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A minimaj-preserving crystal structure on ordered multiset partitions

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Abstract. We provide a crystal structure on the set of ordered multiset partitions, which recently arose in the pursuit of the Delta Conjecture. This conjecture was stated by Haglund, Remmel and Wilson as a generalization of the Shuffle Conjecture. Various statistics on ordered multiset partitions arise in the combinatorial analysis of the Delta Conjecture, one of them being the minimaj statistic, which is a variant of the major index statistic on words. Our crystal has the property that the minimaj statistic is constant on connected components of the crystal. In particular, this yields another proof of the Schur positivity of the graded Frobenius series of the generalization $R_{n,k}$ due to Haglund, Rhoades and Shimozono of the coinvariant algebra R_n. The crystal structure also yields a bijective proof of the equidistributivity of the minimaj statistic with the major index statistic on ordered multiset partitions.

Keywords: Delta Conjecture, ordered multiset partitions, minimaj statistic, crystal bases, equidistribution of statistics

1 Introduction

The Shuffle Conjecture [6], now a theorem due to Carlsson and Mellit [3], provides an explicit combinatorial description of the bigraded Frobenius characteristic of the S_n-module of diagonal harmonic polynomials. Recently, Haglund, Remmel and Wilson [4]

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introduced a generalization of the Shuffle Theorem, coined the Delta Conjecture. The Delta Conjecture involves two quasisymmetric functions $\mathsf{Rise}_{n,k}(\mathbf{x};q,t)$ and $\mathsf{Val}_{n,k}(\mathbf{x};q,t)$, which have combinatorial expressions in terms of labelled Dyck paths. In this paper, we are only concerned with the specializations $q=0$ or $t=0$, in which case [4, Theorem 4.1] and [9, Theorem 1.3] show

$$\mathsf{Rise}_{n,k}(\mathbf{x};0,t) = \mathsf{Rise}_{n,k}(\mathbf{x};t,0) = \mathsf{Val}_{n,k}(\mathbf{x};0,t) = \mathsf{Val}_{n,k}(\mathbf{x};t,0).$$

It was proven in [4, Proposition 4.1] that

$$\mathsf{Val}_{n,k}(\mathbf{x};0,t) = \sum_{\pi \in \mathcal{OP}_{n,k+1}} t^{\mathsf{minimaj}(\pi)} \mathbf{x}^{\mathsf{wt}(\pi)}, \tag{1.1}$$

where $\mathcal{OP}_{n,k+1}$ is the set of ordered multiset partitions of the multiset $\{1^{\nu_1}, 2^{\nu_2}, \ldots\}$ into $k+1$ nonempty blocks and $\nu = (\nu_1, \nu_2, \ldots)$ ranges over all weak compositions of n. The weak composition ν is also called the weight of π, denoted $\mathsf{wt}(\pi) = \nu$, and $\mathbf{x}^{\mathsf{wt}(\nu)} = x_1^{\nu_1} x_2^{\nu_2} \cdots$. In addition, $\mathsf{minimaj}(\pi)$ is the minimum value of the major index of the set partition π over all possible ways to order the elements in each block of π. The symmetric function $\mathsf{Val}_{n,k}(\mathbf{x};0,t)$ has an expansion as a sum of Schur functions with coefficients that are polynomials in t with nonnegative integer coefficients [12, 9].

In this paper, we provide a crystal structure on the set of ordered multiset partitions $\mathcal{OP}_{n,k}$. Crystal bases are $q \to 0$ shadows of representations for quantum groups $U_q(\mathfrak{g})$ [7, 8], though they can also be understood from a purely combinatorial perspective [11, 2]. In type A, the character of a connected crystal component with highest weight element of highest weight λ is the Schur function s_λ. Hence, having a crystal structure on a combinatorial set ($\mathcal{OP}_{n,k}$ in our case) naturally yields the Schur expansion of the associated symmetric function. Furthermore, if the statistic ($\mathsf{minimaj}$ in our case) is constant on connected components, then the graded character can be computed using the crystal.

Haglund, Rhoades and Shimozono [5] introduced a generalization $R_{n,k}$ for $k \leqslant n$ of the coinvariant algebra R_n, with $R_{n,n} = R_n$. Just as the combinatorics of R_n is governed by permutations in S_n, the combinatorics of $R_{n,k}$ is controlled by ordered set partitions of $\{1, 2 \ldots, n\}$ with k blocks. The graded Frobenius series of $R_{n,k}$ is (up to a minor twist) equal to $\mathsf{Val}_{n,k}(\mathbf{x};0,t)$. It is still an open problem to find a bigraded S_n-module whose Frobenius image is $\mathsf{Val}_{n,k}(\mathbf{x};q,t)$. Our crystal provides another representation-theoretic interpretation of $\mathsf{Val}_{n,k}(\mathbf{x};0,t)$ as a crystal character.

Wilson [12] analyzed various statistics on ordered multiset partitions, including inv, dinv, maj, and $\mathsf{minimaj}$. In particular, he gave a Carlitz type bijection, which proves equidistributivity of inv, dinv, maj on $\mathcal{OP}_{n,k}$. Rhoades [9] provided a non-bijective proof that these statistics are also equidistributed with $\mathsf{minimaj}$. Using our new crystal, we can give a bijective proof of the equidistributivity of the $\mathsf{minimaj}$ statistic and the maj statistic on ordered multiset partitions.

This extended abstract is organized as follows. In Section 2 we define ordered multiset partitions and the $\mathsf{minimaj}$ and maj statistics on them. In Section 3 we provide a bijection φ from ordered multiset partitions to tuples of semistandard Young tableaux that will be used in Section 4 to define a minimaj-preserving crystal structure. We conclude in Section 5 by showing that the $\mathsf{minimaj}$ and maj statistics are equidistributed using the same bijection φ. We refer the reader to [1] for further details and proofs of our results.

2 Ordered multiset partitions and two statistics

We consider *ordered multiset partitions* of order n with k blocks. Given a weak composition $\nu = (\nu_1, \nu_2, \ldots)$ of n into nonnegative integer parts, which we denote $\nu \models n$, let $\mathcal{OP}_{\nu,k}$ be the set of partitions of the multiset $\{i^{\nu_i} \mid i \geqslant 1\}$ into k nonempty ordered blocks, such that the elements within each block are distinct. The weak composition ν is also called the *weight* $\mathsf{wt}(\pi)$ of $\pi \in \mathcal{OP}_{\nu,k}$. Let

$$\mathcal{OP}_{n,k} = \bigcup_{\nu \models n} \mathcal{OP}_{\nu,k}.$$

The *minimaj order*, first defined in [4], is a particular reading order for an ordered multiset partition $\pi = (\pi_1 \mid \pi_2 \mid \ldots \mid \pi_k) \in \mathcal{OP}_{n,k}$ with blocks π_i, obtained as follows. Start by writing the last block π_k in increasing order, and continue to order the remaining blocks from right to left as follows. Assume π_{i+1} has been ordered and let b_{i+1} be the leftmost letter of that block. If every element of π_i is less than or equal to b_{i+1}, then write π_i in the form $\pi_i = b_i\beta_i$ where $b_i \in \mathbb{Z}_{>0}$ and β_i is an increasing sequence such that $b_i < \beta_i \leq b_{i+1}$. (For two sequences α, β of integers, we write $\alpha < \beta$ to mean that each element of α is less than every element of β.) Otherwise, π_i can be written in the form $\pi_i = b_i\alpha_i\beta_i$ where $b_i \in \mathbb{Z}_{>0}$ and α_i, β_i are (possibly empty) sequences of increasing integers such that $\beta_i \leq b_{i+1} < b_i < \alpha_i$. Continue until every block has been ordered. We consider the last block to be in the form $\pi_k = b_k\alpha_k$. See Example 2.1.

A sequence or word $w_1 w_2 \cdots w_n$ has a *descent* in position $1 \leqslant i < n$ if $w_i > w_{i+1}$. Let $\pi \in \mathcal{OP}_{n,k}$ be in minimaj order. Observe that a descent is either between the largest and smallest elements of π_i or between the last element of π_i and the first element of π_{i+1}. Suppose that π in minimaj order has descents in positions

$$\mathsf{D}(\pi) = \{d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_\ell\}$$

for some $\ell \in [0, k-1]$. The *minimaj statistic* $\mathsf{minimaj}(\pi)$ of $\pi \in \mathcal{OP}_{n,k}$ as given by [4] is

$$\mathsf{minimaj}(\pi) = \sum_{d \in \mathsf{D}(\pi)} d = \sum_{j=1}^{\ell} (\ell + 1 - j) d_j. \tag{2.1}$$

Example 2.1. For $\pi = (157 \mid 24 \mid 56 \mid 468 \mid 13 \mid 123) \in \mathcal{OP}_{15,6}$, the minimaj order of π is $\pi = (571 \mid 24 \mid 56 \mid 468 \mid 31 \mid 123)$. With respect to the minimaj order, we have

$$\begin{array}{lll} b_1 = 5, \alpha_1 = 7, \beta_1 = 1 & b_2 = 2, \alpha_2 = \emptyset, \beta_2 = 4 & b_3 = 5, \alpha_3 = 6, \beta_3 = \emptyset \\ b_4 = 4, \alpha_4 = 68, \beta_4 = \emptyset & b_5 = 3, \alpha_5 = \emptyset, \beta_5 = 1 & b_6 = 1, \alpha_6 = 23, \beta_6 = \emptyset. \end{array}$$

The descents for the multiset partition $\pi = (57.1 \mid 24 \mid 56. \mid 468. \mid 3.1 \mid 123)$ occur at positions $\mathsf{D}(\pi) = \{2, 7, 10, 11\}$ and are designated with periods. Hence $\ell = 4$, $d_1 = 2$, $d_2 = 5$, $d_3 = 3$, $d_4 = 1$ and $d_5 = 4$, and $\mathsf{minimaj}(\pi) = 2 + 7 + 10 + 11 = 30$.

To define the *major index* of $\pi \in \mathcal{OP}_{n,k}$, we consider the word w obtained by ordering each block π_i in decreasing order, called the *major index order* [12]. The last element in the block π_j is assigned the index j, and the remaining elements in π_j are assigned the index $j-1$, for $j = 1, \ldots, k$. Writing the indices from left to right creates a word v. Then

$$\mathsf{maj}(\pi) = \sum_{j \colon w_j > w_{j+1}} v_j. \tag{2.2}$$

Example 2.2. Continuing Example 2.1, note that the major index order of $\pi = (157 \mid 24 \mid 56 \mid 468 \mid 13 \mid 123) \in \mathcal{OP}_{15,6}$ is $\pi = (751 \mid 42 \mid 65 \mid 864 \mid 31 \mid 321)$. Writing the word v underneath w (omitting $v_0 = 0$), we obtain

$$\begin{aligned} w &= 751 \mid 42 \mid 65 \mid 864 \mid 31 \mid 321 \\ v &= 001 \mid 12 \mid 23 \mid 334 \mid 45 \mid 556, \end{aligned}$$

so that $\mathsf{maj}(\pi) = 0 + 0 + 1 + 2 + 3 + 3 + 4 + 4 + 5 + 5 = 27$.

Note that throughout this section, we could have also restricted ourselves to ordered multiset partitions with letters in $\{1, 2, \ldots, r\}$ instead of $\mathbb{Z}_{>0}$. That is, let $\nu = (\nu_1, \ldots, \nu_r)$ be a weak composition of n and let $\mathcal{OP}^{(r)}_{\nu,k}$ be the set of partitions of the multiset $\{i^{\nu_i} \mid 1 \leqslant i \leqslant r\}$ into k nonempty ordered blocks, such that the elements within each block are distinct. Let

$$\mathcal{OP}^{(r)}_{n,k} = \bigcup_{\nu \models n} \mathcal{OP}^{(r)}_{\nu,k}.$$

This restriction will be important when we discuss the crystal structure on ordered multiset partitions.

3 Bijection with tuples of semistandard Young tableaux

In this section, we describe a bijection from ordered multiset partitions to tuples of semistandard Young tableaux that allows us to impose a crystal structure on the set of ordered multiset partitions in Section 4.

Recall that a *semistandard Young tableau* T is a filling of a (skew) Young diagram (also called the *shape* of T) with positive integers that weakly increase across rows and strictly increase down columns. The *weight* of T is the tuple $\mathsf{wt}(T) = (a_1, a_2, \ldots)$, where a_i records the number of letters i in T. The set of semistandard Young tableaux of shape λ, where λ is a (skew) partition, is denoted by $\mathsf{SSYT}(\lambda)$. If we want to restrict the entries in the semistandard Young tableau from $\mathbb{Z}_{>0}$ to a finite alphabet $\{1, 2, \ldots, r\}$, we denote the set by $\mathsf{SSYT}^{(r)}(\lambda)$.

To state our bijection, we need the following notation. For fixed positive integers n and k, assume $\mathrm{D} = \{d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_\ell\} \subseteq \{1, 2, \ldots, n-1\}$ and $\mathrm{I} = \{i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots + i_\ell\} \subseteq \{1, 2, \ldots, k-1\}$ are sets of ℓ distinct elements each. Define $d_{\ell+1} := n - (d_1 + \cdots + d_\ell)$ and $i_{\ell+1} := k - (i_1 + \cdots + i_\ell)$.

Proposition 3.1. *For fixed positive integers n and k and sets D and I as above, let*

$$\mathrm{M}(\mathrm{D}, \mathrm{I}) = \{\pi \in \mathcal{OP}_{n,k} \mid \mathsf{D}(\pi) = \mathrm{D}, \text{ and the descents occur in } \pi_i \text{ for } i \in \mathrm{I}\}.$$

Then the following map is a weight-preserving bijection:

$$\begin{aligned} \varphi \colon \mathrm{M}(\mathrm{D}, \mathrm{I}) &\to \mathsf{SSYT}(1^{c_1}) \times \cdots \times \mathsf{SSYT}(1^{c_\ell}) \times \mathsf{SSYT}(\gamma) \\ \pi &\mapsto T_1 \times \cdots \times T_\ell \times T_{\ell+1} \end{aligned} \tag{3.1}$$

where

(i) $\gamma = (1^{d_1 - i_1}, i_1, \ldots, i_{\ell+1})$ *is of skew ribbon shape and* $c_j = d_{\ell+2-j} - i_{\ell+2-j}$ *for* $1 \leqslant j \leqslant \ell$.

(ii) *The skew ribbon tableau $T_{\ell+1}$ of shape γ is constructed as follows:*

- *The entries in the first column of $T_{\ell+1}$ beneath the first box are the first $d_1 - i_1$ elements of π in increasing order from top to bottom, excluding any b_j in that range.*
- *The remaining rows $d_1 - i_1 + j$ of $T_{\ell+1}$ for $1 \leqslant j \leqslant \ell + 1$ are filled with* $b_{i_1+\cdots+i_{j-1}+1}, b_{i_1+\cdots+i_{j-1}+2}, \ldots, b_{i_1+\cdots+i_j}$.

(iii) *The column tableau T_j for $1 \leqslant j \leqslant \ell$ of shape 1^{c_j} is filled with the elements of π from the positions $d_1 + d_2 + \cdots + d_{\ell-j+1} + 1$ through and including position $d_1 + d_2 + \cdots + d_{\ell-j+2}$, but excluding any b_i in that range.*

Note that in item (ii), the rows of γ are assumed to be numbered from bottom to top and are filled starting with row $d_1 - i_1 + 1$ and ending with row $d_1 - i_1 + \ell + 1$ at the top. Also observe that since the entries of π are mapped bijectively to the entries of $T_1 \times T_2 \times \cdots \times T_{\ell+1}$, the map φ preserves the total weight $\mathsf{wt}(\pi) = (p_1, p_2, \ldots) \mapsto \mathsf{wt}(T)$, where p_i is the number of entries i in π for $i \in \mathbb{Z}_{>0}$, so it can be restricted to a bijection

$$\varphi \colon \mathrm{M}(\mathrm{D}, \mathrm{I})^{(r)} \to \mathsf{SSYT}^{(r)}(1^{c_1}) \times \cdots \times \mathsf{SSYT}^{(r)}(1^{c_\ell}) \times \mathsf{SSYT}^{(r)}(\gamma),$$

where $\mathrm{M}(\mathrm{D}, \mathrm{I})^{(r)} = \mathrm{M}(\mathrm{D}, \mathrm{I}) \cap \mathcal{OP}^{(r)}_{n,k}$.

The next example illustrates the map φ.

Example 3.2. The ordered multiset partition $\pi = (124 \mid 45. \mid 3 \mid 46.1 \mid 23.1 \mid 1 \mid 25) \in \mathcal{OP}_{15,7}$ in minimaj order has the following data: $\ell = 3$, $(d_1, d_2, d_3, d_4) = (5,3,3,4)$ and $(i_1, i_2, i_3, i_4) = (2,2,1,2)$. Then

$$\pi = (124 \mid 45. \mid 3 \mid 46.1 \mid 23.1 \mid 1 \mid 25) \mapsto \begin{ytableau} 1 \\ 5 \end{ytableau} \times \begin{ytableau} 1 \\ 3 \end{ytableau} \times \begin{ytableau} 6 \end{ytableau} \times \begin{ytableau} \none & \none & 1 & 2 \\ \none & \none & 2 \\ \none & 3 & 4 \\ 1 & 4 \\ 2 \\ 4 \\ 5 \end{ytableau}.$$

Outline of the proof of Proposition 3.1. We explain how the map works for $\ell = 0$. In this case, π has no descents and the map φ takes π to the semistandard tableau $T = T_1$ of hook shape $\gamma = (1^{n-k}, k)$ where the first row consists of the numbers $b_1, \ldots, b_k$, and the remainder of the first column is filled by the sequences $\beta_1, \beta_2, \ldots, \beta_{k-1}, \alpha_k$ in that order.

To reverse this map, suppose T is a hook shape tableau with entries $b_1, \ldots, b_k$ in its first row and $b_1, t_1, \ldots, t_{n-k}$ in its first column. The inverse φ^{-1} maps T to the set partition π whose first block is $\pi_1 = b_1\beta_1$ where $\beta_1 = t_1, \ldots, t_{m_1}$ such that $t_1 < \cdots < t_{m_1} \leq b_2$. Continue in this way for each block π_i and set the last block to $\pi_k = b_k\alpha_k$ where $\alpha_k = t_{m_{k-1}+1}, \ldots, t_{n-k}$, so that π is an ordered multiset partition with no descents.

In the general case for $\ell \geqslant 1$, we describe how φ is reversed. The leftmost entry of each block of $\pi = \varphi^{-1}(T)$ is recovered from the entries of the skew ribbon tableau $T_{\ell+1}$ (excluding the bottom $d_1 - i_1$ entries in the first column of γ), read from left to right. Now, the remaining entries in the $(\ell+1)$-tuple of tableaux $T_1 \times \cdots \times T_{\ell+1}$ are arranged in columns, and the last entry in each column is a member of α_j of some block $\pi_j = b_j\beta_j\alpha_j$. The set I specifies that the bottommost entry in $T_{\ell+1-j}$ belongs to the block $\pi_{i_1+\cdots+i_j}$, and that entry is the position where a descent occurs in that block. Finally, since there are no more descent positions in π, there is a unique way to split up the remaining entries in $T_{\ell+1-j}$ by filling the blocks of π from left to right, to complete the reconstruction of the ordered multiset partition $\pi = (\pi_1 \mid \pi_2 \mid \cdots \mid \pi_k)$. $\square$

For a partition λ, the *Schur function* $\mathsf{s}_\lambda(\mathbf{x})$ is defined as

$$\mathsf{s}_\lambda(\mathbf{x}) = \sum_{T \in \mathsf{SSYT}(\lambda)} \mathbf{x}^{\mathsf{wt}(T)}. \tag{3.2}$$

Similarly for $m \geqslant 1$, the *m-th elementary symmetric function* $\mathsf{e}_m(\mathbf{x})$ is given by

$$\mathsf{e}_m(\mathbf{x}) = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_m} x_{j_1} x_{j_2} \cdots x_{j_m}.$$

As an immediate consequence of Proposition 3.1, we have the following symmetric function identity.

Corollary 3.3. *Assume* $\mathrm{D} \subseteq \{1, 2, \dots, n-1\}$ *and* $\mathrm{I} \subseteq \{1, 2, \dots, k-1\}$ *are sets of* ℓ *distinct elements each and let* $\mathrm{M}(\mathrm{D}, \mathrm{I})$*,* γ *and* c_j *for* $1 \leqslant j \leqslant \ell$ *be as in Proposition 3.1. Then*

$$\sum_{\pi \in \mathrm{M}(\mathrm{D},\mathrm{I})} \mathbf{x}^{\mathsf{wt}(\pi)} = \mathsf{s}_{\gamma}(\mathbf{x}) \prod_{j=1}^{\ell} \mathsf{e}_{c_j}(\mathbf{x}).$$

4 Crystal on ordered multiset partitions

4.1 Crystal structure

Denote the set of words of length n over the alphabet $\{1, 2, \dots, r\}$ by $\mathcal{W}_n^{(r)}$. The set $\mathcal{W}_n^{(r)}$ can be endowed with an $\mathfrak{sl}_r$-crystal structure as follows. The weight $\mathsf{wt}(w)$ of $w \in \mathcal{W}_n^{(r)}$ is the tuple $(a_1, \dots, a_r)$, where a_i is the number of letters i in w. The *Kashiwara raising* and *lowering operators*

$$e_i, f_i \colon \mathcal{W}_n^{(r)} \to \mathcal{W}_n^{(r)} \cup \{0\} \qquad \text{for } 1 \leqslant i < r$$

are defined as follows. Associate to each letter i in w a closed parenthesis ")" and to each letter $i+1$ in w an open parenthesis "(". Then e_i changes the $i+1$ associated to the leftmost unmatched "(" to an i; if there is no such letter, $e_i(w) = 0$. Similarly, f_i changes the i associated to the rightmost unmatched ")" to an $i+1$; if there is no such letter, $f_i(w) = 0$.

For λ a (skew) partition, the $\mathfrak{sl}_r$-crystal action on $\mathsf{SSYT}^{(r)}(\lambda)$ is induced by the crystal on $\mathcal{W}_{|\lambda|}^{(r)}$, where $|\lambda|$ is the number of boxes in λ, by considering the row-reading word $\mathsf{row}(T)$ of $T \in \mathsf{SSYT}^{(r)}(\lambda)$, which is the word obtained from T by reading the rows from bottom to top, left to right. In the same spirit, an $\mathfrak{sl}_r$-crystal structure can be imposed on

$$\mathsf{SSYT}^{(r)}(1^{c_1}, \dots, 1^{c_\ell}, \gamma) := \mathsf{SSYT}^{(r)}(1^{c_1}) \times \cdots \times \mathsf{SSYT}^{(r)}(1^{c_\ell}) \times \mathsf{SSYT}^{(r)}(\gamma)$$

by concatenating the reading words of the tableaux in the tuple. This yields crystal operators

$$e_i, f_i \colon \mathsf{SSYT}^{(r)}(1^{c_1}, \dots, 1^{c_\ell}, \gamma) \to \mathsf{SSYT}^{(r)}(1^{c_1}, \dots, 1^{c_\ell}, \gamma) \cup \{0\}.$$

Via the bijection φ of Proposition 3.1, this also imposes crystal operators on ordered multiset partitions

$$\tilde{e}_i, \tilde{f}_i \colon \mathcal{OP}_{n,k}^{(r)} \to \mathcal{OP}_{n,k}^{(r)} \cup \{0\}$$

as $\tilde{e}_i = \varphi^{-1} \circ e_i \circ \varphi$ and $\tilde{f}_i = \varphi^{-1} \circ f_i \circ \varphi$.

An example of a crystal structure on $\mathcal{OP}_{n,k}^{(r)}$ is given in Figure 1.

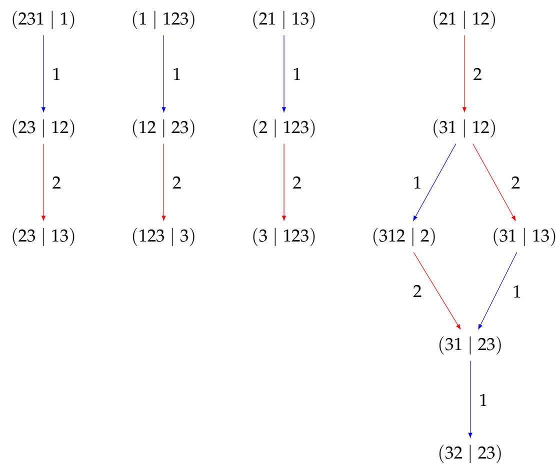

Figure 1: The crystal structure on $\mathcal{OP}^{(3)}_{4,2}$. The minimaj of the connected components are $2, 0, 1, 1$ from left to right.

Theorem 4.1. *The operators $\tilde{e}_i$, $\tilde{f}_i$, and* wt *impose an $\mathfrak{sl}_r$-crystal structure on $\mathcal{OP}^{(r)}_{n,k}$. In addition, $\tilde{e}_i$ and $\tilde{f}_i$ preserve the* minimaj *statistic.*

Proof. By construction, the operators $\tilde{e}_i$, $\tilde{f}_i$, and wt impose an $\mathfrak{sl}_r$-crystal structure since φ is a weight-preserving bijection. The Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ preserve the minimaj statistic, since by Proposition 3.1, the bijection φ restricts to $\mathrm{M(D,I)}^{(r)}$ which fixes the descents of the ordered multiset partitions in minimaj order. □

As shown in [1], the crystal operators $\tilde{e}_i, \tilde{f}_i$ can also be explicitly described on $\mathcal{OP}_{n,k}$.

4.2 Schur expansion

The *character* of a crystal B is defined as $\mathrm{ch} B = \sum_{b \in B} \mathbf{x}^{\mathrm{wt}(b)}$. Denote by $B(\lambda)$ the $\mathfrak{sl}_\infty$-crystal on $\mathsf{SSYT}(\lambda)$ defined above. This is a connected highest weight crystal with highest weight λ, and the character $\mathrm{ch} B(\lambda) = \mathsf{s}_\lambda(\mathbf{x})$ is the Schur function defined in Equation (3.2). Similarly, denoting by $B^{(r)}(\lambda)$ the $\mathfrak{sl}_r$-crystal on $\mathsf{SSYT}^{(r)}(\lambda)$, its character is the Schur polynomial $\mathrm{ch} B^{(r)}(\lambda) = \mathsf{s}_\lambda(x_1, \ldots, x_r)$. Let us define

$$\mathsf{Val}^{(r)}_{n,k}(\mathbf{x}; 0, t) = \sum_{\pi \in \mathcal{OP}^{(r)}_{n,k+1}} t^{\mathsf{minimaj}(\pi)} \mathbf{x}^{\mathsf{wt}(\pi)},$$

which satisfies $\mathsf{Val}_{n,k}(\mathbf{x}; 0, t) = \mathsf{Val}^{(r)}_{n,k}(\mathbf{x}; 0, t)$ for $r \geqslant n$, where $\mathsf{Val}_{n,k}(\mathbf{x}; 0, t)$ is as in (1.1).

As a consequence of Theorem 4.1, we now obtain the Schur expansion of $\mathsf{Val}^{(r)}_{n,k}(\mathbf{x}; 0, t)$.

Corollary 4.2. *We have*

$$\mathsf{Val}^{(r)}_{n,k-1}(\mathbf{x};0,t) = \sum_{\substack{\pi \in \mathcal{OP}^{(r)}_{n,k} \\ \tilde{e}_i(\pi)=0 \ \forall\, 1 \leqslant i < r}} t^{\mathsf{minimaj}(\pi)} \mathsf{s}_{\mathsf{wt}(\pi)}.$$

Example 4.3. The crystal $\mathcal{OP}^{(3)}_{4,2}$, displayed in Figure 1, has four highest weight elements with weights $(2,1,1)$, $(2,1,1)$, $(2,1,1)$, $(2,2)$ from left to right. Hence, we obtain the Schur expansion

$$\mathsf{Val}^{(3)}_{4,1}(\mathbf{x};0,t) = (1+t+t^2)\, \mathsf{s}_{(2,1,1)}(\mathbf{x}) + t\, \mathsf{s}_{(2,2)}(\mathbf{x}).$$

5 Equidistributivity of the minimaj and maj statistics

In this section, we describe a bijection $\psi\colon \mathcal{OP}_{n,k} \to \mathcal{OP}_{n,k}$ in Theorem 5.8 with the property that $\mathsf{minimaj}(\pi) = \mathsf{maj}(\psi(\pi))$ for $\pi \in \mathcal{OP}_{n,k}$. This proves the link between minimaj and maj that was missing in [12]. We can interpret ψ as a crystal isomorphism, where $\mathcal{OP}_{n,k}$ on the left is the minimaj crystal of Section 4 and $\mathcal{OP}_{n,k}$ on the right is viewed as a crystal of k columns with elements written in major index order.

The bijection ψ (see Theorem 5.8) is the composition of φ of Proposition 3.1 with a certain shift operator L (see Definition 5.2). When applying φ to $\pi \in \mathcal{OP}_{n,k}$, we obtain the tuple $T^\bullet = T_1 \times \cdots \times T_{\ell+1}$. We would like to view each column in the tuple of tableaux as a block of a new ordered multiset partition. However, note that some columns could be empty, namely if $c_j = d_{\ell+2-j} - i_{\ell+2-j}$ in Proposition 3.1 is zero for some $1 \leqslant j \leqslant \ell$. For this reason, let us introduce the set of *weak ordered multiset partitions* $\mathcal{WOP}_{n,k}$, where we relax the condition that all blocks need to be nonempty sets.

Define $\mathsf{read}(T^\bullet)$ as the weak ordered multiset partition whose blocks are obtained from $T^\bullet$ by reading the columns from the left to the right and from the bottom to the top. Note that given $\pi = (\pi_1|\pi_2|\cdots|\pi_k) \in \mathcal{OP}_{n,k}$ in minimaj order, $\mathsf{read}(\varphi(\pi))$ is a weak ordered multiset partition in major index order. The map read is invertible.

Example 5.1. For $\pi = (1 \mid 56. \mid 4. \mid 37.12 \mid 2.1 \mid 1 \mid 34) \in \mathcal{OP}_{13,7}$, in minimaj order, we have $\mathsf{minimaj}(\pi) = 22$ and

$$T^\bullet = \varphi(\pi) = \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline \end{array} \times \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \times \begin{array}{|c|}\hline 7 \\ \hline \end{array} \times \varnothing \times \begin{array}{ccc} & 1 & 3 \\ & 2 & \\ & 3 & \\ & 4 & \\ 1 & 5 & \\ 6 & & \end{array}.$$

Then $\pi' = \mathsf{read}(T^\bullet) = (4.1 \mid 2.1 \mid 7. \mid \varnothing \mid 6.1 \mid 5.4.3.2.1 \mid 3)$.

Definition 5.2. We define the *left shift operation* L on $\pi' \in \mathcal{I} = \{\mathsf{read}(\varphi(\pi)) \mid \pi \in \mathcal{OP}_{n,k}\}$ as follows. Suppose π' has $m \geqslant 0$ blocks $\pi'_{p_m}, \ldots, \pi'_{p_1}$ that are either empty or have a descent at the end, and $1 \leqslant p_m < \cdots < p_2 < p_1 < k$. Set $\mathsf{L}(\pi') = \mathsf{L}^{(m)}(\pi')$, where $\mathsf{L}^{(i)}$ for $0 \leqslant i \leqslant m$ are defined as follows:

1. Set $\mathsf{L}^{(0)}(\pi') = \pi'$.
2. Suppose $\mathsf{L}^{(i-1)}(\pi')$ for $1 \leqslant i \leqslant m$ is defined. By induction, the p_i-th block of $\mathsf{L}^{(i-1)}(\pi')$ is π'_{p_i}. Let S_i be the sequence of elements starting immediately to the right of block π'_{p_i} in $\mathsf{L}^{(i-1)}(\pi')$ up to and including the p_i-th descent after the block π'_{p_i}. Let $\mathsf{L}^{(i)}(\pi')$ be the weak ordered multiset partition obtained by moving each element in S_i one block to its left.

Example 5.3. Continuing Example 5.1, we have $\pi' = (4.1 \mid 2.1 \mid 7. \mid \emptyset \mid 6.1 \mid 5.4.3.2.1 \mid 3)$, in major index order. We have $m = 2$ with $p_2 = 3 < 4 = p_1$, $S_1 = 61543$, $S_2 = 6154$ and

$$\mathsf{L}^{(1)}(\pi') = (4.1 \mid 2.1 \mid 7. \mid 6.1 \mid 5.4.3. \mid 2.1 \mid 3),$$
$$\mathsf{L}(\pi') = \mathsf{L}^{(2)}(\pi') = (4.1 \mid 2.1 \mid 7.6.1 \mid 5.4. \mid 3. \mid 2.1 \mid 3).$$

Note that $\mathsf{maj}(\pi') = 28$, $\mathsf{maj}(\mathsf{L}^{(1)}(\pi')) = 25$, and $\mathsf{maj}(\mathsf{L}(\pi')) = 22 = \mathsf{minimaj}(\pi)$.

Proposition 5.4. *The left shift operation* $\mathsf{L}\colon \mathcal{I} \to \mathcal{OP}_{n,k}$ *is well defined.*

Definition 5.5. We define the *right shift operation* R on $\mu \in \mathcal{OP}_{n,k}$ in major index order as follows. Suppose μ has $m \geqslant 0$ blocks $\mu_{q_1}, \ldots, \mu_{q_m}$ that have a descent at the end and $q_1 < q_2 < \cdots < q_m$. Set $\mathsf{R}(\mu) = \mathsf{R}^{(m)}(\mu)$, where $\mathsf{R}^{(i)}$ for $0 \leqslant i \leqslant m$ are defined as follows:

1. Set $\mathsf{R}^{(0)}(\mu) = \mu$.
2. Suppose $\mathsf{R}^{(i-1)}(\mu)$ for $1 \leqslant i \leqslant m$ is defined. Let U_i be the sequence of q_i elements to the left of, and including, the last element in the q_i-th block of $\mathsf{R}^{(i-1)}(\mu)$. Let $\mathsf{R}^{(i)}(\mu)$ be the weak ordered multiset partition obtained by moving each element in U_i one block to its right. Note that all blocks to the right of the $(q_i + 1)$-th block are the same in μ and $\mathsf{R}^{(i)}(\mu)$.

Example 5.6. Continuing Example 5.3, let $\mu = \mathsf{L}(\pi') = (4.1 \mid 2.1 \mid 7.6.1 \mid 5.4. \mid 3. \mid 2.1 \mid 3)$. We have $m = 2$ with $q_1 = 4 < 5 = q_2$, $U_1 = 6154$, $U_2 = 61543$ and

$$\mathsf{R}^{(1)}(\mu) = (4.1 \mid 2.1 \mid 7. \mid 6.1 \mid 5.4.3. \mid 2.1 \mid 3),$$
$$\mathsf{R}(\mu) = \mathsf{R}^{(2)}(\mu) = (4.1 \mid 2.1 \mid 7. \mid \emptyset \mid 6.1 \mid 5.4.3.2.1 \mid 3),$$

which is the same as π' in Example 5.3.

Proposition 5.7. *The right shift operation* R *is well defined and is the inverse of* L.

We extend the definition of the major index in a natural way to the set $\mathcal{WOP}_{n,k}$: the last element in a nonempty block π'_j is assigned the index j, and the remaining elements in π'_j are assigned the index $j-1$, for $j = 1, \ldots, k$ where $\pi'_j \neq \emptyset$.

Theorem 5.8. *Let* $\psi \colon \mathcal{OP}_{n,k} \to \mathcal{OP}_{n,k}$ *be the map defined by*

$$\psi(\pi) = \mathsf{L}(\mathsf{read}(\varphi(\pi))) \qquad \text{for } \pi \in \mathcal{OP}_{n,k} \text{ in minimaj order.}$$

Then ψ *is a bijection that maps ordered multiset partitions in minimaj order to ordered multiset partitions in major index order. Furthermore,* $\mathsf{minimaj}(\pi) = \mathsf{maj}(\psi(\pi))$.

Proof outline. The map ψ is a bijection since each of the maps φ, read, and L is invertible, so it remains to show that $\mathsf{minimaj}(\pi) = \mathsf{maj}(\psi(\pi))$ for $\pi \in \mathcal{OP}_{n,k}$ in minimaj order.

In the simple case that $\pi' = \mathsf{read}(\varphi(\pi))$ has no empty blocks and no descents at the end of any block, then $\mathsf{L}(\pi') = \pi'$, so that $\pi' = \psi(\pi)$. Then

$$\mathsf{maj}(\pi') = \sum_{j=1}^{\ell} (\ell + 1 - j)(d_j - i_j - 1) + \ell + \sum_{j=1}^{\ell} (\ell + \eta_j - j), \tag{5.1}$$

where $d_j, i_j, \eta_j = i_1 + \cdots + i_j$ are defined in Proposition 3.1 for π. Comparing with (2.1),

$$\mathsf{maj}(\pi') = \mathsf{minimaj}(\pi) - \binom{\ell+1}{2} - \sum_{j=1}^{\ell} (\ell + 1 - j) i_j + \binom{\ell+1}{2} + \sum_{j=1}^{\ell} \eta_j = \mathsf{minimaj}(\pi),$$

proving the claim.

In the general case that $\pi' = \mathsf{read}(\varphi(\pi))$ has a descent at the end of block π'_p (respectively if $\pi'_p = \emptyset$), this will contribute an extra p to the major index in (5.1) (respectively $p-1$). Hence, with the notation of Definition 5.2, we have $\mathsf{maj}(\pi') = \mathsf{minimaj}(\pi) + \sum_{i=1}^{m} p_i - e$, where e is the number of empty blocks in π'. Since $\psi(\pi) = \mathsf{L}(\pi')$, then noting that we have for $1 \leqslant i \leqslant m$

$$\mathsf{maj}(\mathsf{L}^{(i)}(\pi')) = \begin{cases} \mathsf{maj}(\mathsf{L}^{(i-1)}(\pi')) - p_i + 1, & \text{if } \pi'_{p_i} = \emptyset, \\ \mathsf{maj}(\mathsf{L}^{(i-1)}(\pi')) - p_i, & \text{if } \pi'_{p_i} \text{ has a descent at the end of its block,} \end{cases}$$

the claim follows. □

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