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Ordered set partitions and the 0-Hecke algebra

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Abstract. Haglund, Rhoades, and Shimozono recently introduced a quotient $R_{n,k}$ of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ depending on two positive integers $k \leq n$, which reduces to the classical coinvariant algebra of the symmetric group $\mathfrak{S}_n$ if $k = n$. They determined the graded $\mathfrak{S}_n$-module structure of $R_{n,k}$ and related it to the Delta Conjecture in the theory of Macdonald polynomials. We introduce an analogous quotient $S_{n,k}$ and determine its structure as a graded module over the (type A) 0-Hecke algebra $H_n(0)$, a deformation of the group algebra of $\mathfrak{S}_n$. When $k = n$ we recover earlier results of the first author regarding the $H_n(0)$-action on the coinvariant algebra.

Keywords: Hecke algebra, set partition, coinvariant algebra

1 Introduction

The symmetric group $\mathfrak{S}_n$ acts on the polynomial ring $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \ldots, x_n]$ by variable permutation. The corresponding *invariant subring* is generated by the *elementary symmetric functions* $e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n)$. The *coinvariant algebra* $R_n := \mathbb{Q}[\mathbf{x}_n]/I_n$, where $I_n := \langle e_1(\mathbf{x}_n), \ldots, e_n(\mathbf{x}_n) \rangle$, plays an important role in algebraic and geometric combinatorics, with properties closely tied to the combinatorics of permutations. It has $\mathbb{Q}$-dimension $n!$ and has various $\mathbb{Q}$-bases constructed by Artin [1], Garsia–Stanton [7], and others. Chevalley [4] proved that R_n is isomorphic to the regular representation $\mathbb{Q}[\mathfrak{S}_n]$ of $\mathfrak{S}_n$. Lusztig (unpublished) and Stanley [19] described the *graded* $\mathfrak{S}_n$-module structure of R_n using the major index statistic on standard Young tableaux.

Let $k \leq n$ be two positive integers. Haglund, Rhoades, and Shimozono [12] introduced a homogeneous ideal $I_{n,k} := \langle x_1^k, x_2^k, \ldots, x_n^k, e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n) \rangle$ of the polynomial ring $\mathbb{Q}[\mathbf{x}_n]$ which is stable under the $\mathfrak{S}_n$-action. They studied the quotient $R_{n,k} := \mathbb{Q}[\mathbf{x}_n]/I_{n,k}$ which is a graded $\mathfrak{S}_n$-module reducing to the coinvariant algebra R_n when $k = n$. They generalized the Artin basis and the Garsia–Stanton basis of R_n to $R_{n,k}$. They showed that $R_{n,k}$ is isomorphic to $\mathbb{Q}[\mathcal{OP}_{n,k}]$ as an ungraded $\mathfrak{S}_n$-module, where $\mathbb{Q}[\mathcal{OP}_{n,k}]$ has a basis $\mathcal{OP}_{n,k}$ consisting of *ordered set partitions* of $[n] := \{1, 2, \ldots, n\}$ with k blocks and admits an $\mathfrak{S}_n$-action by permuting $1, \ldots, n$; consequently the dimension

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of $R_{n,k}$ is $|\mathcal{OP}_{n,k}| = k! \cdot \mathrm{Stir}(n,k)$, where $\mathrm{Stir}(n,k)$ is the *(signless) Stirling number of the second kind* counting k-block set partitions of $[n]$. They provided explicit descriptions of the *graded* $\mathfrak{S}_n$-module structure of $R_{n,k}$, generalizing the work of Lusztig–Stanley.

The symmetric group $\mathfrak{S}_n$ has an interesting deformation called the *(type A) 0-Hecke algebra* and denoted by $H_n(0)$. Norton [16] studied the representation theory of $H_n(0)$ over an arbitrary field $\mathbb{F}$. Krob and Thibon [15] introduced two characteristic maps from representations of $H_n(0)$ to quasisymmetric functions and noncommutative symmetric functions, which are similar to the Frobenius correspondence from representations of symmetric groups to symmetric functions.

Finding 0-Hecke analogs of results on $\mathfrak{S}_n$-representations has received a great deal of recent attention in algebraic combinatorics [2, 13, 14, 20]. In particular, the first author [13] showed that the coinvariant algebra R_n is a graded $H_n(0)$-module isomorphic to the regular representation of $H_n(0)$ and has bigraded quasisymmetric characteristic given by a generating function for the pair of Mahonian statistics (inv, maj) on $\mathfrak{S}_n$.

There is an $H_n(0)$-action on the polynomial ring $\mathbb{F}[\mathbf{x}_n] := \mathbb{F}[x_1, \ldots, x_n]$ by the *isobaric Demazure operators*; see Equation (2.2). However, the ideal $I_{n,k}$ is not closed under this action, so that $R_{n,k}$ does not have an $H_n(0)$-module structure. We remedy this situation as follows.

Definition 1.1. Given positive integers $k \leq n$, define $S_{n,k} := \mathbb{F}[\mathbf{x}_n]/J_{n,k}$, where $J_{n,k}$ is the ideal of $\mathbb{F}[\mathbf{x}_n]$ generated by $e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n)$ together with the *complete homogeneous symmetric functions* $h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n)$.

The ideal $J_{n,k}$ is closed under the action of $H_n(0)$, so that $S_{n,k}$ is a graded $H_n(0)$-module. The polynomials $h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n)$ have span isomorphic to the defining representation of $H_n(0)$; this is analogous to the generators $x_1^k, x_2^k, \ldots, x_n^k$ of the ideal $I_{n,k}$ under the $\mathfrak{S}_n$-action. We show that $S_{n,k}$ has algebraic and combinatorial properties analogous to those of $R_{n,k}$, including a connection to the *Delta Conjecture* of Haglund, Remmel, and Wilson [11] in the theory of Macdonald polynomials.

The remainder of the paper is structured as follows. In Section 2 we give background on representations of the symmetric groups and 0-Hecke algebras. In Section 3 we study $S_{n,k}$ as a graded vector space. In Section 4 we study $S_{n,k}$ as a module over $H_n(0)$ (both graded and ungraded). In Section 5 we connect our results to the Delta Conjecture.

2 Background

The symmetric group $\mathfrak{S}_n$ consists of all permutations on the set $[n]$. It is generated by $s_1, s_2, \ldots, s_{n-1}$, where s_i is the adjacent transposition $s_i := (i, i+1)$, subject to the quadratic relations $s_i^2 = 1$ for all $i \in [n-1]$ and the braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for all $i \in [n-2]$ and $s_i s_j = s_j s_i$ for all $i, j \in [n-1]$ with $|i-j| > 1$. A permutation

$w \in \mathfrak{S}_n$ can be expressed as $w = s_{i_1} \cdots s_{i_k}$ in terms of the generators $s_1, \ldots, s_{n-1}$; such an express is *reduced* if k is as small as possible, and the smallest k is the *length* $\ell(w)$ of w.

A permutation $w \in \mathfrak{S}_n$ can be written in one-line notation $w = w(1) \cdots w(n)$. Define $\mathrm{Des}(w) := \{i \in [n-1] : w(i) > w(i+1)\}$, $\mathrm{des}(w) := |\mathrm{Des}(w)|$, $\mathrm{maj}(w) := \sum_{i \in \mathrm{Des}(w)} i$, and $\mathrm{inv}(w) := |\{(i,j) : 1 \le i < j \le n-1,\ w(i) > w(j)\}|$; one has $\mathrm{inv}(w) = \ell(w)$. It is well known that $\sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)} = [n]!_q$, where $[n]!_q := [n]_q[n-1]_q \cdots [1]_q$ and $[n]_q := 1 + q + \cdots + q^{n-1}$. Any statistic on $\mathfrak{S}_n$ with this distribution is *Mahonian*.

We review the (ordinary) representation theory of $\mathfrak{S}_n$. Irreducible $\mathbb{Q}[\mathfrak{S}_n]$-modules S^λ are indexed by partitions $\lambda \vdash n$ and form a free $\mathbb{Z}$-basis for the *Grothendieck group* $G_0(\mathbb{Q}[\mathfrak{S}_n])$ of $\mathfrak{S}_n$. As $\mathbb{Q}[\mathfrak{S}_n]$ is *semisimple*, any finite-dimensional $\mathbb{Q}[\mathfrak{S}_n]$-module M can be written as a direct sum of irreducible submodules, hence an element of $G_0(\mathbb{Q}[\mathfrak{S}_n])$; sending each irreducible S^λ to the *Schur function* s_λ gives the *Frobenius character* $\mathrm{Frob}(M)$. This is an isomorphism of self-dual graded Hopf algebras between the *Grothendieck group* $G_0(\mathbb{Q}[\mathfrak{S}_*]) := \bigoplus_{n \ge 0} G_0(\mathbb{Q}[\mathfrak{S}_n])$ of the tower $\mathbb{Q}[\mathfrak{S}_*] : \mathbb{Q}[\mathfrak{S}_0] \hookrightarrow \mathbb{Q}[\mathfrak{S}_1] \hookrightarrow \mathbb{Q}[\mathfrak{S}_2] \hookrightarrow \cdots$ of algebras and the ring Sym of symmetric functions (see, e.g., Grinberg and Reiner [10, Section 4.4]). Moreover, a graded $\mathbb{Q}[\mathfrak{S}_n]$-module $V = \bigoplus_{d \ge 0} V_d$ with each component V_d finite-dimensional has *graded Frobenius series* $\mathrm{grFrob}(V; q) := \sum_{d \ge 0} \mathrm{Frob}(V_d) \cdot q^d$.

Now let $\mathbb{F}$ be an arbitrary field. The *(type A) 0-Hecke algebra* $H_n(0)$ is a unital associative $\mathbb{F}$-algebra with generators $\pi_1, \ldots, \pi_{n-1}$ subject to quadratic relations $\pi_i^2 = \pi_i$ for all $i \in [n-1]$ and the same braid relations as the generators $s_1, \ldots, s_{n-1}$ of $\mathfrak{S}_n$. One can realize π_i as the *bubble sorting operator* acting on a list of entries $(a_1, \ldots, a_n)$ from a totally ordered alphabet by swapping a_i and a_{i+1} if $a_i > a_{i+1}$ or fixing the list otherwise.

The algebra $H_n(0)$ is also generated by $\overline{\pi}_1, \ldots, \overline{\pi}_{n-1}$, where $\overline{\pi}_i := \pi_i - 1$, with quadratic relations $\overline{\pi}_i^2 = -\overline{\pi}_i$ for all i, and the same braid relations as above. For each permutation $w \in \mathfrak{S}_n$ with any reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, the elements $\pi_w := \pi_{i_1} \cdots \pi_{i_\ell}$ and $\overline{\pi}_w := \overline{\pi}_{i_1} \cdots \overline{\pi}_{i_\ell}$ are well-defined. The sets $\{\pi_w : w \in \mathfrak{S}_n\}$ and $\{\overline{\pi}_w : w \in \mathfrak{S}_n\}$ are both $\mathbb{F}$-bases for $H_n(0)$. In particular, $H_n(0)$ has dimension $n!$.

We recall some notations before reviewing the representation theory of $H_n(0)$. A sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of positive integers is a *composition* of $n = |\alpha| := \alpha_1 + \cdots + \alpha_\ell$; this is denoted by $\alpha \models n$. We call $\alpha_1, \ldots, \alpha_\ell$ the *parts* of α and define the *length* of α to be $\ell(\alpha) := \ell$. The *descent set* of α is $\mathrm{Des}(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\}$. The map $\alpha \mapsto \mathrm{Des}(\alpha)$ is a bijection from compositions of n to subsets of $[n-1]$. The *major index* of α is $\mathrm{maj}(\alpha) := \sum_{i \in \mathrm{Des}(\alpha)} i$. Given two compositions $\alpha, \beta \models n$, we write $\alpha \preceq \beta$ if $\mathrm{Des}(\alpha) \subseteq \mathrm{Des}(\beta)$, i.e., if α is refined by β. The *complement* α^c of $\alpha \models n$ is the unique composition of n which satisfies $\mathrm{Des}(\alpha^c) = [n-1] \setminus \mathrm{Des}(\alpha)$. For $\alpha = (2,3,1,2) \models 8$ we have $\ell(\alpha) = 4$, $\mathrm{Des}(\alpha) = \{2,5,6\}$, $\mathrm{maj}(\alpha) = 2+5+6 = 13$, $\alpha^c = (1,2,1,3,1) \models 8$, and $\mathrm{Des}(\alpha^c) = \{1,3,4,7\} = [7] \setminus \{2,5,6\}$.

The *descent class* of a composition $\alpha \models n$ consists of permutations $w \in \mathfrak{S}_n$ with $\mathrm{Des}(w) = \mathrm{Des}(\alpha)$; it is an interval under the left weak order of $\mathfrak{S}_n$ whose unique minimal element $w_0(\alpha)$ is the longest element in the parabolic subgroup of $\mathfrak{S}_n$ generated by

$\{s_i : i \in \mathrm{Des}(\alpha)\}$. For example, if $\alpha = (2,3,1,2) \models 8$ then $w_0(\alpha) = 13247658 \in \mathfrak{S}_8$.

Norton [16] showed that, for each $\alpha \models n$, the module $P_\alpha := H_n(0)\overline{\pi}_{w_0(\alpha)}\pi_{w_0(\alpha^c)}$ has a basis $\{\overline{\pi}_w \pi_{w_0(\alpha^c)} : w \in \mathfrak{S}_n,\ \mathrm{Des}(w) = \mathrm{Des}(\alpha)\}$ and a unique maximal submodule spanned by all elements in this basis except the cyclic generator $\overline{\pi}_{w_0(\alpha)}\pi_{w_0(\alpha^c)}$. The quotient of P_α by this maximal submodule, denoted by C_α, is one-dimensional and admits an $H_n(0)$-action by $\overline{\pi}_i = -1$ for all $i \in \mathrm{Des}(\alpha)$ and $\overline{\pi}_i = 0$ for all $i \in \mathrm{Des}(\alpha^c)$.

The algebra $H_n(0)$ is *non-semisimple*. The set $\{P_\alpha : \alpha \models n\}$ (or $\{C_\alpha : \alpha \models n\}$, resp.) is a complete list of nonisomorphic projective indecomposable (or irreducible, resp.) $H_n(0)$-modules, and gives a $\mathbb{Z}$-basis for the *Grothendieck group* $G_0(H_n(0))$ (or $K_0(H_n(0))$, resp.) of $H_n(0)$. A finite-dimensional $H_n(0)$-module M is identified with the sum of its composition factors (with multiplicities) in $G_0(H_n(0))$. If M is also projective then it is a direct sum of projective indecomposable submodules, hence an element of $K_0(H_n(0))$. With certain product and coproduct, the *Grothendieck groups* $G_0(H_*(0)) := \bigoplus_{n\ge 0} G_0(H_n(0))$ and $K_0(H_*(0)) := \bigoplus_{n\ge 0} K_0(H_n(0))$ of the tower $H_*(0) : H_0(0) \hookrightarrow H_1(0) \hookrightarrow H_2(0) \hookrightarrow \cdots$ of algebras become graded Hopf algebras dual to each other via the pairing defined by $\langle P_\alpha, C_\alpha \rangle := \delta_{\alpha,\beta}$ (Kronecker delta) for all compositions α and β.

Recall that the ring QSym of quasisymmetric functions (or the ring **NSym** of noncommutative symmetric functions, resp.) has a basis consisting of the *fundamental quasisymmetric functions* F_α (or the *noncommutative ribbon Schur functions* $\mathbf{s}_\alpha$, resp.) for all compositions α. Krob and Thibon [15] defined two isomorphisms of graded Hopf algebras, the *quasisymmetric characteristic* $\mathrm{Ch} : G_0(H_*) \to \mathrm{QSym}$ and the *noncommutative characteristic* $\mathbf{ch} : K_0(H_*) \to \mathbf{NSym}$ by $\mathrm{Ch}(C_\alpha) := F_\alpha$ and $\mathbf{ch}(P_\alpha) := \mathbf{s}_\alpha$, respectively.

Let $V = \bigoplus_{d\ge 0} V_d$ be a graded $H_n(0)$-module with finite-dimensional components V_d. It has *graded quasisymmetric characteristic* $\mathrm{Ch}_t(V) := \sum_{d\ge 0} \mathrm{Ch}(V_d) \cdot t^d$. If V is projective, then it has a *graded noncommutative characteristic* $\mathbf{ch}_t(V) := \sum_{d\ge 0} \mathbf{ch}(V_d) \cdot t^d$. Moreover, the *length filtration* $H_n(0)^{(0)} \supseteq H_n(0)^{(1)} \supseteq H_n(0)^{(2)} \supseteq \cdots$, where $H_n(0)^{(\ell)}$ is the span of $\{\pi_w : w \in \mathfrak{S}_n,\ \ell(w) \ge \ell\}$ for all $\ell \ge 0$, induces a filtration for any cyclic $H_n(0)$-module $H_n(0)v$. Thus if V is a direct sum of cyclic $H_n(0)$-modules, then its has a bi-filtration by $V^{(\ell)} \cap V_d$ for $\ell, d \ge 0$. Following earlier work [13], we define the *(length-degree-)bigraded quasisymmetric characteristic* of V below, which specializes to $\mathrm{Ch}_{1,t}(V) = \mathrm{Ch}_t(V)$:

$$\mathrm{Ch}_{q,t}(V) := \sum_{\ell,d\ge 0} \mathrm{Ch}\left((V^{(\ell)} \cap V_d) \big/ (V^{(\ell+1)} \cap V_d) \right) \cdot q^\ell t^d. \tag{2.1}$$

The algebra $H_n(0)$ acts on the polynomial ring $\mathbb{F}[\mathbf{x}_n]$ by the *isobaric Demazure operators*:

$$\pi_i(f) := \frac{x_i f - x_{i+1}(s_i(f))}{x_i - x_{i+1}}, \quad \forall f \in \mathbb{F}[\mathbf{x}_n], \quad 1 \le i \le n-1. \tag{2.2}$$

The quotient algebra $S_{n,k} := \mathbb{F}[\mathbf{x}_n]/J_{n,k}$ defined in Section 1 is a graded $H_n(0)$-module as one can verify that the ideal $J_{n,k}$ is homogeneous and stable under the $H_n(0)$-action on $\mathbb{F}[\mathbf{x}_n]$ using the 'Leibniz Rule' $\overline{\pi}_i(fg) = \overline{\pi}_i(f)g + s_i(f)\overline{\pi}_i(g)$ and the observation that

$\pi_i(h_k(x_1,\ldots,x_i)) = h_k(x_1,\ldots,x_i,x_{i+1})$ for all $i \in [n-1]$. This observation also implies that the span of $\{h_k(x_1,\ldots,x_i) : i \in [n]\}$ is isomorphic to the *defining representation* of $H_n(0)$ on the span of $[n]$ by $\pi_i(i) = i+1$ and $\pi_i(j) = j$ for all $i \in [n-1]$ and $j \in [n] \setminus \{i\}$.

We have $J_{n,1} = \langle x_1, x_2, \ldots, x_n \rangle$, so that $S_{n,1} \cong \mathbb{F}$ is the trivial $H_n(0)$-module in degree 0. It can be shown that $J_{n,n} = I_n$, so that $S_{n,n} = R_n$ (over $\mathbb{F} = \mathbb{Q}$) is the classical coinvariant algebra. The first author [13] proved that R_n is isomorphic to the regular representation of $H_n(0)$ and obtained its length-degree-bigraded quasisymmetric characteristic (with $F_{\mathrm{Des}(w^{-1})} := F_\alpha$ for the composition $\alpha \models n$ satisfying $\mathrm{Des}(\alpha) = \mathrm{Des}(w^{-1})$)

$$\mathrm{Ch}_{q,t}(R_n) = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w)} F_{\mathrm{Des}(w^{-1})}.$$

To study $R_{n,k}$ we need to use ordered set partitions. An *ordered set partition σ* of size n is a set partition of $[n]$ with a total order on its blocks. Let $\mathcal{OP}_{n,k}$ denote the collection of ordered set partitions of size n with k blocks. In particular, we may identify $\mathcal{OP}_{n,n}$ with $\mathfrak{S}_n$. We can write $\sigma \in \mathcal{OP}_{n,k}$ as a permutation in $\mathfrak{S}_n$ with k blocks separated by $k-1$ bars such that letters within each block are increasing and blocks are ordered from left to right. For example, we have $\sigma = (245|6|13) \in \mathcal{OP}_{6,3}$. The *shape* of an ordered set partition $\sigma = (B_1|\cdots|B_k) \in \mathcal{OP}_{n,k}$ is the composition $\alpha = (|B_1|,\ldots,|B_k|) \models n$. For example, $\sigma = (245|6|13)$ has shape $(3,1,2) \models 6$. Given $\alpha \models n$, let $\mathcal{OP}_\alpha$ denote the collection of ordered set partitions of size n with shape α. We can represent $\sigma \in \mathcal{OP}_\alpha$ as the pair (w,α), where $w = w(1)\cdots w(n) \in \mathfrak{S}_n$ is obtained by erasing the bars in σ. For example, $\sigma = (245|6|13) = (245613, (3,1,2))$. This notation establishes a bijection between $\mathcal{OP}_{n,k}$ and pairs (w,α) where $\alpha \models n$, $\ell(\alpha) = k$, $w \in \mathfrak{S}_n$, $\mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha)$.

The algebra $H_n(0)$ acts on the $\mathbb{F}$-vector space $\mathbb{F}[\mathcal{OP}_{n,k}]$ with basis $\mathcal{OP}_{n,k}$ by the rule

$$\overline{\pi}_i.\sigma := \begin{cases} -\sigma, & \text{if } i+1 \text{ appears in a block to the left of } i \text{ in } \sigma, \\ s_i(\sigma) & \text{if } i+1 \text{ appears in a block to the right of } i \text{ in } \sigma. \\ 0, & \text{if } i+1 \text{ appears in the same block as } i \text{ in } \sigma, \end{cases} \tag{2.3}$$

For example, if $\sigma = (25|6|134)$ then $\overline{\pi}_1(\sigma) = -\sigma$, $\overline{\pi}_2(\sigma) = (35|6|124)$, and $\overline{\pi}_3(\sigma) = 0$. This $H_n(0)$-action preserves $\mathbb{F}[\mathcal{OP}_\alpha]$ for each $\alpha \models n$ and turns out to be a special case of an $H_n(0)$-action on *generalized ribbon tableaux* introduced by the first author [14].

We next extend the major index from permutations to ordered set partitions in a 'reverse' way to some other work [12, 17]. For $\sigma = (B_1|\cdots|B_k) = (w,\alpha) \in \mathcal{OP}_{n,k}$, define

$$\mathrm{maj}(\sigma) = \mathrm{maj}(w,\alpha) := \mathrm{maj}(w) + \sum_{i\,:\,\max(B_i) < \min(B_{i+1})} (\alpha_1 + \cdots + \alpha_i - i). \tag{2.4}$$

For example, $\mathrm{maj}(24|57|136|8) = \mathrm{maj}(24571368) + (2-1) + (2+2+3-3) = 4+5 = 9$.

Let rev_q be the operator on polynomials in q that reverses coefficient sequences. For example, $\mathrm{rev}_q(3q^3 + 2q^2 + 1) = q^3 + 2q + 3$. The *$q$-Stirling number* $\mathrm{Stir}_q(n,k)$ is defined by $\mathrm{Stir}_q(0,k) := \delta_{0,k}$ and $\mathrm{Stir}_q(n,k) := \mathrm{Stir}_q(n-1,k-1) + [k]_q \cdot \mathrm{Stir}_q(n-1,k)$ for $n \geq 1$.

Proposition 2.1. *For $n \geq k \geq 1$ we have $\sum_{\sigma \in \mathcal{OP}_{n,k}} q^{\mathrm{maj}(\sigma)} = \mathrm{rev}_q([k]!_q \cdot \mathrm{Stir}_q(n,k))$.*

3 Graded vector space structure

In this section we give the Hilbert series and describe the standard monomial basis for $S_{n,k} := \mathbb{F}[\mathbf{x}_n]/J_{n,k}$ as a graded vector space, where $k \leq n$ are two positive integers.

We endow monomials in $\mathbb{F}[\mathbf{x}_n]$ with the *negative lexicographical term order* $<$ defined by $x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n}$ if and only if there exists $j \in [n]$ such that $a_{j+1} = b_{j+1}, \ldots, a_n = b_n$, and $a_j < b_j$. Following the notation of SAGE, we denote this term order by `neglex`. For any nonzero $f \in \mathbb{F}[\mathbf{x}_n]$, let $\mathrm{in}_<(f)$ be its leading (i.e., largest) term with respect to $<$. The *initial ideal* of an ideal I of $\mathbb{F}[\mathbf{x}_n]$ is the monomial ideal $\mathrm{in}_<(I) := \langle \mathrm{in}_<(f) : f \in I \setminus \{0\} \rangle$. The set of all monomials $m \in \mathbb{F}[\mathbf{x}_n]$ with $m \notin \mathrm{in}_<(I)$ descends to an $\mathbb{F}$-basis for the quotient $\mathbb{F}[\mathbf{x}_n]/I$; this basis is called the *standard monomial basis* [5, Proposition 1, pp. 230].

Following the notion of *skip monomials* in [12, Definition 3.2], we define the *reverse skip monomial* of $S = \{s_1 < \cdots < s_m\} \subseteq [n]$ as $\mathbf{x}(S)^* := x_{n-s_1+1}^{s_1} x_{n-s_2+1}^{s_2-1} \cdots x_{n-s_m+1}^{s_m-m+1}$. For example, $\mathbf{x}(2578)^* = x_8^2 x_5^4 x_3^5 x_2^5$ if $n = 9$. A monomial $m \in \mathbb{F}[\mathbf{x}_n]$ is (n,k)*-reverse nonskip* if $x_i^k \nmid m$ for all $i \in [n]$ and $\mathbf{x}(S)^* \nmid m$ for all $S \subseteq [n]$ with $|S| = n + k - 1$. Let $\mathcal{C}_{n,k}$ be the set of all (n,k)-reverse nonskip monomials in $\mathbb{F}[\mathbf{x}_n]$.

Theorem 3.1. *For any field $\mathbb{F}$, the dimension of $S_{n,k} = \mathbb{F}[\mathbf{x}_n]/J_{n,k}$ is $|\mathcal{OP}_{n,k}|$ and the set $\mathcal{C}_{n,k}$ is the standard monomial basis of $S_{n,k}$ with respect to the* `neglex` *term order on $\mathbb{F}[\mathbf{x}_n]$.*

If $k = n$ then $\mathcal{C}_{n,n}$ consists of 'sub-staircase' monomials $x_1^{a_1} \cdots x_n^{a_n}$ with $0 \leq a_i \leq n - i$ for all $i \in [n]$; this is the basis for the coinvariant algebra R_n obtained by E. Artin [1] using Galois theory. To generalize this 'staircase' characterization to $\mathcal{C}_{n,k}$, we define an (n,k)*-staircase* to be a shuffle of $(k-1, k-2, \ldots, 1, 0)$ and $(k-1, k-1, \ldots, k-1)$, where the second sequence has $n - k$ copies of $k - 1$. For example, the $(5,3)$-staircases are

$$(2,1,0,2,2),\ (2,1,2,0,2),\ (2,2,1,0,2),\ (2,1,2,2,0),\ (2,2,1,2,0),\ \text{and}\ (2,2,2,1,0).$$

The following result follows from Theorem 3.1 and a previous result [12, Theorem 4.13].

Corollary 3.2. *The standard monomial basis $\mathcal{C}_{n,k}$ of $S_{n,k}$ consists of those monomials $x_1^{a_1} \cdots x_n^{a_n}$ whose exponent sequences $(a_1, \ldots, a_n)$ are componentwise $\leq$ some (n,k)-staircase.*

For example, $(4,2)$-staircases are shuffles of $(1,0)$ and $(1,1)$, i.e., $(1,0,1,1), (1,1,0,1)$, and $(1,1,1,0)$, so

$$\mathcal{C}_{4,2} = \{1, x_1, x_2, x_3, x_4, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_1x_3x_4, x_1x_2x_4, x_1x_2x_3\}.$$

Our next result gives a Gröbner basis of $J_{n,k}$. Recall that a finite set $G = \{g_1, \ldots, g_r\}$ of nonzero polynomials in an ideal I of the polynomial ring $\mathbb{F}[\mathbf{x}_n]$ is a *Gröbner basis* of I

if $\text{in}_<(I) = \langle \text{in}_<(g_1), \ldots, \text{in}_<(g_r) \rangle$; this implies $I = \langle G \rangle$. The reader is referred to Cox, Little, and O'Shea [5] for an introduction to Gröbner theory. Given a *weak composition* (i.e., a sequence of nonnegative integers) γ of length n, let $\kappa_\gamma(\mathbf{x}_n) \in \mathbb{F}[\mathbf{x}_n]$ be the associated *Demazure character* (or *key polynomial*); see e.g. [12, Section 2.4]. For any $S \subseteq [n]$, let $\gamma(S)^*$ be the exponent sequence of the reverse skip monomial $\mathbf{x}(S)^*$.

Theorem 3.3. *Let $k \leq n$ be positive integers and endow monomials in $\mathbb{F}[\mathbf{x}_n]$ with the* `neglex` *term order. Then the ideal $J_{n,k}$ has a Gröbner basis consisting of $h_k(x_1, \ldots, x_i)$ for all $i \in [n]$ and $\kappa_{\gamma(S)^*}(\mathbf{x}_n)$ for all $S \subseteq [n-1]$ with $|S| = n - k + 1$. This Gröbner basis is minimal when $k < n$.*

For example, if $(n, k) = (6, 4)$, a (minimal) Gröbner basis of $J_{6,4} \subseteq \mathbb{F}[\mathbf{x}_6]$ consists of the polynomials $h_4(x_1, x_2, \ldots, x_i)$ for $i = 1, 2, \ldots, 6$ and the Demazure characters

$$\kappa_{(0,0,0,1,1,1)}(\mathbf{x}_6),\ \kappa_{(0,0,2,0,1,1)}(\mathbf{x}_6),\ \kappa_{(0,3,0,0,1,1)}(\mathbf{x}_6),\ \kappa_{(0,0,2,2,0,1)}(\mathbf{x}_6),\ \kappa_{(0,3,0,2,0,1)}(\mathbf{x}_6),$$
$$\kappa_{(0,3,3,0,0,1)}(\mathbf{x}_6),\ \kappa_{(0,0,2,2,2,0)}(\mathbf{x}_6),\ \kappa_{(0,3,0,2,2,0)}(\mathbf{x}_6),\ \kappa_{(0,3,3,0,2,0)}(\mathbf{x}_6),\ \kappa_{(0,3,3,3,0,0)}(\mathbf{x}_6).$$

Recall that the *Hilbert series* of a graded vector space $V = \bigoplus_{d \geq 0} V_d$ with each component V_d finite-dimensional is $\text{Hilb}(V; q) := \sum_{d \geq 0} \dim(V_d) \cdot q^d$.

Theorem 3.4. *Let $k \leq n$ be positive integers. We have* $\text{Hilb}(S_{n,k}; q) = \text{rev}_q([k]!_q \cdot \text{Stir}_q(n, k))$.

The Garsia–Stanton basis is another important basis of the coinvariant algebra R_n. For any composition $\alpha \models n$, let $\mathbf{x}_\alpha := \prod_{j \in \text{Des}(\alpha)} (x_1 x_2 \ldots x_j)$. The *Garsia–Stanton monomial* (or *descent monomial*) of a permutation $w \in \mathfrak{S}_n$ is $gs_w := w(\mathbf{x}_\alpha)$, where $\alpha \models n$ is characterized by $\text{Des}(\alpha) = \text{Des}(w)$. By construction the degree of gs_w is $\text{maj}(w)$. Garsia [6] proved that the set $\mathcal{GS}_n := \{gs_w : w \in \mathfrak{S}_n\}$ of all GS monomials descends to a basis of R_n. Garsia and Stanton [7] later studied $\mathcal{GS}_n$ in the context of Stanley–Reisner theory.

The (n, k)-generalization of the GS monomials is as follows. If $\mathbf{i} = (i_1, \ldots, i_n)$ is a sequence of nonnegative integers and $\alpha \models n$ then define $\mathbf{x}_{\alpha,\mathbf{i}} := \mathbf{x}_\alpha \cdot x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$. For any $w \in \mathfrak{S}_n$, let α be the composition of n with $Des(\alpha) = \text{Des}(w)$ and define a monomial $gs_{w,\mathbf{i}} := w(\mathbf{x}_{\alpha,\mathbf{i}}) \in \mathbb{F}[\mathbf{x}_n]$ of degree $\text{maj}(w) + |\mathbf{i}|$, where $|\mathbf{i}| := i_1 + \cdots + i_n$. Let

$$\mathcal{GS}_{n,k} := \{gs_{w,\mathbf{i}} : w \in \mathfrak{S}_n,\ k - \text{des}(w) > i_1 \geq \cdots \geq i_{n-k} \geq 0 = i_{n-k+1} = \cdots = i_n\}. \quad (3.1)$$

Haglund, Rhoades, and Shimozono [12, Theorem 5.3] proved that $\mathcal{GS}_{n,k}$ descends to a basis of $R_{n,k}$. We generalize this result to $S_{n,k}$. Given a monomial $m = x_1^{a_1} \cdots x_n^{a_n}$, let $\lambda(m)$ be the sequence obtained by sorting the exponent sequence $(a_1, \ldots, a_n)$ of m into weakly decreasing order. Let $\prec$ be the partial order on monomials in $\mathbb{F}[\mathbf{x}_n]$ defined by $m \prec m'$ if and only if $\lambda(m) < \lambda(m')$ in lexicographical order. The next result describes a family of sets of polynomials, including $\mathcal{GS}_{n,k}$, which all descend to bases of $S_{n,k}$.

Theorem 3.5. *A set $\mathcal{B}_{n,k} = \{b_{w,\mathbf{i}}\}$ indexed by pairs $(w, \mathbf{i})$ with $w \in \mathfrak{S}_n$, $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, and $k - \text{des}(w) > i_1 \geq \cdots \geq i_{n-k} \geq 0 = i_{n-k+1} = \cdots = i_n$ descends to a basis of $S_{n,k}$ if each $b_{w,\mathbf{i}} \in \mathcal{B}_{n,k}$ satisfies $b_{w,\mathbf{i}} = gs_{w,\mathbf{i}} + \sum_{m \prec gs_{w,\mathbf{i}}} c_m \cdot m$ where $c_m \in \mathbb{F}$ could depend on $(w, \mathbf{i})$.*

In particular, the set $\mathcal{GS}_{n,k}$ descends to a basis of $S_{n,k}$.

The bases given by Theorem 3.5 are important for studying $S_{n,k}$ as an $H_n(0)$-module.

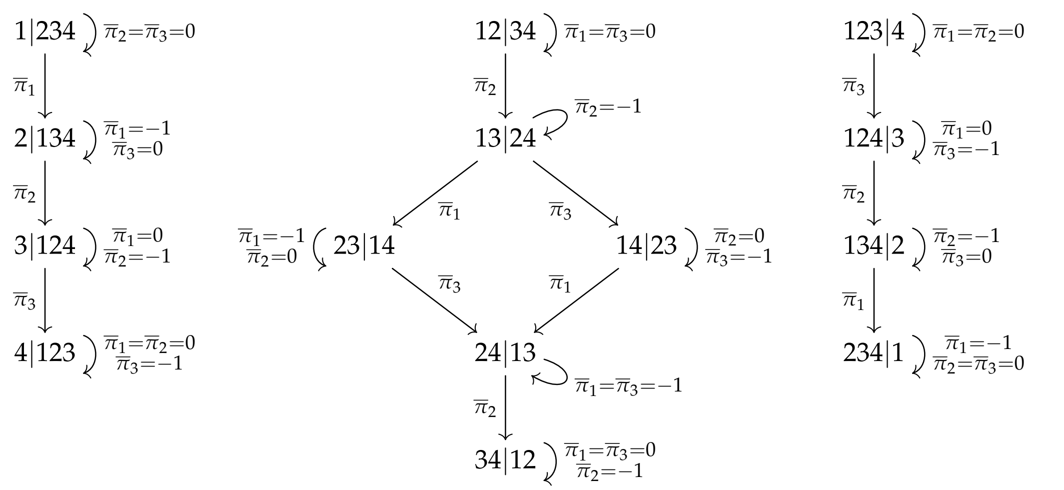
4 Module structure over the 0-Hecke algebra

In this section we study $S_{n,k}$ as a module (ungraded and graded) over the 0-Hecke algebra $H_n(0)$. The ungraded $H_n(0)$-module structure of $S_{n,k}$ is given by the next result.

Theorem 4.1. *Let $k \le n$ be positive integers. As an ungraded $H_n(0)$-module, $S_{n,k}$ is projective and isomorphic to $\mathbb{F}[\mathcal{OP}_{n,k}]$ with the following direct sum decomposition into indecomposables:*

$$S_{n,k} \cong \bigoplus_{\beta \models n} P_\beta^{\oplus \binom{n-\ell(\beta)}{k-\ell(\beta)}}. \tag{4.1}$$

For example, $S_{4,2} \cong P_{(2,2)} \oplus P_{(1,3)} \oplus P_{(3,1)} \oplus P_{(4)}^{\oplus 3} \cong \mathbb{F}[\mathcal{OP}_{4,2}]$. See Figure 1 and 2.



$\mathbb{F}[\mathcal{OP}_{(1,3)}] \cong P_{(4)} \oplus P_{(1,3)}$ $\mathbb{F}[\mathcal{OP}_{(2,2)}] \cong P_{(4)} \oplus P_{(2,2)}$ $\mathbb{F}[\mathcal{OP}_{(3,1)}] \cong P_{(4)} \oplus P_{(3,1)}$

Figure 1: A decomposition of $\mathbb{F}[\mathcal{OP}_{4,2}]$

Proof. (Sketch.) It suffices to show that $\mathbb{F}[\mathcal{OP}_{n,k}]$ and $S_{n,k}$ both have the claimed direct sum decomposition into projective indecomposable modules.

We have a disjoint union decomposition $\mathcal{OP}_{n,k} = \bigsqcup_{\beta \models n, \ell(\beta)=k} \mathcal{OP}_\beta$, giving a direct sum decomposition $\mathbb{F}[\mathcal{OP}_{n,k}] = \bigoplus_{\beta \models n, \ell(\beta)=k} \mathbb{F}[\mathcal{OP}_\beta]$; see Figure 1 when $(n,k) = (4,2)$. One shows that $\mathbb{F}[\mathcal{OP}_\beta] \cong \bigoplus_{\beta \preceq \alpha} P_\alpha$ for any fixed composition β, which leads to the desired decomposition of $\mathbb{F}[\mathcal{OP}_{n,k}]$ into projective indecomposables.

To analyze the 0-Hecke structure of $S_{n,k}$, we use a strategically chosen member of the family of bases of $S_{n,k}$ afforded by Theorem 3.5. Let $A_{n,k}$ be the set of all pairs $(\alpha, \mathbf{i})$, where $\alpha = (\alpha_1, \ldots, \alpha_\ell) \models n$ and $\mathbf{i} = (i_1, \ldots, i_n)$ satisfy $\alpha_1 > n-k$, $i_1, \ldots, i_n \in \mathbb{Z}$, and $k - \ell \ge i_1 \ge \cdots \ge i_{n-k} \ge 0 = i_{n-k+1} = \cdots = i_n$. For each pair $(\alpha, \mathbf{i}) \in A_{n,k}$, the sets $\mathrm{Des}(\alpha)$ and $\mathrm{Des}(\mathbf{i}) := \{1 \le j \le n-1 : i_j > i_{j+1}\}$ are disjoint. The set

$$\{\overline{\pi}_w(\mathbf{x}_{\alpha,\mathbf{i}}) : (\alpha, \mathbf{i}) \in A_{n,k},\, w \in \mathfrak{S}_n,\, \mathrm{Des}(\alpha) \subseteq \mathrm{Des}(w) \subseteq \mathrm{Des}(\alpha) \sqcup \mathrm{Des}(\mathbf{i})\} \tag{4.2}$$

satisfies the conditions of Theorem 3.5, and hence descends to an $\mathbb{F}$-basis of $S_{n,k}$. The $H_n(0)$-action on this basis is easy to describe; see Figure 2 for the case $(n,k) = (4,2)$.

The above basis of $S_{n,k}$ gives a direct sum decomposition $S_{n,k} \cong \bigoplus_{(\alpha,\mathbf{i})\in A_{n,k}} N_{\alpha,\mathbf{i}}$ of $S_{n,k}$ into certain direct summands $N_{\alpha,\mathbf{i}}$ indexed by $(\alpha,\mathbf{i}) \in A_{n,k}$; see Figure 2. The modules $N_{\alpha,\mathbf{i}}$ have explicit decompositions into projective indecomposables; one shows that summing over $(\alpha,\mathbf{i}) \in A_{n,k}$ gives the claimed decomposition of $S_{n,k}$. $\square$

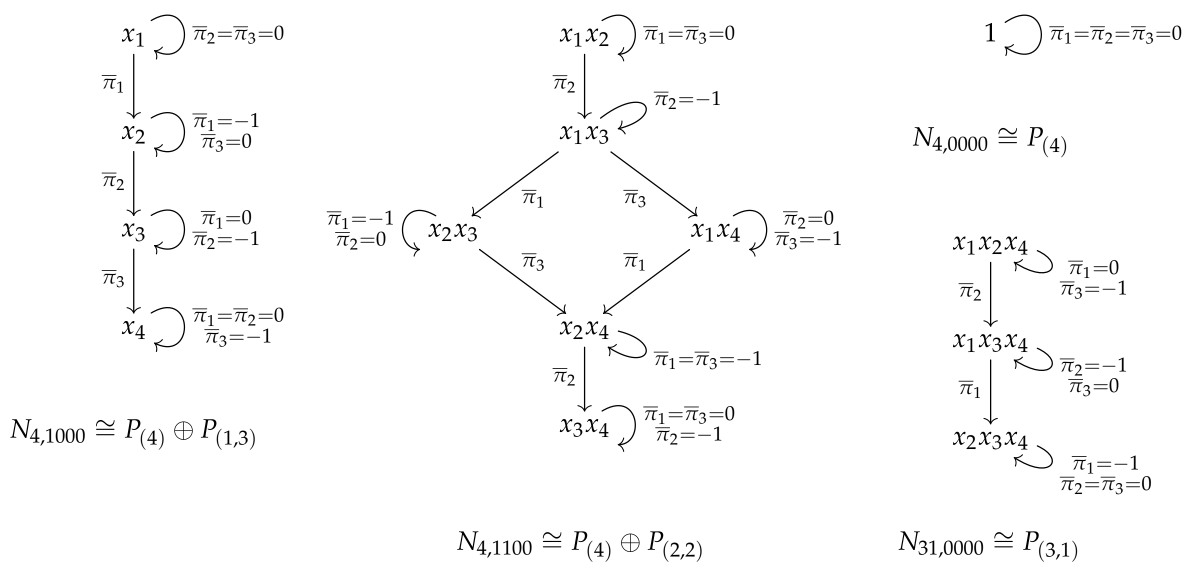


Figure 2: A decomposition of $S_{4,2}$

We next give the graded noncommutative characteristic and bigraded quasisymmetric characteristic of the projective $H_n(0)$-module $S_{n,k}$.

Theorem 4.2. *Let $k \leq n$ be positive integers. We have*

$$\mathbf{ch}_t(S_{n,k}) = \sum_{\alpha\models n} t^{\mathrm{maj}(\alpha)} \begin{bmatrix} n-\ell(\alpha) \\ k-\ell(\alpha) \end{bmatrix}_t \mathbf{s}_\alpha \quad \textit{and} \tag{4.3}$$

$$\mathrm{Ch}_{q,t}(S_{n,k}) = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w)} \begin{bmatrix} n-\mathrm{des}(w)-1 \\ k-\mathrm{des}(w)-1 \end{bmatrix}_t F_{\mathrm{Des}(w^{-1})} \tag{4.4}$$

$$= \sum_{(w,\alpha)\in\mathcal{OP}_{n,k}} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w,\alpha)} F_{\mathrm{Des}(w^{-1})}. \tag{4.5}$$

We define the *length* of $\sigma = (w,\alpha) \in \mathcal{OP}_{n,k}$ to be $\ell(\sigma) := \mathrm{inv}(w)$, since w is the minimal representative of the parabolic coset $w\mathfrak{S}_\alpha = w(\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k})$ corresponding to σ. We have the distributions

$$\sum_{\sigma\in\mathcal{OP}_\alpha} q^{\ell(\sigma)} = \begin{bmatrix} n \\ \alpha_1,\ldots,\alpha_k \end{bmatrix}_q \quad \text{and} \quad \sum_{\sigma\in\mathcal{OP}_{n,k}} q^{\mathrm{maj}(\sigma)} = \mathrm{rev}_q([k]!_q \cdot \mathrm{Stir}_q(n,k)). \tag{4.6}$$

Both distributions equal $[n]!_q$ in the case $k = n$. There is a different extension of the inversion/length statistic on $\mathfrak{S}_n$ to $\mathcal{OP}_{n,k}$ whose distribution is $[k]!_q \cdot \mathrm{Stir}_q(n,k)$ [11, 12, 17, 18, 21]. By Theorem 4.2, $\mathrm{Ch}_{q,t}(S_{n,k})$ is the generating function for the 'biMahonian pair' (ℓ, maj) on $\mathcal{OP}_{n,k}$ with quasisymmetric function weights.

Since $S_{n,k}$ is projective, the graded noncommutative characteristic $\mathrm{Ch}_t(S_{n,k})$ is symmetric. We will expand it in Schur functions. Let $\mathrm{SYT}(n)$ be the set of standard Young tableaux with n boxes. For each $Q \in \mathrm{SYT}(n)$, its *shape* is the corresponding partition $\mathrm{shape}(Q) \vdash n$, its *descent set* $\mathrm{Des}(Q)$ consists of all $i \in [n-1]$ appearing in a row above $i+1$ in Q, and its *major index* is $\mathrm{maj}(Q) := \sum_{i \in \mathrm{Des}(Q)} i$. We also let $\mathrm{des}(Q) := |\mathrm{Des}(Q)|$. The next result follows from Theorem 4.2 and the Robinson–Schensted correspondence.

Corollary 4.3. $\mathrm{Ch}_t(S_{n,k}) = \sum_{Q \in \mathrm{SYT}(n)} t^{\mathrm{maj}(Q)} \begin{bmatrix} n - \mathrm{des}(Q) - 1 \\ k - \mathrm{des}(Q) - 1 \end{bmatrix}_t s_{\mathrm{shape}(Q)}.$

For example, each projective indecomposable P_α in Figure 2 is graded by polynomial degree and corresponds to $\mathbf{s}_\alpha$ and s_α. By Corollary 4.3, the characteristic $\mathrm{Ch}_t(S_{n,k})$ coincides with the Frobenius image of the graded $\mathfrak{S}_n$-module $R_{n,k}$ [12, Corollary 6.13]:

$$\mathrm{Ch}_t(S_{n,k}) = \mathrm{grFrob}(R_{n,k}; t). \tag{4.7}$$

It would be interesting to have a conceptual explanation of this coincidence. One can always use the positive expansion $s_\lambda = \sum_{Q \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(Q)}$ due to Gessel [9] to define a $H_n(0)$-module structure on every $\mathfrak{S}_n$-module, but this is ad-hoc and does not explain such a coincidence for the pair of modules $S_{n,k}$ and $R_{n,k}$.

5 Connection with Macdonald Theory

Our results have the following connection to the theory of Macdonald polynomials. Given a partition μ, let $\widetilde{H}_\mu$ be the corresponding *modified Macdonald symmetric function* and let $B_\mu := \sum q^i t^j$, where (i,j) ranges over the coordinates of the cells of μ. If $F \in \mathrm{Sym}$ is any symmetric function, then the *delta operator* $\Delta'_F : \mathrm{Sym} \to \mathrm{Sym}$ is the Macdonald operator defined (using plethystic notation) by $\Delta'_F : \widetilde{H}_\mu \mapsto F[B_\mu(q,t) - 1] \cdot \widetilde{H}_\mu$.

When $F = e_{k-1}$ is an elementary symmetric function, the *Delta Conjecture* of Haglund, Remmel, and Wilson [11] predicts the monomial expansion of $\Delta'_{e_{k-1}} e_n$: it has the form

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(\mathbf{x}; q, t) = \mathrm{Val}_{n,k}(\mathbf{x}; q, t), \tag{5.1}$$

where $\mathrm{Rise}(\mathbf{x}; q, t)$ and $\mathrm{Val}(\mathbf{x}; q, t)$ are certain combinatorially defined formal power series in the variable set $\mathbf{x} = (x_1, x_2, \dots)$ involving the parameters q and t.

When $k = n$, the Delta Conjecture reduces to the Shuffle Theorem proved recently by Carlsson and Mellit [3]. Although the Delta Conjecture is open for general $k \leq n$, its truth has been established when one of the variables q, t is set to zero by Garsia–Haglund–Remmel–Yoo [8]. Combining this with results of the second author [18] and Wilson [21], we have

$$\begin{aligned} &\Delta'_{e_{k-1}} e_n|_{q=0} = \Delta'_{e_{k-1}} e_n|_{t=0,\, q=t} \\ =&\mathrm{Rise}_{n,k}(\mathbf{x};0,t) = \mathrm{Rise}_{n,k}(\mathbf{x};t,0) = \mathrm{Val}_{n,k}(\mathbf{x};0,t) = \mathrm{Val}_{n,k}(\mathbf{x};t,0). \end{aligned} \tag{5.2}$$

If we let $C_{n,k}(\mathbf{x};t)$ be the common symmetric function in Equation (5.2), then Corollary 4.3 and results from earlier work [11] show that

$$\mathrm{Ch}_t(S_{n,k}) = (\mathrm{rev}_t \circ \omega) C_{n,k}(\mathbf{x};t) \tag{5.3}$$

where ω is the involution on Sym which interchanges e_n and h_n.

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