Combinatorial aspects of an exact sequence that is related to a graph

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Abstract

The five problems of counting component colorings, vertex colorings, arc colorings, cocycles, and switching equivalence classes of a graph with respect to a finite field up to isomorphism are related by an exact sequence that stems from a coboundary operator. This cohomology is presented, and counting formulas are given for each of the five problems.

1 Introduction

Let G be an undirected simple graph with vertex set $V = V(G)$, edge set $E = E(G)$, and automorphism group Γ. Two objects related to G (e.g. vertices, edges, components, vertex colorings,...) are called *isomorphic*, if there is an automorphism of G mapping the one onto the other.

For some prime power q, let $\mathbb{F}_q$ be the finite field of this order. Consider the following problems, which are related by the common use of the field $\mathbb{F}_q$.

Problem 1 *Color the components of G with q colors. What is the number of such colorings up to isomorphism?*

Problem 2 *Count the nonisomorphic vertex colorings with q colors.*

Problem 3 *Let $A = A(G)$ be the set of (directed) arcs of the corresponding symmetric digraph of G. An* alternating coloring *of G is an arc coloring with color set $\mathbb{F}_q$, such that inverse arcs have inverse colors, with respect to addition in $\mathbb{F}_q$. Count the alternating colorings of G up to isomorphism.*

Problem 4 *A* cocycle *of G is an alternating coloring of G such that, whenever $(i, j) \in A$, $i \in V_x$, $j \in V_y$, it follows that the color of (i, j) is $x - y$ for a vertex partition $(V_u)_{u \in \mathbb{F}_q}$ in (possibly empty) pairwise disjoint sets. Enumerate the nonisomorphic cocycles of G.*

Problem 5 *Two alternating colorings of G are called* switching equivalent*, if they differ only by a cocycle of G. What is the number of nonisomorphic switching equivalence classes?*

There is a nice cohomological approach relating these five problems. Our purpose is to present this cohomology and solutions of the problems. But let us first continue with some remarks. Problems 1 and 2 can be easily solved by PÓLYA's theorem. Problem 3 reduces to the problem of coloring the edges of G with q colors up to isomorphism if the field characteristic of $\mathbb{F}_q$ (denoted by $\chi(\mathbb{F}_q)$) equals two; in this case, the enumeration can be done by PÓLYA's theorem again. Problem 4 is solved in [5] in full generality. The last problem was solved in [7] for complete graphs and $q = 2$, and later for arbitrary graphs and $q = 2$ in [3] and [8] independently. In this paper we will present a counting formula for this problem in the general case of arbitrary graphs and finite fields; its proof can be found in [6].

2 Cohomology

All objects considered in Problems 1 - 5 have one property in common: they form vector spaces over $\mathbb{F}_q$. A vertex coloring of G with q colors can be understood as a function $f : V \to \mathbb{F}_q$; let $\mathcal{C}^0(G; \mathbb{F}_q)$ be the vector space of such functions, with pointwise addition and scalar multiplication. In a similar way, an alternating coloring of G can be described by a function $F : A \to \mathbb{F}_q$ such that $F(i,j) = -F(j,i)$ for each arc $(i,j) \in A$; let $\mathcal{C}^1(G; \mathbb{F}_q)$ be the vector space of such functions, with pointwise addition and scalar multiplication again. Next we define the vector space homomorphism

$$\delta : \mathcal{C}^0(G; \mathbb{F}_q) \to \mathcal{C}^1(G; \mathbb{F}_q) \tag{1}$$

by setting

$$\delta(f)(i,j) = f(i) - f(j) \qquad ((i,j) \in A). \tag{2}$$

Let $\mathcal{H}^0(G; \mathbb{F}_q) = ke(\delta)$, the 0-*cohomology space* of G, and let $\mathcal{H}^1(G; \mathbb{F}_q) = \mathcal{C}^1(G; \mathbb{F}_q)/im(\delta)$, the 1-*cohomology space* of G. Let $\delta^0 : \mathcal{H}^0(G; \mathbb{F}_q) \to \mathcal{C}^0(G; \mathbb{F}_q)$ and $\delta^1 : \mathcal{C}^1(G; \mathbb{F}_q) \to \mathcal{H}^1(G; \mathbb{F}_q)$ be the canonical monomorphism and epimorphism, respectively. Then we have an exact sequence

$$0 \longrightarrow \mathcal{H}^0(G; \mathbb{F}_q) \overset{\delta^0}{\longrightarrow} \mathcal{C}^0(G; \mathbb{F}_q) \overset{\delta}{\longrightarrow} \mathcal{C}^1(G; \mathbb{F}_q) \overset{\delta^1}{\longrightarrow} \mathcal{H}^1(G; \mathbb{F}_q) \longrightarrow 0. \tag{3}$$

Since $\mathcal{H}^0(G; \mathbb{F}_q)$ consists of those functions of $\mathcal{C}^0(G; \mathbb{F}_q)$ which are constant on the components of G, the space $\mathcal{H}^0(G; \mathbb{F}_q)$ is the space of component colorings of G with q colors. The set of cocycles of G corresponds to $im(\delta)$, while the set of switching equivalence classes is given by $\mathcal{H}^1(G; \mathbb{F}_q)$.

The following dimension formulas can be easily obtained from elementary counting arguments and exactness of Sequence 3.

Proposition 1 *Let m, n, k be the number of edges, vertices, and components of G. Then*

1. $dim(\mathcal{H}^0(G; \mathbb{F}_q)) = k$;
2. $dim(\mathcal{C}^0(G; \mathbb{F}_q)) = n$;

3. $dim(\mathcal{C}^1(G;I\!F_q)) = m$;

4. $dim(\mathcal{H}^1(G;I\!F_q)) = m - n + k$;

5. $dim(im(\delta)) = n - k$.

3 Automorphisms

The automorphism group Γ of G, considered as a permutation group of the vertices of G, acts as a permutation group on edges, arcs, and components of G via $\gamma[i,j] = [\gamma(i), \gamma(j)]$, $\gamma(i,j) = (\gamma(i), \gamma(j))$, and $\gamma(H) = \tilde{H}$ iff $\gamma(i)$ is a vertex of $\tilde{H}$ for some vertex i of H, for each edge $[i,j]$, arc (i,j), and component H of G. The cycle type of $\gamma \in \Gamma$, considered as a permutation of components, vertices, and edges of G, is denoted by $(\omega_1(\gamma), \ldots, \omega_k(\gamma))$, $(\nu_1(\gamma), \ldots, \nu_n(\gamma))$, and $(\epsilon_1(\gamma), \ldots, \epsilon_m(\gamma))$, respectively. Their corresponding sums are denoted by $\omega(\gamma)$, $\nu(\gamma)$, and $\epsilon(\gamma)$.

The group Γ acts not only on vertices and edges of G, but also on the spaces $\mathcal{C}^0(G;I\!F_q)$ and $\mathcal{C}^1(G;I\!F_q)$ via

$$\begin{array}{rll} \gamma(f) = f \circ \gamma^{-1} & \text{for} & f \in \mathcal{C}^0(G;I\!F_q), \\ \gamma(F) = F \circ \gamma^{-1} & \text{for} & F \in \mathcal{C}^1(G;I\!F_q). \end{array} \tag{4}$$

Proposition 2 *For every $\gamma \in \Gamma$, $\gamma \circ \delta = \delta \circ \gamma$.*

It is Proposition 2 from which we conclude that Γ acts on $\mathcal{H}^0(G;I\!F_q)$ and $im(\delta)$ in an obvious way, and on $\mathcal{H}^1(G;I\!F_q)$ via

$$\gamma(F + im(\delta)) = \gamma(F) + im(\delta). \tag{5}$$

In the sense of Equations 4, every $\gamma \in \Gamma$ establishes vector space automorphisms of the four spaces of Sequence 3 and of $im(\delta)$. Our considerations may be summarized by the observation that the diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{H}^0(G;I\!F_q) & \stackrel{\delta^0}{\longrightarrow} & \mathcal{C}^0(G;I\!F_q) & \stackrel{\delta}{\longrightarrow} & \mathcal{C}^1(G;I\!F_q) & \stackrel{\delta^1}{\longrightarrow} & \mathcal{H}^1(G;I\!F_q) & \longrightarrow 0 \\ & \gamma\downarrow & & \gamma\downarrow & & \gamma\downarrow & & \gamma\downarrow & \\ 0 \longrightarrow & \mathcal{H}^0(G;I\!F_q) & \underset{\delta^0}{\longrightarrow} & \mathcal{C}^0(G;I\!F_q) & \underset{\delta}{\longrightarrow} & \mathcal{C}^1(G;I\!F_q) & \underset{\delta^1}{\longrightarrow} & \mathcal{H}^1(G;I\!F_q) & \longrightarrow 0 \end{array} \tag{6}$$

is commutative.

4 Enumeration

The Problems 1-5 can be restated in the following form: Count the orbits of the actions of Γ on the spaces $\mathcal{H}^0(G; \mathbb{F}_q)$, $\mathcal{C}^0(G; \mathbb{F}_q)$, $\mathcal{C}^1(G; \mathbb{F}_q)$, $im(\delta)$, and $\mathcal{H}^1(G; \mathbb{F}_q)$. By Burnside's lemma (which is in fact due to Cauchy-Frobenius), these problems reduce to the problems of counting the component colorings, vertex colorings, alternating colorings, cocycles, and switching equivalence classes that are fixed under γ, for every $\gamma \in \Gamma$.

We define homomorphisms $\alpha^0_\gamma : \mathcal{C}^0(G; \mathbb{F}_q) \to \mathcal{C}^0(G; \mathbb{F}_q)$ and $\alpha^1_\gamma : \mathcal{C}^1(G; \mathbb{F}_q) \to \mathcal{C}^1(G; \mathbb{F}_q)$ by setting

$$\begin{aligned} \alpha^0_\gamma(f) &= f - \gamma(f), \\ \alpha^1_\gamma(F) &= F - \gamma(F), \end{aligned} \tag{7}$$

for $f \in \mathcal{C}^0(G; \mathbb{F}_q)$ and $F \in \mathcal{C}^1(G; \mathbb{F}_q)$. Set $\beta^0_\gamma = \alpha^0_\gamma | \mathcal{H}^0(G; \mathbb{F}_q)$, and let $\beta^1_\gamma : \mathcal{H}^1(G; \mathbb{F}_q) \to \mathcal{H}^1(G; \mathbb{F}_q)$ be defined by $\beta^1_\gamma(F + im(\delta)) = \alpha^1_\gamma(F) + im(\delta)$. In order to solve the Problems 1-5, it suffices to determine the sizes of $ke(\beta^0_\gamma)$, $ke(\alpha^0_\gamma)$, $ke(\alpha^1_\gamma)$, $ke(\alpha^1_\gamma | im(\delta))$, and $ke(\beta^1_\gamma)$. But in the cases of nonisomorphic component colorings and vertex colorings we can use Pólya's theorem directly.

Theorem 1 *The number of nonisomorphic component colorings of G with q colors is*

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} q^{\omega(\gamma)}. \tag{8}$$

Theorem 2 *The number of nonisomorphic vertex colorings of G with q colors is*

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} q^{\nu(\gamma)}. \tag{9}$$

Let $\gamma \in \Gamma$, and let σ_ν be a vertex cycle of γ. The cycle σ_ν is called *diagonal*, if its size is even, say $|\sigma_\nu| = 2t$, and $[i, \gamma^t(i)] \in E$ for some $i \in \sigma_\nu$. The corresponding edge cycle and arc cycle as well as their edges and arcs are called diagonal, too. Now set

$$\rho(\gamma) = \begin{cases} \text{number of diagonal vertex cycles of } \gamma & , \text{ if } \chi(\mathbb{F}_q) \neq 2, \\ 0 & , \text{ if } \chi(\mathbb{F}_q) = 2. \end{cases} \tag{10}$$

Theorem 3 *The number of nonisomorphic alternating colorings of G with q colors is*

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} q^{\epsilon(\gamma) - \rho(\gamma)}. \tag{11}$$

Now we turn attention on the cocycles of G. Let σ_ω be a component cycle of γ. A vertex cycle σ_ν of γ is *associated* to σ_ω, if σ_ν permutes vertices of components in σ_ω. Let $\kappa(\gamma)$ be the number of component cycles σ_ω of γ that have an associated vertex cycle σ_ν such that

$$\frac{|\sigma_\nu|}{|\sigma_\omega|} \not\equiv 0 \pmod{\chi(I\!\!F_q)}. \tag{12}$$

Note that $\frac{|\sigma_\nu|}{|\sigma_\omega|}$ indicates how many vertices of σ_ν are contained in a component of σ_ω. Recall that k is the number of components of G.

Theorem 4 *The number of nonisomorphic cocycles of G over $I\!\!F_q$ is*

$$\frac{1}{|\Gamma|q^k} \sum_{\gamma\in\Gamma} q^{\nu(\gamma)-\kappa(\gamma)+\omega(\gamma)}. \tag{13}$$

As an application of Theorem 4, let $G = K_n$ be the complete graph with n vertices. Then we have $k = 1$ and $\Gamma = S_n$, the symmetric group on the n vertices. For every $\gamma \in S_n$ we have $\omega(\gamma) = 1$, hence $|\sigma_\omega| = 1$ for the only component cycle of γ. We conclude that

$$\kappa(\gamma) = \begin{cases} 0 & \text{if } |\sigma_\nu| \equiv 0 \pmod{p} \text{ for every vertex cycle } \sigma_\nu \text{ of } \gamma, \\ 1 & \text{otherwise .} \end{cases}$$

Now, by Theorem 4, the number of non-equivalent cocycles of K_n over $I\!\!F_q$ is

$$\frac{1}{n!}\sum q^{\nu(\gamma)} + \frac{1}{q\cdot n!}\sum q^{\nu(\gamma)}, \tag{14}$$

where the first sum extends over all $\gamma \in S_n$ such that $|\sigma_\nu| \equiv 0 \pmod{p}$ for every vertex cycle σ_ν of γ, and the second sum extends over the remaining permutations in S_n.

The *cycle index* of S_n [2] is the polynomial

$$Z(S_n;\mathbf{s}) = \frac{1}{n!}\sum_{\gamma\in S_n} s_1^{\nu_1(\gamma)} \ldots s_n^{\nu_n(\gamma)} \ ,$$

where $\mathbf{s} = (s_1, s_2, s_3, \ldots)$. Set $\mathbf{1} = (1, 1, 1, \ldots)$ and for $r \in I\!\!N$ define $\mathbf{1}[r] = (x_1, x_2, x_3, \ldots)$ by setting

$$x_i = \begin{cases} 1 & \text{if } r \text{ is a divisor of } i \ , \\ 0 & \text{otherwise .} \end{cases}$$

Then it follows from Expression 14 by a short calculation that the number of non-equivalent cocycles of K_n over $I\!\!F_q$ is

$$\frac{1}{q}(Z(S_n; q\cdot\mathbf{1}) + (q-1)Z(S_n; q\cdot\mathbf{1}[p])) \ ,$$

where, as usual, the number p is the field characteristic of $I\!\!F_q$. From this formula we obtained Table 1. The cycle indices of small order symmetric groups are tabulated in [2].

Now we will present a counting formula for the number of nonisomorphic switching equivalence classes of the graph G. We remarked already, that the problem reduces, by Burnside's lemma, to the computation of the size of $ke(\beta^1_\gamma)$, since this space is the set of switching equivalence classes fixed by the automorphism $\gamma \in \Gamma$.

$q\backslash n$	2	3	4	5	6	7	8
2	2	2	3	3	4	4	5
3	2	4	5	7	10	12	15
4	4	5	11	14	24	30	45
5	3	7	14	26	42	66	99
7	4	12	30	66	132	246	429
8	8	15	50	99	232	429	835
9	5	21	55	143	339	715	1430
11	6	26	91	273	728	1768	3978
13	7	30	140	476	1428	3876	9690
16	16	51	276	969	3504	10659	30954
17	9	57	285	1197	4389	14296	43263
19	10	70	385	1771	7084	25300	82225
23	12	100	650	3510	16380	67860	254475
25	13	117	819	4755	23751	105183	420732

Table 1

Consider the fiber product of the homomorphisms δ and α^1_γ, i.e. the space $\mathcal{C}_\gamma(G; I\!\!F_q)$ consisting of all pairs (f, F) such that $\delta(f) = \alpha^1_\gamma(F)$, together with the canonical projections $\mu^0_\gamma : \mathcal{C}_\gamma(G; I\!\!F_q) \to \mathcal{C}^0(G; I\!\!F_q)$ and $\mu^1_\gamma : \mathcal{C}_\gamma(G; I\!\!F_q) \to \mathcal{C}^1(G; I\!\!F_q)$. Set $\mathcal{C}^0_\gamma(G; I\!\!F_q) = im(\mu^0_\gamma)$ and $\mathcal{C}^1_\gamma(G; I\!\!F_q) = im(\mu^1_\gamma)$. Then we have $im(\delta^1|\mathcal{C}^1_\gamma(G; I\!\!F_q)) = ke(\beta^1_\gamma)$. It is clear that we can obtain the size of $ke(\beta^1_\gamma)$ from $dim(\mathcal{C}^0_\gamma(G; I\!\!F_q))$.

For an automorphism γ of G, let G_γ be the *cycle graph* of G with respect to γ, i.e. the simple graph with the vertex cycles of γ as vertices; two different vertices σ_ν, τ_ν of G_γ are adjacent in G_γ iff there are $i \in \sigma_\nu$, $j \in \tau_\nu$ such that $[i, j] \in E(G)$.

We define an evaluation on vertex cycles of γ by setting $\Omega(\sigma_\nu) = s$ if $|\sigma_\nu| = p^s u$, where s is chosen so that p is not a divisor from u. Let V_s be the set of vertex cycles of γ that satisfy $\Omega(\sigma_\nu) = s$. Then $G_\gamma < V_s >$ denotes the subgraph of G_γ induced by V_s. A component of $G_\gamma < V_s >$ is called *minimal* in G_γ if it does not contain a vertex that is adjacent in G_γ to a vertex σ_ν with $\Omega(\sigma_\nu) < s$. Let X_γ be the subgraph consisting of the minimal components of all graphs $G_\gamma < V_s >$ in G_γ if $p \neq 2$, respectively the subgraph consisting of such components that do not contain a vertex that is a diagonal vertex cycle of γ if $p = 2$. Let $\xi(\gamma)$ be the number of components of X_γ.

Recall that the number of vertex cycles of γ is denoted by $\nu(\gamma)$.

Theorem 5 *The number of nonisomorphic switching equivalence classes of G is*

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} q^{\epsilon(\gamma) - \nu(\gamma) + \xi(\gamma) - \rho(\gamma)}. \tag{15}$$

In order to illustrate the last theorem, consider again the complete graph $G = K_n$ with n vertices, and remember the notations about its automorphism group given above.

Let $\gamma \in S_n$, and let $(\nu_1, \ldots, \nu_n)$ be the cycle type of γ. Then $\nu(\gamma) = \sum_{i=1}^n \nu_i$. From the cycle index of the *pair group* $S_n^{(2)}$ (see [2]) we obtain $\epsilon(\gamma)$; the cycle indices of pair groups are tabulated in [2] for $n \leq 10$.

$q \backslash n$	2	3	4	5	6	7
2	1	2	3	7	16	54
3	1	2	4	14	120	3222
4	1	4	11	100	2200	242064
5	1	3	10	155	14030	6099115
7	1	4	21	1036	395283	943185908
8	1	8	50	3088	1557536	7022450816

Table 2

Every vertex cycle of γ of even length is diagonal, hence

$$\xi(\gamma) = \begin{cases} 0 & , \text{ if every vertex cycle of } \gamma \text{ is of even length and } p = 2, \\ 1 & , \text{ otherwise.} \end{cases} \quad (16)$$

Recall that p is the field characteristic of $\mathbb{F}_q$. Furthermore,

$$\rho(\gamma) = \begin{cases} \text{Number of vertex cycles of } \gamma \text{ of even length} & , \text{ if } p \neq 2, \\ 0 & , \text{ otherwise.} \end{cases} \quad (17)$$

Using these facts, we obtained Table 2 presenting the numbers of nonisomorphic switching equivalence classes of K_n for some small values of n and q.

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