# A NOTE ON THE PARITY OF THE SUM-OF-DIGITS FUNCTION

Peter J. Grabner

## 1. Introduction

Let for the following $\nu(n)$ be the binary sum-of-digits function, i.e.

$$\nu\left(\sum_{l=0}^{L}\varepsilon_l 2^l\right)=\sum_{l=0}^{L}\varepsilon_l.$$

Newman [Ne] proved that

$$S(N)=\sum_{n<N}(-1)^{\nu(3n)}$$

is always positive and of exact order of magnitude $N^{\log_4 3}$. Coquet [Co] observed that

$$S(N)=N^{\log_4 3}F(\log_4 N)+\frac{\eta(N)}{3},\tag{1.1}$$

where $F(x)$ is a continuous, nowhere differentiable periodic function of period 1 (to speak of continuity makes sense, because the values $\log_4 N$ are dense modulo 1) and $\eta(N)$ only takes the values $0,\pm 1$. He also gave the extreme values of the function $F$. In [FGKPT] the mean value of $F$ was computed.

It is now natural to ask how the function

$$\sum_{n<N}(-1)^{\nu(pn)}$$

behaves for given odd $p$. Numerical studies show that for most values of $p$ this function takes positive and negative values. The asymptotic behaviour like a power of $N$ times a periodic function persists (cf. [GKS], [Gr]). In a concluding section we want to give some examples and state conjectures in this context.

We want to investigate

$$T(N)=\sum_{n<N}(-1)^{\nu(5n)}$$

and will prove

The author is supported by the Austrian Science Foundation, Project Nr.P8274-PHY

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

**Theorem 1.** *The function $T(N)$ is positive for $N>0$ and satisfies*

$$T(N)=N^\alpha\Phi(\log_{16}N)+\frac{\eta_5(N)}{5} \tag{1.2}$$

*with a continuous nowhere differentiable periodic function $\Phi$ of period 1,*

$$\eta_5(N)=\begin{cases}0 & \text{for } N \text{ even}\\ (-1)^{\nu(5N-1)} & \text{for } N \text{ odd,}\end{cases}$$

*and $\alpha=\frac{\log 5}{\log 16}$. The function $\Phi$ satisfies*

$$0.83808514\ldots=\Phi\left(\log_{16}\frac{176}{15}\right)=\frac{7}{10}\left(\frac{15}{11}\right)^\alpha\leq\Phi(x)$$
$$\leq\frac{9}{10}\left(\frac{60}{13}\right)^\alpha=\Phi\left(\log_{16}\frac{52}{15}\right)=2.18677074\ldots$$

*and*

$$\int_0^1\Phi(x)\,dx=5^{\alpha-1}\frac{c_1+c_2+c_3+c_4}{\Gamma(\alpha+1)\log 16}=1.56205765115\ldots$$

*with*

$$c_k=\int_0^\infty\Big(g_k(1)e^{-x}+\cdots+g_k(15)e^{-15x}+$$
$$\left(1+g_k(1)e^{-x}+\cdots+g_k(15)e^{-15x}-5\right)\left(G_k(e^{-16x})-1\right)\Big)x^{\alpha-1}dx,$$

*where $g_k(n)=e^{\frac{2kn\pi i}{5}}(-1)^{\nu(n)}$ and*

$$G_k(z)=\prod_{m=0}^\infty\left(1+g_k(1)z^{16^m}+\cdots+g_k(15)z^{15\cdot16^m}\right).$$

## 2. Proof of the Theorem

Let for the following $\xi_k=\exp(\frac{2k\pi i}{5})$ for $k=0,\ldots,4$. Then it is an immediate consequence of $16^n\equiv 1\mod 5$ that

$$g_k(n)=\xi_k^n(-1)^{\nu(n)} \tag{2.1}$$

satisfies

$$g_k(16n+b)=g_k(n)g_k(b)\quad\text{for } 0\leq b\leq 15. \tag{2.2}$$

This property is called "complete 16-multiplicativity" and immediately yields

$$g_k\left(\sum_{l=0}^L a_l16^l\right)=\prod_{l=0}^L g_k(a_l). \tag{2.3}$$

Thus the value of $g_k(n)$ only depends on the digit expansion of $n$ to the base 16.

Setting $G_k(M) = \sum_{n<M} g_k(n)$ we have

$$T(N) = \frac{1}{5}G_0(5N) + \frac{1}{5}\sum_{k=1}^{4} G_k(5N) = \frac{\eta_5(N)}{5} + \frac{1}{5}\sum_{k=1}^{4} G_k(5N). \tag{2.4}$$

We will now investigate the asymptotic behaviour of $G_k(M)$, $k = 1, \ldots, 4$: Let $M = \sum_{l=0}^{L} a_l 16^l$ be the 16-adic expansion of $M$ and set $M_p = \sum_{l=p}^{L} a_l 16^l$. Then we have

(2.5)

$$G_k(M) = \sum_{n<M_L} g_k(n) + \sum_{p=0}^{L-1} \sum_{n=M_{p+1}}^{M_p-1} g_k(n) = G_k(a_L 16^L) + \sum_{p=0}^{L} g_k(M_{p+1}) G_k(a_p 16^p).$$

Thus we have reduced the problem to the computation of $G_k(a16^l)$:

$$G_k(a16^l) = \sum_{\varepsilon<a} g_k(\varepsilon) G_k(16^l) = G_k(a) G_k(16)^l.$$

Notice that

$$G_k(16) = \sum_{n=0}^{15} \xi_k^n (-1)^{\nu(n)} = \prod_{l=0}^{3}\left(1 - \xi_k^{2^l}\right) = 5. \tag{2.6}$$

This holds because 2 is a primitive root mod 5 and therefore the product can be rewritten as $\prod_{l=1}^{4}(1-\xi_k^l)$. (We will refer to this argument later in the concluding remarks.)

We rewrite (2.5)

$$G_k(M) = 5^L \sum_{p=0}^{L} 5^{p-L} G_k(a_p) \prod_{l=p+1}^{L} g_k(a_l) \tag{2.7}$$

and set

$$\varphi_k\left(\sum_{l=0}^{\infty} a_l 16^{-l}\right) = \sum_{l=0}^{\infty}\prod_{p=0}^{l-1} g_k(a_p) G_k(a_l) 5^{-l}. \tag{2.8}$$

Notice that these functions are well-defined and continuous (this is proved in a more general setting in [Gr]) and $\varphi_k(1) = 1$, $\varphi_k(16) = 5$.

Inserting the definition of $\varphi_k$ into (2.7) yields

$$G_k(M) = 5^{[\log_{16} M]} \varphi_k\left(\frac{M}{16^{[\log_{16} M]}}\right) = M^\alpha 5^{-\{\log_{16} M\}} \varphi_k\left(16^{\{\log_{16} M\}}\right), \tag{2.9}$$

where $[x]$ and $\{x\}$ denote the integer and the fractional part of $x$ as usual. We set now $\psi_k(x) = \varphi_k(x) x^{-\alpha}$ for $1 \le x \le 16$ and observe that

$$\Psi(x) = \frac{1}{5}\sum_{k=1}^{4} \psi_k(x)$$

is a continuous function which can be continued periodically (with period 1). Then we have

$$T(N) = (5N)^{\alpha}\Psi(5N) + \frac{\eta_5(N)}{5}.$$

and $\Phi(y) = 5^{\alpha}\Psi(5 \cdot 16^{y})$.

In order to compute the extremal values of $\Phi$ we derive an explicit formula for $\varphi(x) = \frac{1}{5}\sum_{k=1}^{4}\varphi_k(x)$. For this purpose we introduce some notations:

$$\alpha_1(l,x) = \#\{p < l : a_p = 1 \vee a_p = 11\}$$
$$\alpha_2(l,x) = \#\{p < l : a_p = 2 \vee a_p = 7\}$$
$$\alpha_3(l,x) = \#\{p < l : a_p = 3\}$$
$$\alpha_4(l,x) = \#\{p < l : a_p = 4 \vee a_p = 14\}$$
$$\alpha_5(l,x) = \#\{p < l : a_p = 6\}$$
$$\alpha_6(l,x) = \#\{p < l : a_p = 8 \vee a_p = 13\}$$
$$\alpha_7(l,x) = \#\{p < l : a_p = 9\}$$
$$\alpha_8(l,x) = \#\{p < l : a_p = 12\}$$

for $x = \sum_{p=0}^{\infty} \frac{a_p}{16^p}$ (from now on we will omit the dependence on $x$)

$$A(l) = \alpha_1(l) + 2\alpha_2(l) + 3\alpha_3(l) + 4\alpha_4(l) + \alpha_5(l) + 3\alpha_6(l) + 4\alpha_7(l) + 2\alpha_8(l)$$
$$B(l) = \alpha_1(l) + \alpha_2(l) + \alpha_4(l) + \alpha_6(l)$$

and

| | $A(l)$ mod 5 | | | | |
|---|---|---|---|---|---|
| $d(a_l, A(l))$ | 0 | 1 | 2 | 3 | 4 |
| 0 | $0$ | $0$ | $0$ | $0$ | $0$ |
| 1 | $\frac{4}{5}$ | $-\frac{1}{5}$ | $-\frac{1}{5}$ | $-\frac{1}{5}$ | $-\frac{1}{5}$ |
| 2 | $1$ | $0$ | $0$ | $0$ | $-1$ |
| 3 | $\frac{6}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $-\frac{4}{5}$ | $-\frac{4}{5}$ |
| 4 | $1$ | $0$ | $1$ | $-1$ | $-1$ |
| 5 | $\frac{6}{5}$ | $-\frac{4}{5}$ | $\frac{6}{5}$ | $-\frac{4}{5}$ | $-\frac{4}{5}$ |
| 6 | $2$ | $-1$ | $1$ | $-1$ | $-1$ |
| 7 | $\frac{9}{5}$ | $-\frac{6}{5}$ | $\frac{4}{5}$ | $-\frac{6}{5}$ | $-\frac{1}{5}$ |
| 8 | $2$ | $-1$ | $1$ | $-2$ | $0$ |
| 9 | $\frac{11}{5}$ | $-\frac{4}{5}$ | $\frac{1}{5}$ | $-\frac{9}{5}$ | $\frac{1}{5}$ |
| 10 | $2$ | $0$ | $0$ | $-2$ | $0$ |
| 11 | $\frac{14}{5}$ | $-\frac{1}{5}$ | $-\frac{1}{5}$ | $-\frac{11}{5}$ | $-\frac{1}{5}$ |
| 12 | $3$ | $0$ | $0$ | $-2$ | $-1$ |
| 13 | $\frac{14}{5}$ | $-\frac{1}{5}$ | $-\frac{1}{5}$ | $-\frac{6}{5}$ | $-\frac{6}{5}$ |
| 14 | $3$ | $0$ | $-1$ | $-1$ | $-1$ |
| 15 | $\frac{16}{5}$ | $-\frac{4}{5}$ | $-\frac{4}{5}$ | $-\frac{4}{5}$ | $-\frac{4}{5}$ |

We are now able to write

$$\varphi(x) = \sum_{l=0}^{\infty}(-1)^{B(l)}\frac{d(a_l, A(l))}{5^l}. \tag{2.10}$$

Detailed investigation of the entries of $d(a, A)$ yields $\frac{7}{10} \leq \varphi \leq 4$ and also estimates for $\varphi(x)$, $x \in [\frac{k}{16^l}, \frac{k+1}{16^l}]$, $16^l \leq k < 16^{l+1}$:

$$\begin{aligned}(2.11)\qquad &\varphi\left(\frac{k}{16^l}\right) + (-1)^{B(l+1)} m(B(l+1)+1, A(l+1)) 5^{-l-1} \leq \varphi(x) \leq \\ &\varphi\left(\frac{k}{16^l}\right) + (-1)^{B(l+1)} m(B(l+1), A(l+1)) 5^{-l-1},\end{aligned}$$

where $m(B, A)$ is given by

| | $m(B(l), A(l))$ | $A(l)$ mod 5: 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| $B(l)$ mod 2 | 0 | 0 | $-\frac{61}{50}$ | $-\frac{11}{10}$ | $-\frac{5}{2}$ | $-\frac{3}{2}$ |
| | 1 | 4 | $\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{1}{10}$ | $\frac{11}{50}$ |

Outside the interval $[1, 2]$ it can be proved by trivial estimates that $\Psi(x) < \frac{9}{10}(\frac{12}{13})^\alpha =: M$. The interval $[1, 2]$ has to be splitted into several parts to prove that the maximum of $\Psi$ is attained at $x = \frac{13}{12}$.

(1) $1 \leq x \leq \frac{17}{16}$: $\varphi(x) \leq \frac{1061}{1250}$ and $\Psi(x) < \frac{1061}{1250} < M$.
(2) $\frac{13}{12} - \frac{1}{3}16^{-k} \leq x \leq \frac{13}{12} - \frac{1}{3}16^{-k-1}$ for $k \geq 1$: $\varphi(x) \leq \frac{9}{10} - 32 \cdot 5^{-k-2}$ and $\Psi(x) \leq (\frac{9}{10} - 32 \cdot 5^{-k-2})(\frac{13}{12} - \frac{1}{3}16^{-k})^{-\alpha} < M$.
(3) $\frac{13}{12} \leq x \leq \frac{5}{4}$: $\varphi(x) \leq \frac{9}{10}$ and $\Psi(x) \leq M$
(4) $\frac{5}{4} \leq x \leq \frac{21}{16}$: $\varphi(x) \leq \frac{24}{25}$ and $\Psi(x) < M$
(5) $\frac{21}{16} \leq x \leq \frac{23}{16}$: in this interval some local extrema are attained which are only $\sim \frac{1}{100}$ smaller than $M$; therefore this interval has to be split into 32 intervals of length $\frac{1}{256}$ to prove $\Psi(x) < M$.
(6) $\frac{23}{16} \leq x \leq 2$: $\varphi(x) \leq \frac{261}{250}$ and $\Psi(x) \leq \frac{261}{250}(\frac{16}{23})^\alpha < M$.

In order to prove that $\Psi(x) \geq \frac{7}{10}(\frac{3}{11})^\alpha =: m$ we note first that outside of the interval $[3, 4]$ this inequality can be obtained by trivial estimates. The interval $[3, 4]$ again has to be split:

(1) $3 \leq x \leq \frac{11}{3}$: $\varphi(x) \geq \frac{7}{10}$ and $\Psi(x) \geq m$
(2) $\frac{11}{3} + \frac{1}{3}16^{-k-1} \leq x \leq \frac{11}{3} + \frac{1}{3}16^{-k}$: $\varphi(x) \geq \frac{7}{10} + 32 \cdot 5^{-k-3}$ and $\Psi(x) \geq (\frac{7}{10} + 32 \cdot 5^{-k-3})(\frac{11}{3} + \frac{1}{3}16^{-k})^{-\alpha} > m$.
(3) $\frac{59}{16} \leq x \leq 4$: $\varphi(x) \geq \frac{939}{1250}$ and $\Psi(x) \geq \frac{939}{1250\sqrt{5}} > m$

After rescaling this yields the extremal values stated in the theorem.

It is an immediate consequence of (2.11) that for every $x \in [0, 1]$ and every $l > 0$ there exists a $y$ with $|x - y| \leq 16^{-l}$, such that $|\varphi(x) - \varphi(y)| \geq \frac{43}{50}5^{-l-1}$. Thus $\varphi$ is nowhere differentiable.

It remains to compute the mean value of $\Phi$. For this purpose we note that in [Gr] a formula for the Fourier coefficients of a fractal function occurring in the context of $q$-multiplicative functions is developed. Inserting the 16-multiplicative functions $g_k$ into this formula yields the mean value stated in the theorem. □

## 3. Concluding Remarks

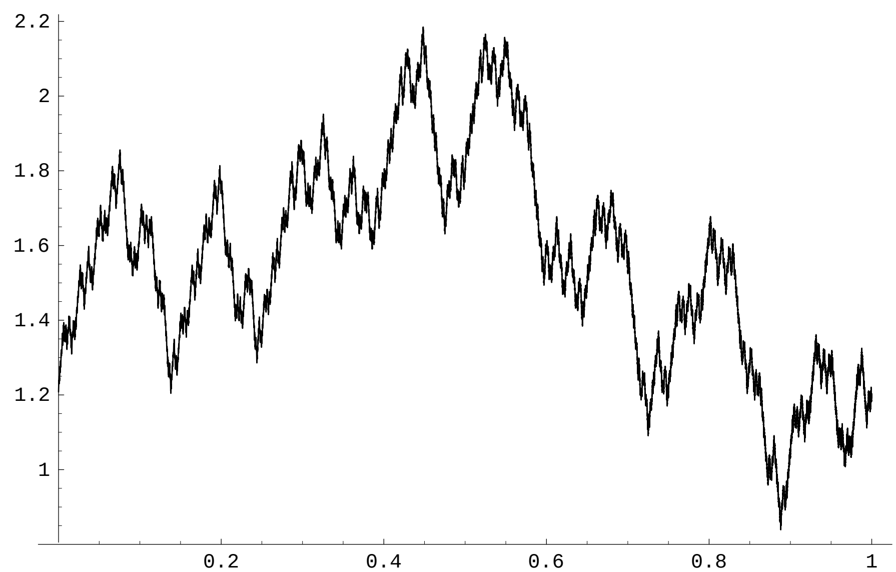
In the recent paper [GKS] the asymptotic behaviour of the summatory function

$$\sum_{n<N} (-1)^{\nu(pn+q)}$$

for prime numbers $p$ and $0 \leq q < p$ is investigated. It turns out that for all these functions the asymptotic behaviour resembles that discussed in the previous section; however it seems to be difficult to determine the value of the exponent of $N$ in the asymptotic formula, because it depends on the value

$$\sum_{n<2^s} \zeta^n (-1)^{\nu(n)} = \prod_{k=0}^{s-1} \left(1 - \zeta^{2^k}\right),$$

where $\zeta$ is a $p$-th root of unity and $s$ is the multiplicative order of $2 \mod p$. In the cases $s = p-1$ and $s = \frac{p-1}{2}$ it is possible to derive general formulæ for this expression (cf. [GKS]).




THE GRAPH OF $\Phi(x)$.

By an immediate generalization of the method used above it is possible to describe the behaviour of $\sum_{n<N} (-1)^{\nu(p^r n)}$. The cases $p = 3$ and $p = 5$ are the easiest, because 2 is a primitive root $\mod 3^r$ and $\mod 5^r$. Here the asymptotic behaviour of the summands of the formula corresponding to (2.4) depends on the order of the root $\exp(\frac{2k\pi i}{p^r})$. The main term originates from the primitive $3^{\text{rd}}$ ($5^{\text{th}}$ resp.) roots of unity. This gives asymptotic formulæ

$$\begin{aligned}
S_r(N) = \sum_{n<N} (-1)^{\nu(3^r n)} &= \frac{1}{3^{r-1}} (3^r N)^\beta F\left(\log_4 3^{r-1} N\right) \\
&+ N^{\frac{\beta}{3}} F_1\left(\frac{1}{3} \log_4 N\right) + \cdots + N^{\frac{\beta}{3^{r-1}}} F_{r-1}\left(\frac{1}{3^{r-1}} \log_4 N\right) + \frac{\eta_{3^r}(N)}{3^r} \\
T_r(N) = \sum_{n<N} (-1)^{\nu(5^r n)} &= \frac{1}{5^{r-1}} (5^r N)^\alpha \Phi\left(\log_{16} 5^{r-1} N\right) \\
&+ N^{\frac{\alpha}{5}} \Phi_1\left(\frac{1}{5} \log_{16} N\right) + \cdots + N^{\frac{\beta}{5^{r-1}}} \Phi_{r-1}\left(\frac{1}{5^{r-1}} \log_{16} N\right) + \frac{\eta_{5^r}(N)}{5^r},
\end{aligned}$$

where $\beta = \log_4 3$ and $F$ is the fractal function studied in Coquet's paper [Co]; $\alpha = \log_{16} 5$ and $\Phi$ is the fractal function of Theorem 1 (this is the reason for the cumbersome notation of the two leading terms). The other functions occurring in the formulæ are also continuous and periodic of period 1, the $\eta$'s only take the values 0, $\pm 1$. Therefore these two sums only take at most finitely many negative values.

Let us conclude with some remarks on the sum $U_{rs}(N) = \sum_{n<N} (-1)^{\nu(3^r 5^s n)}$. The order of 2 $\mod 3^r 5^s$ is $4 \cdot 3^{r-1} 5^{s-1}$. Thus 2 generates half of $\mathbb{Z}^*_{3^r 5^s}$ and it is not too difficult to compute the possible values for the exponent: If $\zeta$ is a primitive $3^k 5^l$-th root of unity ($0 < k \le r$, $0 < l \le s$) we have

$$P(\zeta) = \prod_{t=0}^{4 \cdot 3^{k-1} 5^{l-1}} \left(1 - \zeta^{2^t}\right) = \pm 1,$$

because $P(\zeta) = P(\bar{\zeta})$ and $P(\zeta)P(\bar{\zeta}) = C_{3^k 5^l}(1) = 1$, where $C_q$ is the cyclotomic polynomial of order $q$ (these terms only contribute $O(\log N)$ to $U_{rs}$). Therefore the asymptotic behaviour of $U_{rs}(N)$ is determined by those terms in the formula analogous to (2.4), which correspond to primitive $3^k$-th and $5^l$-th roots of unity. But these terms just constitute the sums $S_r$ and $T_s$. This gives

$$U_{rs}(N) = \frac{1}{3^r 5^s} \left(3^r S_r(5^s N) + 5^s T_s(3^r N)\right) + O(\log N)$$

and again we have that $U_{rs}$ only takes at most finitely many negative values. It remains as a question, for which primes $p$ the sum $\sum_{n<N} (-1)^{\nu(pn)}$ is always positive. Numerical studies show that 17, 43 and 101 are possible candidates for this property, but this is far from a proof. The method used to prove this for $p = 3$ and $p = 5$ could be applied to $p = 17$, but would require immense computations for larger primes.

## References


[Co] J. Coquet, *A Summation Formula Related to the Binary Digits*, Invent. math. **73** (1983), 107–115.

[FGKPT] P. Flajolet, P.J. Grabner, P. Kirschenhofer, H. Prodinger and R.F. Tichy, *Mellin Transforms and Asymptotics: Digital Sums*, Theor. Comput. Sci. (1993) (to appear).

[GKS] S. Goldstein, K.A. Kelly and E.S. Speer, *The Fractal Structure of Rarefied Sums of the Thue-Morse Sequence*, J. Number Th. **42** (1992), 1–19.

[Gr] P.J. Grabner, *Completely q-Multiplicative Functions: the Mellin-Transform Approach*, Acta Arith. (to appear).

[Ne] D.J. Newman, *On the Number of Binary Digits in a Multiple of Three*, Proc. A.M.S. **21** (1969), 719–721.



Institut für Mathematik A, Technische Universität Graz, Steyergasse 30, 8010 Graz, Austria

*E-mail address*: grabner@weyl.math.tu-graz.ac.at