinant A_α clearly vanishes if any two of the α_i are equal, and hence (up to sign) we may assume that $\alpha_1 > \cdots > \alpha_n \geq 0$, i.e., that $\alpha = \lambda + \delta$ where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of length $\leq n$. It follows from what we have just proved that

(7.5) $$S_\lambda(x_1, \ldots, x_n) = A_{\lambda+\delta} \;/\; A_\delta$$

is a polynomial, i.e., an element of $\mathbf{S}(V)$, homogeneous of degree

$$\sum_{i=1}^{n} (q^{\lambda_i} - 1) q^{n-i}.$$

These polynomials S_λ (and their skew analogues $S_{\lambda/\mu}$ that we shall define later) constitute our 7 th Variation. Clearly they are symmetric in $x_1, \ldots, x_n$; but they are in fact invariant under a larger group, namely the group $GL_n(F)$ (or $GL(V)$).

For if $g = (g_{ij}) \in GL_n(F)$, we have

$$g x_i = \sum_{k=1}^{n} g_{ki} x_k$$

and therefore

$$(g x_i)^{q^r} = \sum_k g_{ki} x_k^{q^r}$$

for all integers $r \geq 0$, from which it follows that $g A_\alpha = (\det g) A_\alpha$ and hence that

$$S_\lambda(g x_1, \ldots, g x_n) = S_\lambda(x_1, \ldots, x_n).$$

Consequently $S_\lambda(x_1, \dots, x_n)$ depends only on (λ and) the vector space V, and not on the particular basis $x_1, \dots, x_n$ of V, and accordingly we shall write $S_\lambda(V)$ in place of $S_\lambda(x_1, \dots, x_n)$ from now on.

When $\lambda = (r)$ we shall write

$$H_r(V) = S_{(r)}(V) \qquad (r \geq 0)$$

with the usual convention that $H_r(V) = 0$ if $r < 0$; and when $\lambda = (1^r)$ $(0 \leq r \leq n)$ we shall write

$$E_r(V) = S_{(1^r)}(V) \qquad (0 \leq r \leq n)$$

with the convention that $E_r(V) = 0$ if $r < 0$ or $r > n$.

A well-known theorem of Dickson states that the subalgebra of $GL(V)$-invariant elements of $\mathbf{S}(V)$ is a polynomial algebra over F, generated by the $E_r(V)$ $(1 \leq r \leq n)$. But by contrast with the classical situation, the $S_\lambda(V)$ do not form an F-basis of $S(V)^{GL(V)}$, as one sees already in the simplest case $n = 1$.

Let t be another indeterminate and let

$$f_V(t) = \prod_{v \in V} (t + v). \tag{7.6}$$

From (7.3) and (7.4) it follows that

$$\begin{aligned} f_V(t) &= P(t, x_1, \dots, x_n)/P(x_1, \dots, x_n) \\ &= A_{\delta_{n+1}}(t, x_1, \dots, x_n)/A_{\delta_n}(x_1, \dots, x_n). \end{aligned}$$

By expanding the determinant $A_{\delta_{n+1}}$ along the top row, we shall obtain

$$f_V(t) = t^{q^n} - E_1(V)t^{q^{n-1}} + \cdots + (-1)^n E_n(V)t. \tag{7.7}$$

Since $(at + bu)^{q^r} = at^{q^r} + bu^{q^r}$ for all $a, b \in F$ and integers $r \geq 0$ (t, u being indeterminates) it follows from (7.7) that

$$f_V(at + bu) = af_V(t) + bf_V(u), \tag{7.8}$$

i.e., that f_V is an *additive* (or Ore) polynomial.

Let $\varphi : \mathbf{S}(V) \to \mathbf{S}(V)$ denote the Frobenius map, namely

$$\varphi(u) = u^q \qquad (u \in \mathbf{S}(V)).$$

The mapping φ is an F-algebra endomorphism of $\mathbf{S}(V)$, its image being $F[x_1^q, \ldots, x_n^q]$. Since we shall later encounter negative powers of φ, it is convenient to introduce

$$\widehat{\mathbf{S}}(V) = \bigcup_{r \geq 0} \mathbf{S}(V)^{q^{-r}}$$

where $\mathbf{S}(V)^{q^{-r}} = F[x_1^{q^{-r}}, \ldots, x_n^{q^{-r}}]$. On $\widehat{\mathbf{S}}(V)$, φ is an automorphism.

Let $\mathbf{E}(V)$, $\mathbf{H}(V)$ be the (infinite) matrices

$$\mathbf{H}(V) = \big(\varphi^{i+1} H_{j-i}(V)\big)_{i,j \in \mathbb{Z}},$$
$$\mathbf{E}(V) = \big((-1)^{j-i} \varphi^j E_{j-i}(V)\big)_{i,j \in \mathbb{Z}}.$$

Both are upper triangular, with 1's on the diagonal. They are related by

(7.9) $$\mathbf{E}(V) = \mathbf{H}(V)^{-1}.$$

Proof. — We have to show that

$$\sum_j (-1)^{k-j} \varphi^k (E_{k-j}) \varphi^{i+1} (H_{j-i}) = \delta_{ik}$$

for all i, k. This is clear if $i \geq k$. If $i < k$, we may argue as follows : since $f_V(x_i) = 0$ it follows from (7.7) that

$$\varphi^n(x_i) - E_1 \varphi^{n-1}(x_i) + \cdots + (-1)^n E_n x_i = 0$$

and hence that

(1) $$\varphi^{n+r-1}(x_i) - \varphi^{r-1}(E_1) \varphi^{n+r-2}(x_i) + \cdots + (-1)^n \varphi^{r-1}(E_n) \varphi^{r-1}(x_i) = 0$$

for all $r \geq 0$ and $1 \leq i \leq n$. On the other hand, by expanding the determinant $A_{(r)+\delta}$ down the first column, it is clear that $H_r = H_r(V)$ is of the form

(2) $$H_r = \sum_{i=1}^n u_i \varphi^{n+r-1}(x_i)$$

with coefficients $u_i \in F(x_1, \ldots, x_n)$ independent of r. From (1) and (2) it follows that

(3) $$H_r - \varphi^{r-1}(E_1)H_{r-1} + \cdots + (-1)^n \varphi^{r-1}(E_n)H_{r-n} = 0$$

for each $r \geq 0$. Putting $r = k - i$ and operating on (3) with φ^{i+1}, we obtain

$$\sum_{i \leq j \leq k} (-1)^{k-j} \varphi^k(E_{k-j})\varphi^{i+1}(H_{j-i}) = 0$$

as required. ∎

Next, we have analogues of the Jacobi-Trudi and Nägelsbach-Kostka formulas (0.2), (0.3) :

(7.10) *Let λ be a partition of length $\leq n = \dim V$. Then*

$$\begin{aligned} S_\lambda(V) &= \det\bigl(\varphi^{1-j} H_{\lambda_i - i + j}(V)\bigr) \\ &= \det\bigl(\varphi^{j-1} E_{\lambda'_i - i + j}(V)\bigr). \end{aligned}$$

Proof. — Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. From equation (2) above we have

$$\varphi^{1-j}(H_{\alpha_i - n + j}) = \sum_{k=1}^{n} \varphi^{\alpha_i}(x_k)\varphi^{1-j}(u_k) \qquad (1 \leq i, j \leq n)$$

which shows that the matrix $\bigl(\varphi^{1-j} H_{\alpha_i - n + j}\bigr)_{i,j}$ is the product of the matrices $\bigl(\varphi^{\alpha_i} x_k\bigr)_{i,k}$ and $\bigl(\varphi^{1-j} u_k\bigr)_{k,j}$. On taking determinants it follows that

(1) $$\det\bigl(\varphi^{1-j} H_{\alpha_i - n + j}\bigr) = A_\alpha B$$

where $B = \det\bigl(\varphi^{1-j} u_k\bigr)$.

In particular, taking $\alpha = \delta$ (so that $\alpha_i - n + j = j - i$), the left-hand side of (1) becomes equal to 1, so that $A_\delta B = 1$ and therefore

$$\det\bigl(\varphi^{1-j} H_{\alpha_i - n + j}\bigr) = A_\alpha \,/\, A_\delta$$

for all $\alpha \in \mathbb{N}^n$. Taking $\alpha = \lambda + \delta$, we obtain the first of the formulas (7.10). The second formula (involving the E's) is then deduced from it and (7.9), exactly as in the case of Schur functions ([M$_1$], Ch. I § 2). ∎

More generally, if λ and μ are partitions we define

(7.11) $$S_{\lambda/\mu}(V) = \det\bigl(\varphi^{\mu_j - j + 1} H_{\lambda_i - \mu_j - i + j}(V)\bigr)$$

and then it follows as above from (7.9) that also

$$S_{\lambda/\mu}(V) = \det\big(\varphi^{-\mu'_j+j-1} E_{\lambda'_i-\mu'_j-i+j}(V)\big). \tag{7.11$'$}$$

Moreover

(7.12) $S_{\lambda/\mu} = 0$ *unless* $0 \le \lambda'_i - \mu'_i \le n$ *for all* $i \ge 1$.

The proof is the same as for Schur functions ([M$_1$], Ch. I § 5).

The hook and ribbon formulas (0.4), (0.5) remain valid in the present context : if $\lambda = (\alpha_1, \dots, \alpha_p \,|\, \beta_1, \dots, \beta_p)$ in Frobenius notation, then

$$\begin{aligned} S_\lambda(V) &= \det\big(S_{(\alpha_i\,|\,\beta_j)}(V)\big)_{1\le i,j\le p} \\ &= \det\big(S_{[\alpha_i\,|\,\beta_j]}(V)\big)_{1\le i,j\le p} \end{aligned}$$

in the notation of (0.5). These identities are formal consequences of the definition (7.11) : see § 9 below.

Remark. — Since $H_r(V)$ has degree $(q^r-1)q^{n-1}$, it follows that the degree of $\varphi^{\mu_j-j+1}H_{\lambda_i-\mu_j-i+j}(V)$ is

$$q^{\mu_j-j+1}(q^{\lambda_i-\mu_j-i+j}-1)q^{n-1} = q^{\lambda_i+n-i} - q^{\mu_j+n-j}.$$

Hence the determinant (7.11) (and likewise (7.12)) is isobaric, and $S_{\lambda/\mu}(V)$ is homogeneous of degree

$$\sum_{i=1}^n (q^{\lambda_i} - q^{\mu_i})q^{n-i}. \tag{7.13}$$

We shall next consider analogues of the "addition formula"

$$s_{\lambda/\mu}(x,y) = \sum_\nu s_{\lambda/\mu}(x) s_{\nu/\mu}(y)$$

for Schur functions ([M$_1$], Ch. I § 5). For this purpose let U be an F-vector subspace of V. From (7.8) it follows that

$$f_U : v \mapsto f_U(v) = \prod_{u\in U}(v+u)$$

is an F-linear mapping of V into $\mathbf{S}(V)$ with kernel U. Hence $f_U(V)$ is isomorphic to the quotient of V by U, and we shall write

$$f_U(V) = V/U \tag{7.14}$$

thereby reserving the notation V/U for this particular embedding of the quotient space in $\mathbf{S}(V)$. If U' is a vector space complement of U in V, the elements of V/U are the products $f_U(v) = \prod_{u\in U}(v+u)$ as v runs through U', that is to say, they are the products in $\mathbf{S}(V)$ of the elements of the cosets of U in V.

With this notation we have

$$(7.15) \qquad f_V = f_{V/U} \circ f_U \qquad \big[i.e.,\ f_V(t) = f_{V/U}(f_U(t))\big].$$

Proof. — We have

$$\begin{aligned} f_{V/U}\big(f_U(t)\big) &= \prod_{w\in V/U} \big(f_U(t) + w\big) \\ &= \prod_{u'\in U'} \big(f_U(t) + f_U(u')\big) \\ &= \prod_{u'\in U'} f_U(t+u') && \text{[by (7.8)]} \\ &= \prod_{\substack{u\in U \\ u'\in U'}} (t+u+u') = f_V(t). \quad \square \end{aligned}$$

(7.16) *Let T be a vector subspace of U. Then*

$$V/U = (V/T)\big/(U/T)$$

(with equality, not merely isomorphism).

Proof. — By definition

$$\begin{aligned} (V/T)/(U/T) &= f_{U/T}(V/T) = f_{U/T}(f_T(V)) \\ &= f_U(V) = V/U. \quad \square && \text{[by (7.15)]} \end{aligned}$$

$$\begin{aligned} (7.17) \quad &\text{(i)} \quad \mathbf{E}(V) = \varphi^{\dim(V/U)}\big(\mathbf{E}(U)\big)\cdot \mathbf{E}(V/U), \\ &\text{(ii)} \quad \mathbf{H}(V) = \mathbf{H}(V/U)\cdot \varphi^{\dim(V/U)}\big(\mathbf{H}(U)\big). \end{aligned}$$

Proof. — It is enough to prove (i), since (ii) then follows by taking inverses and using (7.9). From (7.7) and (7.15) we have ($\dim V = n$, $\dim U = m$)

$$\sum_{i\geq 0}(-1)^i E_i(V)\varphi^{n-i}(t) = f_V(t) = f_{V/U}\big(f_U(t)\big)$$
$$= \sum_{j\geq 0}(-1)^j E_j(V/U)\varphi^{n-m-j}\Big(\sum_{k\geq 0}(-1)^k E_k(U)\varphi^{m-k}(t)\Big)$$
$$= \sum_{j,k\geq 0}(-1)^{j+k} E_j(V/U)\varphi^{n-m-j}\big(E_k(U)\big)\varphi^{n-j-k}(t)$$

and therefore

$$E_i(V) = \sum_{j+k=i} E_j(V/U)\varphi^{n-m-j}\big(E_k(U)\big).$$

With a change of notation this can be written in the form

$$E_{c-a}(V) = \sum_{a\leq b\leq c} \varphi^{n-m+b-c}\big(E_{b-a}(U)\big)E_{c-b}(V/U)$$

or equivalently

$$(-1)^{c-a}\varphi^c\big(E_{c-a}(V)\big)$$
$$= \sum_b (-1)^{b-a}\varphi^{n-m+b}\big(E_{b-a}(U)\big)(-1)^{c-b}\varphi^c\big(E_{c-b}(V/U)\big),$$

proving (i). ∎

(7.18) *Let λ, μ be partitions. Then*

$$S_{\lambda/\mu}(V) = \sum_\nu S_{\nu/\mu}(V/U)\cdot\varphi^{\dim(V/U)}\big(S_{\lambda/\mu}(U)\big).$$

Proof. — Suppose $r \geq \max\big(l(\lambda), l(\mu)\big)$. By definition (7.11), $S_{\lambda/\mu}(V)$ is the $r\times r$ minor of $\mathbf{H}(V)$ corresponding to the row indices $\mu_1 - 1, \ldots, \mu_r - r$ and the column indices $\lambda_1 - 1, \ldots, \lambda_r - r$, that is to say, it is the element of $\bigwedge^r \mathbf{H}(V)$ indexed by these sets of indices. The formula (7.18) now follows from (7.17) (ii) and the functoriality of exterior powers, which together imply that $\bigwedge^r \mathbf{H}(V) = \bigwedge^r \mathbf{H}(V/U)\cdot\varphi^{\dim(V/U)}\bigwedge^r \mathbf{H}(U)$. ∎

By iteration of (7.18) (and making use of (7.16)) we obtain

(7.19) *Let $V_0 > V_1 > \cdots > V_r$ be a chain of subspaces of V. Then*

$$S_{\lambda/\mu}(V_0/V_r) = \sum_{(\nu)}\prod_{i=1}^r \varphi^{\dim(V_0/V_{i-1})}S_{\nu^{(i)}/\nu^{(i-1)}}(V_{i-1}/V_i)$$

summed over all sequences $(\nu) = (\nu^{(0)}, \nu^{(1)}, \dots, \nu^{(r)})$ *of partitions such that* $\mu = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(r)} = \lambda$. □

As in the case of Schur functions, we shall use (7.19) to express $S_{\lambda/\mu}(V)$ as a sum over column-strict tableaux of shape $\lambda - \mu$.

If U is any finite-dimensional subspace of $F(V)$, let

$$\pi(U) = \prod_{\substack{u \in U \\ u \neq 0}} u.$$

From (7.7) it follows that

$$\pi(U) = (-1)^d E_d(U) \tag{7.20}$$

if $\dim U = d$.

If U' is a subspace of U, then U/U' is defined by (7.14) as a subspace of $\mathbf{S}(U) \subset \mathbf{S}(V)$, and its elements are the products (in $\mathbf{S}(U)$ or $\mathbf{S}(V)$) of the elements of the cosets of U' in U. From this it follows that

$$\pi(U/U') = \pi(U)/\pi(U') = \prod_{u \in U - U'} u. \tag{7.21}$$

We now consider the case where U is 1-dimensional.

(7.22) *Let* U *be a* 1*-dimensional subspace of* $\mathbf{S}(V)$ *and let* λ, μ *be partitions. Then*

$$S_{\lambda/\mu}(U) = (-1)^{|\lambda-\mu|} \prod_{s \in \lambda - \mu} \varphi^{c(s)} \pi(U)$$

if $\lambda - \mu$ *is a horizontal strip, and is zero otherwise.*
(Here $c(s)$ is the *content* of s : $c(s) = j - i$ if $s = (i, j)$.)

Proof. — Since $\dim U = 1$ we have $E_r(U) = 0$ for $r \geq 2$ and hence, by (7.10),

$$\begin{aligned} H_r(U) = S_{(r)}(U) &= \prod_{j=1}^{r} \varphi^{j-1} E_1(U) \\ &= (-1)^r \prod_{j=1}^{r} \varphi^{j-1} \pi(U) \end{aligned}$$

by (7.20). From (7.12), we have $S_{\lambda/\mu}(U) = 0$ unless $0 \leq \lambda'_i - \mu'_i \leq 1$ for all i, that is to say, unless $\lambda - \mu$ is a horizontal strip. If on the other hand

$\lambda - \mu$ is a horizontal strip, we have

$$\begin{aligned} S_{\lambda/\mu}(U) &= \det\big(\varphi^{\mu_j - j+1} H_{\lambda_i - \mu_j - i + j}(U)\big) \\ &= \prod_{i\geq 1} \varphi^{\mu_i - i+1} H_{\lambda_i - \mu_i}(U) \\ &= (-1)^{|\lambda-\mu|} \prod_{i\geq 1} \varphi^{\mu_i - i+1} \prod_{j=1}^{\lambda_i - \mu_i} \varphi^{j-1}\pi(U) \\ &= (-1)^{|\lambda-\mu|} \prod_{s\in\lambda-\mu} \varphi^{c(s)}\pi(U). \quad \square \end{aligned}$$

Now let

$$\mathfrak{V}: \quad V = V_0 > V_1 > \cdots > V_n = 0$$

be a (full) flag in V, so that V_i is a vector subspace of V of dimension $(n-i)$, for each i. Let

$$\pi_i(\mathfrak{V}) = \pi(V_{i-1}/V_i) = \prod_{v\in V_{i-1}-V_i} v$$

for $1 \leq i \leq n$.

From (7.19) and (7.22) it follows that

$$\begin{aligned} S_{\lambda/\mu} &= \sum_{(\nu)} \prod_{i=1}^{n} \varphi^{i-1} S_{\nu^{(i)}/\nu^{(i-1)}}(V_{i-1}/V_i) \\ &= (-1)^{|\lambda-\mu|} \sum_{(\nu)} \prod_{i=1}^{n} \prod_{s\in\nu^{(i)}/\nu^{(i-1)}} \varphi^{i-1+c(s)}\pi_i(\mathfrak{V}) \end{aligned}$$

summed over all sequences $(\nu) = (\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(n)})$ of partitions such that $\mu = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(n)} = \lambda$ and each $\nu^{(i)} - \nu^{(i-1)}$ is a horizontal strip. Such sequences are in one-one correspondence with column-strict tableaux $T : \lambda - \mu \to [1, n]$, and hence we obtain

$$S_{\lambda/\mu}(V) = (-1)^{|\lambda-\mu|} \sum_{T} \psi(T, \mathfrak{V}) \tag{7.23}$$

summed over column strict tableaux $T : \lambda - \mu \to [1, n]$, *where*

$$\psi(T, \mathfrak{V}) = \prod_{s\in\lambda-\mu} \varphi^{T^*(s)-1} \pi_{T(s)}(\mathfrak{V}),$$

and $T^*(i,j) = T(i,j) + j - i$ (as in (6.16)).

Remark. — The degree of $\psi(T,\mathfrak{V})$ is

$$\sum_{s\in\lambda-\mu} q^{c(s)+T(s)-1}(q^{n-T(s)+1}-q^{n-T(s)}) = \sum_{s\in\lambda-\mu} q^{c(s)}(q^n-q^{n-1})$$

which is easily seen to be equal to $\sum_{i=1}^{n}(q^{\lambda_i}-q^{\mu_i})q^{n-i}$, in agreement with (7.13).

In the formula (7.23) the flag $\mathfrak{V}$ is fixed and the sum is over tableaux T. Since $S_{\lambda/\mu}(V)$ is $GL(V)$-invariant, and since the number of flags $\mathfrak{V}$ in V is congruent to 1 modulo q, we may sum over all flags as well :

$$S_{\lambda/\mu}(V) = (-1)^{|\lambda-\mu|}\sum_{T,\mathfrak{V}}\psi(T,\mathfrak{V}), \tag{7.23$'$}$$

It seems plausible that (when $\mu=0$) there should be another expression for $S_\lambda(V)$ as a sum over flags, namely

$$S_\lambda(V) = (-1)^{|\lambda|}\sum_{\mathfrak{V}}\psi(T_0,\mathfrak{V}) \tag{7.24 ?}$$

where T_0 is the tableau defined by $T_0(i,j)=i$ for all $(i,j)\in\lambda$. For this tableau we have

$$\begin{aligned}\psi(T_0,\mathfrak{V}) &= \prod_{(i,j)\in\lambda}\psi^{j-1}\pi_i(\mathfrak{V})\\ &= (-1)^{|\lambda|}\prod_{i\geq 1}H_{\lambda_i}(V_{i-1}/V_i).\end{aligned}$$

The formula (7.24 ?) is true for example when $\lambda=(1^r)$ $(1\leq r\leq n)$, and in some other cases; but I do not know whether it is true generally. It would be enough to show that, if $l(\lambda)<n$,

$$S_\lambda(V) = \sum_L S_\lambda(V/L) \tag{7.25 ?}$$

summed over all lines (*i.e.*, 1-*dimensional subspaces*) L *in* V.

Finally, we shall indicate an analogue of the dual Cauchy formula (0.11$'$). Let V (resp. W) be the F-vector space spanned by $x_1, \ldots, x_n$ (resp. $y_1, \ldots, y_m$), the x's and y's being independent indeterminates over F. Let

$$\pi(V,W) = \prod_{L,M}\big(\pi(L)+\pi(M)\big)$$

where the product on the right is over all lines L in V and M in W. Then we have (with the notation of (0.11′))

(7.26) $$\pi(V,W)=\sum_{\lambda}S_{\lambda}(V)S_{\widehat{\lambda'}}(W).$$

Proof. — Consider the quotient

$$A_{\delta_{m+n}}(x,y)\;/\;A_{\delta_n}(x)A_{\delta_m}(y)$$

which by (7.4) is equal to the product

$$\prod_{\substack{v\in V_0\\ w\in W\\ w\neq 0}}(v+w)=\prod_{\substack{w\in V_0\\ w\in W_0}}(v^{q-1}-w^{q-1})$$

$$=\prod_{L,M}\big(\pi(L)-\pi(M)\big) \qquad (1)$$

(product over lines $L<V$ and $M<W$). On the other hand, by Laplace expansion of the determinant $A_{\delta_{m+n}}(x,y)$, we obtain

$$A_{\delta_{m+n}}(x,y)=\sum_{\lambda\subset(m^n)}(-1)^{|\widehat{\lambda}|}A_{\lambda+\delta_n}(x)A_{\widehat{\lambda'}+\delta_m}(y). \qquad (2)$$

From (1) and (2) it follows that

$$\prod_{L,M}\big(\pi(L)-\pi(M)\big)=\sum_{\lambda\subset(m^n)}(-1)^{|\widehat{\lambda}|}S_{\lambda}(V)S_{\widehat{\lambda'}}(W).$$

Finally, to get rid of the minus signs, replace each y_j by ωy_j, where ω lies in an extension field of F and satisfies $\omega^{q-1}=-1$. ∎

8 th Variation : flagged Schur functions

Let x_1, x_2, ... be independent variables. For all positive integers a, b, r define $h_r(a,b)$ (resp. $e_r(a,b)$) to be the complete (resp. elementary) symmetric function of degree r in the variables x_a, x_{a+1}, ... , x_b if $a\le b$, and to be zero if $a>b$; also define $h_0(a,b)=e_0(a,b)=1$ for all a,b, and $h_r(a,b)=e_r(a,b)=0$ for all a, b when $r<0$.

Let λ, μ be partitions of length $\le n$ and let $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)$ be sequences of positive integers. The *row-flagged Schur function* $s_{\lambda/\mu}(a,b)$ with row flags a, b is defined [W] to be

(8.1) $$s_{\lambda/\mu}(a,b)=\det\big(h_{\lambda_i-\mu_j-i+j}(a_j,b_i)\big)_{1\le i,j\le n}$$

It is zero unless $\lambda\supset\mu$, which we assume henceforth.

Let $m \geq \lambda_1$ (so that $\mu \subset \lambda \subset (m^n)$) and let $a' = (a'_1, \ldots, a'_m)$, $b' = (b'_1, \ldots, b'_m)$ be sequences of positive integers. The notion dual to (8.1) is that of the *column-flagged Schur function* $s'_{\lambda/\mu}(a', b')$* with column flags (a', b') :

$$s'_{\lambda/\mu}(a', b') = \det\bigl(e_{\lambda'_i - \mu'_j - i + j}(a'_j, b'_i)\bigr)_{1 \leq i,j \leq m} \tag{8.1$'$}$$

where λ', μ' are the partitions conjugate to λ, μ respectively.

With mild restrictions on the flags these Schur functions can be expressed as sums over tableaux :

(8.2) *Suppose that* $a_i \leq a_{i+1}$ *and* $b_i \leq b_{i+1}$ *whenever* $\mu_i < \lambda_{i+1}$ (i.e., the sequences a, b are increasing on each connected component of $\lambda - \mu$). *Then*

$$s_{\lambda/\mu}(a, b) = \sum_T x^T$$

summed over column-strict tableaux T *of shape* $\lambda - \mu$ *such that* $a_i \leq T(i, j) \leq b_i$ *for all* $(i, j) \in \lambda - \mu$, *where as usual* $x^T = \prod_{s \in \lambda - \mu} x_{T(s)}$.

(8.2$'$) *Suppose that* $a'_i - \mu'_i \leq a'_{i+1} - \mu'_{i+1} + 1$ *and* $b'_i - \lambda'_i \leq b'_{i+1} - \lambda'_{i+1} + 1$ *whenever* $\mu'_i < \lambda'_{i+1}$. *Then*

$$s'_{\lambda/\mu}(a', b') = \sum_T x^T$$

summed over column-strict tableaux T *of shape* $\lambda - \mu$ *such that* $a'_j \leq T(i, j) \leq b'_j$ *for all* $(i, j) \in \lambda - \mu$.

Both these results are proved in [W].

In general (i.e., for arbitrary choices of the flags a, b) the row flagged Schur function $s_{\lambda/\mu}(a, b)$ will not be equal to any column-flagged $s'_{\lambda/\mu}(a', b')$. However, there is the following duality theorem :

(8.3) *Let* λ, μ *be partitions such that* $\mu \subset \lambda \subset (m^n)$ *and let* α, β *be positive integers such that* $\alpha > m$ *and* $\beta - \alpha \geq m$. *Let*

$$a_i = \alpha + i - \mu_i - 1, \qquad b_i = \beta + i - \lambda_i \qquad (1 \leq i \leq n),$$
$$a'_j = \alpha + \mu'_j - j + 1, \qquad b'_j = \beta + \lambda'_j - j \qquad (1 \leq j \leq m).$$

Then $s_{\lambda/\mu}(a, b) = s'_{\lambda/\mu}(a', b')$.

This is a particular case of (9.6$'$) in the next section; alternatively, it is not hard to verify that with these choices of a, b, a', b' a column-strict tableau T of shape $\lambda - \mu$ satisfies the row restrictions of (8.2) if and only if it satisfies the column restrictions of (8.2$'$).

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* Our notation here differs from that of [W].

9 th Variation

In this, our last variation, let h_{rs} ($r \geq 1$, $s \in \mathbb{Z}$) be independent indeterminates over $\mathbb{Z}$. Also, for convenience, define $h_{0s} = 1$ and $h_{rs} = 0$ for $r < 0$ and all $s \in \mathbb{Z}$. Define an automorphism of the ring R generated by the h_{rs} by $\varphi(h_{rs}) = h_{r,s+1}$ for all r, s. Thus $h_{rs} = \varphi^s h_r$, where $h_r = h_{r0}$, and we shall generally use this alternative notation.

Now define, for any two partitions λ, μ of length $\leq n$,

$$s_{\lambda/\mu} = \det\big(\varphi^{\mu_j - j+1} h_{\lambda_i - \mu_j - i+j}\big)_{1\leq i,j\leq n} \tag{9.1}$$

and in particular ($\mu = 0$)

$$s_\lambda = \det\big(\varphi^{-j+1} h_{\lambda_i - i+j}\big)_{1\leq i,j\leq n} \tag{9.1$'$}$$

As in the case of ordinary Schur functions ([M$_1$], Chap. I, § 5) we have

$$s_{\lambda/\mu} = 0 \qquad \text{unless } \lambda \supset \mu. \tag{9.2}$$

The "Schur functions" defined by (9.1) include as special cases Variations 4, 5, 6, 7 and 8 (in part). Namely for Variation 4 we specialize $h_{rs} = w_r(z-s)$; for Variation 5, $h_{rs} = w_r(\alpha; z-s)$; for Variation 6, $h_{rs} = h_r(x\,|\,\tau^s a)$; for Variation 7, $h_{rs} = \varphi^s H_r(V)$; and for Variation 8, $h_{rs} = h_r(\alpha + r + s - 1, \beta + s)$.

From (9.1$'$) it follows that

$$h_r = s_{(r)} \qquad (r \geq 0)$$

and we define

$$e_r = s_{(1^r)}$$

for all $r \geq 0$, and $e_r = 0$ for $r < 0$.

(9.3) *Let I be any interval in $\mathbb{Z}$. Then the matrices*

$$H = H_I = \big(\varphi^{1-j} h_{j-i}\big)_{i,j\in I}$$
$$E = E_I = \big((-1)^{j-i}\varphi^{-i} e_{j-i}\big)_{i,j\in I}$$

are inverses of each other.

Proof. — Both H and E are upper unitriangular, hence so also are HE and EH. Hence it is enough to show that the (i,k) element of HE is zero whenever $i,k \in I$ and $i < k$, i.e., that

$$\sum_j \varphi^{1-j}(h_{j-i})(-1)^{k-j}\varphi^{-j}(e_{k-j}) = 0$$

or equivalently that

$$\sum_j \varphi^{1+i-j}(h_{j-i})(-1)^{k-j}\varphi^{i-j}(e_{k-j}) = 0.$$

If we put $k - i = r > 0$, this is equivalent to

$$\sum_{j=0}^{r} (-1)^j \varphi^{1-j}(h_j)\varphi^{-j}(e_{r-j}) = 0. \tag{1}$$

Now by definition

$$e_r = \det\big(\varphi^{1-j}(h_{1-i+j})\big)_{1 \le i,j \le r}$$

and expansion of this determinant along the top row gives (1), as required. ∎

Let λ, μ be partitions such that $\lambda \supset \mu$, and let $\theta = \lambda - \mu$. The function $s_\theta = s_{\lambda/\mu}$ depends not only on the skew shape θ but also on its location in the lattice plane. For each $(p, q) \in \mathbb{Z}^2$ let $\tau_{p,q}$ denote the translation $(i, j) \mapsto (i + p, j + q)$. Then it follows immediately from (9.1) that

$$s_{\tau_{0,1}(\theta)} = \varphi s_\theta, \qquad s_{\tau_{1,0}(\theta)} = \varphi^{-1} s_\theta$$

and hence that

$$s_{\tau_{p,q}(\theta)} = \varphi^{q-p} s_\theta. \tag{9.4}$$

In particular, s_θ is invariant under diagonal translation ($p = q$).

Next let $\hat{\theta}$ be the result of rotating θ through 180^o about a point on the main diagonal. Then we have

$$s_{\hat{\theta}} = \varepsilon s_\theta \tag{9.5}$$

where ε is the involution defined by $\varphi^s h_r \mapsto \varphi^{1-r-s} h_r$.

Proof. — We may assume that λ, μ are both contained in the square (n^n). Let $\widehat{\lambda}$, $\widehat{\mu}$ be their respective complements in this square, so that $\widehat{\lambda}_i = n - \lambda_{n+1-i}$, $\widehat{\mu}_i = n - \mu_{n+1-i}$. Then we may take $\widehat{\theta} = \widehat{\mu} - \widehat{\lambda}$, so that

$$s_{\widehat{\theta}} = \det\big(\varphi^{\widehat{\lambda}_i - i + 1} h_{\widehat{\mu}_j - \widehat{\lambda}_i + i + j}\big)$$

which is easily seen to be equal to $\det\big(\varphi^{i-\lambda_i} h_{\lambda_i - \mu_j - i + j}\big) = \varepsilon s_{\lambda/\mu}$. ∎

Let $\theta' = \lambda' - \mu'$ be the reflection of θ in the main diagonal. Then we have

$$s_{\theta'} = \omega s_\theta \tag{9.6}$$

where ω is the involution defined by $\varphi^s h_r \mapsto \varphi^{-s} e_r$ for all r, s. Equivalently,

$$s_{\lambda/\mu} = \det\big(\varphi^{-\mu'_j+j-1} e_{\lambda'_i-\mu'_j-i+j}\big). \tag{9.6'}$$

Proof. — Since the matrices H, E (over an appropriate interval of $\mathbb{Z}$) are unitriangular, we have $\det H = \det E = 1$; and since by (9.3) they are inverses of each other, it follows that each minor of H is equal to the complementary cofactor of the transpose of E. This leads to (9.6′), exactly as in the case of classical Schur functions ([M$_1$], Ch. I, (2.9)). ∎

All the other determinant formulas for Schur functions have their analogues in the present context, and the proofs are essentially the same. First, if $\lambda = (\alpha_1, \dots, \alpha_r \,|\, \beta_1, \dots, \beta_r)$ in Frobenius notation, we have

$$s_\lambda = \det\big(s_{(\alpha_i\,|\,\beta_j)}\big)_{1\le i,j\le r}. \tag{9.7}$$

Proof (*cf.* [M$_1$], Ch. I, § 3, Ex. 9). — If a, $b \ge 0$, so that $(a|b) = (a+1, 1^b)$, the definition (9.1′) gives

$$s_{(a\,|\,b)} = \sum_{i=0}^{b} (-1)^i \varphi^{-i}(h_{a+i+1})\varphi^{-i-1}(e_{b-i})$$

on expansion of the determinant along the top row. If $a < 0$ (but $b \ge 0$) we *define* $s_{(a\,|\,b)}$ by this formula : this definition gives $s_{(a\,|\,b)} = 0$ except when $a = -1-b$, in which $s_{(a\,|\,b)} = (-1)^b$.

Suppose that $l(\lambda) \le n$ and consider the $n \times n$ matrices A, B where

$$A_{ij} = \varphi^{1-j}(h_{\lambda_i-i+j}), \qquad B_{jk} = (-1)^{j-1}\varphi^{-j}(e_{n+1-j-k}).$$

We have then $(AB)_{ik} = s_{(\lambda_i-i\,|\,n-k)}$ for $1 \le i,k \le n$; and since $\det A = s_\lambda$ and $\det B = 1$, it follows that

$$s_\lambda = \det\big(s_{(\lambda_i-i\,|\,n-k)}\big)_{1\le i,k\le n},$$

which reduces to $\det\big(s_{(\alpha_i\,|\,\beta_j)}\big)_{1\le i,j\le r}$ exactly as in *loc. cit.* ∎

Next we consider the generalization of (9.7) to skew functions $s_{\lambda/\mu}$. Let $\lambda = (\alpha_1, \dots, \alpha_r \,|\, \beta_1, \dots, \beta_r)$ as before, and let $\mu = (\gamma_1, \dots, \gamma_s \,|\, \varepsilon_1, \dots \varepsilon_s)$. Then we have

$$s_{\lambda/\mu} = (-1)^s \det\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \tag{9.8}$$

where $A = \big(s_{(\alpha_i\,|\,\beta_j)}\big)_{1\le i,j\le r}$, $\quad B = \big(\varphi^{\gamma_j+1}h_{\alpha_i-\gamma_j}\big)_{1\le i\le r, 1\le j\le s}$, $C = \big(\varphi^{-\varepsilon_i-1}e_{\beta_j-\varepsilon_i}\big)_{1\le i\le s, 1\le j\le r}$.

The proof of (9.8) is the same as that of Lascoux and Pragacz [LP_1] for classical Schur functions. (Observe that $\varphi^{\gamma_j+1}h_{\alpha_i-\gamma_j} = s_{\theta_{ij}}$, where $\theta_{ij} = (\alpha_i\,|\,0) - (\gamma_j\,|\,0)$, and likewise that $\varphi^{-\varepsilon_i-1}e_{\beta_j-\varepsilon_i} = s_{\varphi_{ij}}$, where $s_{\varphi_{ij}} = (0\,|\,\beta_j) - (0\,|\,\varepsilon_i)$.)

Finally we have, with the notation of (0.5)

$$s_\lambda = \det\big(s_{[\alpha_i\,|\,\beta_j]}\big)_{1\le i,j\le r} \tag{9.9}$$

exactly as in the classical case. Again, the proof is the same as in [LP_2] : indeed, both proofs given there apply in the present context, and we therefore omit the details. There is also a “skew” version of (9.9) in [LP_2] which is likewise valid in the present context : but it is rather complicated to state in full generality, and we shall therefore leave its precise formulation to the conscientious reader.

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