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Cyclically Symmetric Lozenge Tilings of a Hexagon with Four Holes

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Abstract. Mills, Robbins, and Rumsey's work on cyclically symmetric plane partitions yields a simple product formula for the number of lozenge tilings of a regular hexagon, which are invariant under rotation by 120°. In this extended abstract, we generalize this result by enumerating the cyclically symmetric lozenge tilings of a hexagon in which four triangles have been removed in a symmetric way.

Keywords: perfect matchings, plane partitions, lozenge tilings, graphical condensation

1 Introduction

A *plane partition* can be defined to be a rectangular array of non-negative integers with weakly decreasing rows and columns. Plane partitions with bounded entries are in bijection with lozenge tilings of a hexagon on the triangular lattice. Here a *lozenge* is a union of any two unit equilateral triangles sharing an edge, and a *lozenge tiling* of a region is a covering of the region by lozenges so that there are no gaps or overlaps. MacMahon [10] proved that these lozenge tilings are enumerated by

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}. \tag{1.1}$$

It has been shown that 10 symmetry classes of lozenge tilings of a hexagon are all given by simple product formulas (see e.g. the classical paper by Stanley [11], or the excellent survey by Krattenthaler [7]). Macdonald [9] conjectured a q-enumeration for the *cyclically symmetric tilings* (i.e. the tilings invariant under 120° rotation) of a regular hexagon. Andrew succeeded in proving the $q = 1$ case of the conjecture [1]; the full conjecture was later proved by Mills, Robbins, and Rumsey [12]. This result implies the following formula for number of the cyclically symmetric tilings of a regular hexagon of side a:

$$\prod_{i=1}^{a}\left(\frac{3i-1}{3i-2}\prod_{j=i}^{a}\frac{a+i+j-1}{2i+j-1}\right). \tag{1.2}$$

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Generalizing MacMahon's classical tiling formula (1.1) is an important topic in the study of plane partitions and the study of enumeration of tilings. A natural way to generalize MacMahon's tiling formula is to enumerate lozenge tilings of a hexagon with certain 'defects'. In particular, we are interested in hexagons with several triangles removed. Even though the enumeration of tilings of defected hexagons has been investigated extensively, the study of their cyclically symmetric tilings is very limited. One of such a few results is Ciucu and Krattenthaler's formula for the number of cyclically symmetric tilings of a hexagon with a triangle removed in the center [5]. It is worth noticing that Krattenthaler [6] proved an one-to-one correspondence between the lozenge tilings in Ciucu and Krattenthaler's formula (with the central triangular hole of size 2) and the *descending plane partitions*.

This paper is devoted to the study of cyclically symmetric lozenge tilings of two new families of defected hexagons as follows. Unlike most known defects in hexagons that are either a single triangle or a cluster of adjacent or aligned triangles, the defects in our regions are four non-aligned, non-adjacent triangular holes.

Let x, y, z, a be non-negative integers. Our first family of defected hexagons consists of the hexagons with side-lengths[1] $t + x + 3a,\ t,\ t + x + 3a,\ t,\ t + x + 3a,\ t$, where an up-pointing triangle of side-length x has been removed from the center, and three satellite up-pointing triangles of side-length a have been removed equally along the three intervals connecting the center of the hexagon to the midpoints of its southern, northeastern, and northwestern sides. In addition, we set to $2y$ the distance from the central triangular hole to each of three satellite holes. Denote by $\mathcal{H}_{t,y}(a, x)$ the resulting defected hexagon (see Figure 1 for an example; the black triangles indicate the triangles that have been removed). The second family, denoted by $\overline{\mathcal{H}}_{t,y}(a, x)$, is similar to the first one, the only difference is that the satellite holes are now on the opposite side of the central hole as in Figure 2.

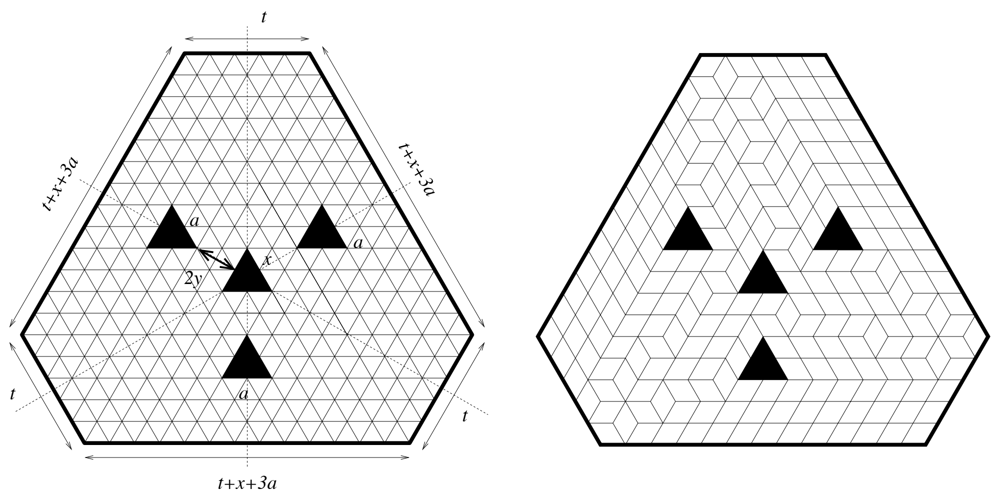
As in the case of the ordinary hexagons, we are interested in cyclically symmetric tilings of the defected hexagons $\mathcal{H}_{t,y}(a, x)$ and $\overline{\mathcal{H}}_{t,y}(a, x)$ (see Figures 1(b) and 2(b) for examples).

Theorem 1.1. *The number of cyclically symmetric lozenge tilings of the hexagon with four holes $\mathcal{H}_{t,y}(a, x)$ is given by a simple product formula when x is even.*

Theorem 1.2. *The number of cyclically symmetric lozenge tilings of the hexagon with four holes $\overline{\mathcal{H}_{t,y}}(a, x)$ is given by a simple product formula when a is even.*

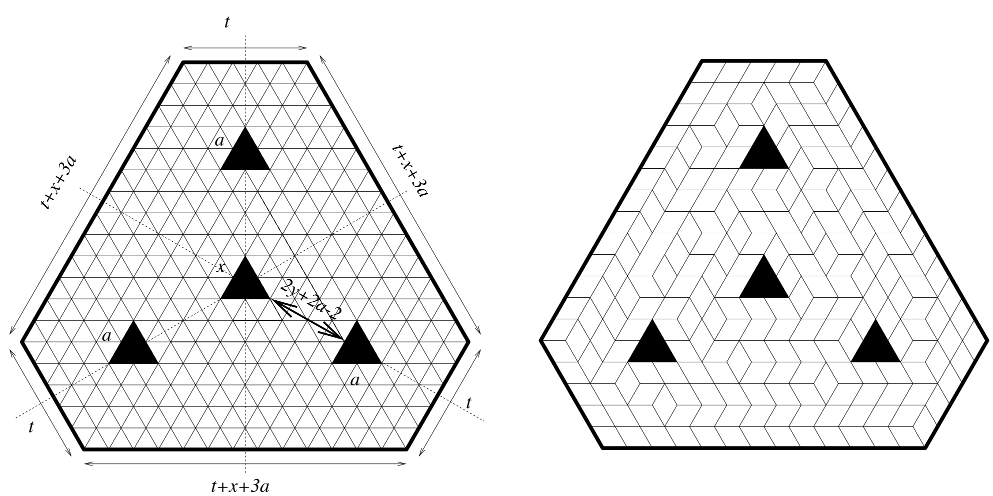
The rest of this extended abstract is organized as follows. In Section 2, we state precisely our main result. Then we present a sketched proof of the main result in Section 3.

[1] From now on, we always list the side-lengths of a hexagon in clockwise order, starting from the northwestern side.



(a) (b)

Figure 1: (a) The hexagon with four holes $\mathcal{H}_{5,1}(2,2)$. (b) A cyclically symmetric tiling of $\mathcal{H}_{5,1}(2,2)$.



(a) (b)

Figure 2: (a) The hexagon with four holes $\overline{\mathcal{H}}_{5,1}(2,2)$. (b) A cyclically symmetric tiling of $\overline{\mathcal{H}}_{5,1}(2,2)$.

2 Precise statement of the main result

In this section, we show an explicit formula for the number of cyclically symmetric tilings of $\mathcal{H}_{t,y}(a,x)$ in Theorem 1.1. The explicit tiling formula for Theorem 1.2 is similar, and will be omitted here.

For non-negative integer n, the *Pochhammer symbol* $(x)_n$ is defined by

$$(x)_n = \begin{cases} x(x+1)\dots(x+n-1) & \text{if } n>0; \\ 1 & \text{if } n=0; \\ \frac{1}{(x-1)(x-2)\dots(x+n)} & \text{if } n<0, \end{cases} \tag{2.1}$$

and its “skipping” version is

$$[x]_n = \begin{cases} x(x+2)(x+4)\dots(x+2(n-1)) & \text{if } n>0; \\ 1 & \text{if } n=0; \\ \frac{1}{(x-2)(x-4)\dots(x+2n)} & \text{if } n<0. \end{cases} \tag{2.2}$$

Next, we define four products as follows:

$$P_1(x,y,z,a) := \frac{1}{2^{y+z}} \prod_{i=1}^{y+z} \frac{(2x+6a+2i)_i[2x+6a+4i+1]_{i-1}}{(i)_i[2x+6a+2i+1]_{i-1}} \times \tag{2.3}$$
$$\prod_{i=1}^{a} \frac{(z+i)_{y+a-2i+1}(x+y+2z+2a+2i)_{2y+2a-4i+2}(x+3i-2)_{y-i+1}(x+3y+2i-1)_{i-1}}{(i)_y(y+2z+2i-1)_{y+2a-4i+3}(2z+2i)_{y+2a-4i+1}(x+y+z+2a+i)_{y+a-2i+1}}.$$

$$P_2(x,y,z,a) := \frac{[x+3y]_a(x+2y+z+2a)_a}{2^{2(ay+z)}[x+3y+2z+2a+1]_a} \prod_{i=1}^{y+z} \frac{(2x+6a+2i-2)_{i-1}[2x+6a+4i-1]_i}{(i)_i[2x+6a+2i-1]_{i-1}} \times$$
$$\prod_{i=1}^{a} \frac{(z+i)_{y+a-2i+1}(x+y+2z+2a+2i-1)_{2y+2a-4i+3}(x+3i-2)_{y-i}(x+3y+2i-1)_{i-1}}{(i)_y(y+2z+2i-1)_{y+2a-4i+3}(2z+2i)_{y+2a-4i+1}(x+y+z+2a+i-1)_{y+a-2i+2}}. \tag{2.4}$$

$$\begin{aligned} F_1(x,y,z,a) = {} & \frac{1}{2^{ya+z}} \frac{[x+y+2z+2a+1]_y}{[x+y+2a-1]_y} \frac{\prod_{i=1}^{\lfloor \frac{a}{3} \rfloor}(x+3y+6i-3)_{3a-9i+1}}{\prod_{i=1}^{\lfloor \frac{a-1}{3} \rfloor}(x+3y+6i-2)} \\ & \times \prod_{i=1}^{y+z} \frac{i!(x+3a+i-3)!(2x+6a+2i-4)_i(x+3a+2i-2)_i(2x+6a+3i-4)}{(x+3a+2i-2)!(2i)!} \\ & \times \prod_{i=1}^{a-1}(x+3i-2)_{y-i+1}(x+y+2z+2a+2i)_{2y+2a-4i} \\ & \times \prod_{i=1}^{y} \frac{[2i+3]_{z-1}(x+3a+3i-5)_{2y+z-a-4i+5}}{(a+i+1)_{z-1}(i)_{a+1}[2i+3]_{a-2}[2x+6a+6i-7]_{z+2y-4i+3}}. \end{aligned} \tag{2.5}$$

$$F_2(x,y,z,a) = \frac{1}{2^{y(a+2)+2a+z+1}} \frac{\prod_{i=1}^{\lfloor \frac{y+1}{3} \rfloor}(x+3i-2)_{3y-9i+4}}{\prod_{i=1}^{\lfloor \frac{y}{3} \rfloor}(x+3y-6i)}$$
$$\times \prod_{i=1}^{y+z} \frac{i!(x+3a+i-1)!(2x+6a+2i)_i(x+3a+2i)_i(x+3a+3i)}{(x+3a+2i)!(2i)!} \tag{2.6}$$
$$\times \prod_{i=1}^{y} \frac{[2i+3]_{z-1}(x+y+2a+2i-1)_{y+z-3i+2}(x+y+2z+2a+2i)_{2y+2a-4i+3}}{(a+i+2)_{z-1}(i)_{a+2}[2i+3]_{a-1}[2x+6a+6i-1]_{2y+z-4i+2}}.$$

The number of cyclically symmetric tilings of $\mathcal{H}_{t,y}(a,x)$ is given by a simple product formula when x is even. However, for odd x, the region does *not* yield a simple product formula.

Theorem 2.1. *For non-negative integers y,t,a,x*

$$\mathrm{CS}(\mathcal{H}_{2t+1,y}(2a,2x)) = 2^{2t+4a+1}P_1(x+1,y,t-y,a)P_2(x+1,y,t-y,a), \tag{2.7}$$

$$\mathrm{CS}(\mathcal{H}_{2t,y}(2a,2x)) = 2^{2t+4a}P_1(x+1,y,t-y-1,a)P_2(x+1,y,t-y,a), \tag{2.8}$$

$$\mathrm{CS}(\mathcal{H}_{2t+1,y}(2a+1,2x)) = 2^{2t+4a+3}F_1(x+1,y,t-y,a+1)F_2(x+1,y,t-y,a), \tag{2.9}$$

$$\mathrm{CS}(\mathcal{H}_{2t,y}(2a+1,2x)) = 2^{2t+4a+2}F_1(x+1,y,t-y-1,a+1)F_2(x+1,y,t-y,a), \tag{2.10}$$

where we use the notation $\mathrm{CS}(R)$ *for the number of cyclically symmetric tilings of a region R.*

3 Sketched Proof of the main result

A *(perfect) matching* of a graph is a collection of vertex-disjoint edges covering all the vertices of the graph. We use the notation $\mathrm{M}(G)$ for the number of matchings of G, or the weighted sum of the matchings in the weighted case. Here the *weight* of a matching is the product of weights of its constituent edges.

The lozenge tilings of a region (on the triangular lattice) are in bijection with matchings of its *(planar) dual graph* (the graph whose vertices are unit equilateral triangles in the region and whose edges connect precisely two unit equilateral triangles sharing an edge). In this point of view, we use the notation $\mathrm{M}(R)$ for the number (or weighted sum in the weighted case) of tilings of a region R.

One of our main tools is the following powerful *graphical condensation* of Kuo [8], that is usually referred to as *Kuo condensation*, and a factorization due to Ciucu [2].

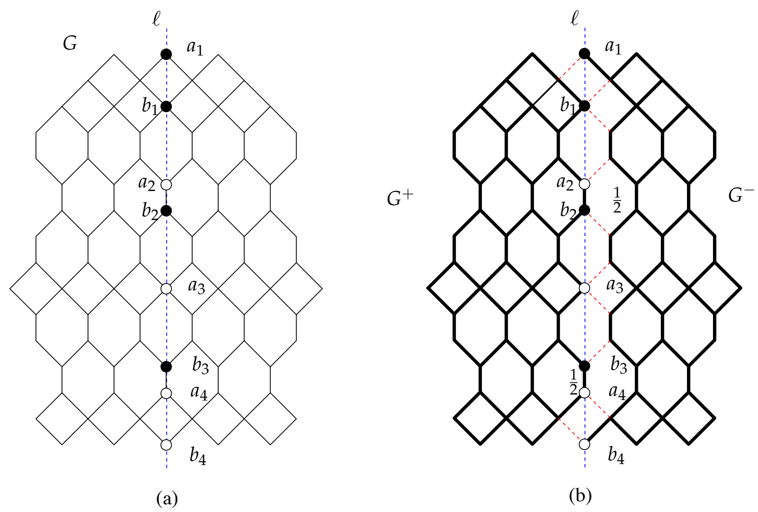


Figure 3: An illustration of Ciucu's factorization theorem. The removed edges are given by the dotted lines to the right and left of ℓ.

Lemma 3.1 (Kuo Condensation [8])**.** *Assume that $G = (V_1, V_2, E)$ is a weighted bipartite planar graph with $|V_1| = |V_2| + 1$. Assume that u, v, w, s are four vertices appearing in this cyclic order on a face of G, such that $u, v, w \in V_1$ and $s \in V_2$. Then*

$$\begin{aligned} \mathrm{M}(G - \{v\})\,\mathrm{M}(G - \{u, w, s\}) = &\,\mathrm{M}(G - \{u\})\,\mathrm{M}(G - \{v, w, s\}) \\ &+ \mathrm{M}(G - \{w\})\,\mathrm{M}(G - \{u, v, s\}). \end{aligned} \tag{3.1}$$

Lemma 3.2 (Ciucu's Factorization Theorem [2])**.** *Let $G = (V_1, V_2, E)$ be a weighted bipartite planar graph with a vertical symmetry axis ℓ. Assume that $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ are all the vertices of G on ℓ appearing in this order from top to bottom*[2]*. Assume in addition that the vertices of G on ℓ form a cut set of G (i.e. the removal of those vertices separates G into two disjoint subgraphs). We reduce the weights of all edges of G lying on ℓ by half and keep the other edge-weights unchanged. Next, we color the two vertex classes of G by black and white, without loss of generality, assume that a_1 is black. Finally, we remove all edges on the left of ℓ which are adjacent to a black a_i or a white b_j; we also remove the edges on the right of ℓ which are adjacent to a white a_i or a black b_j. This way, G is divided into two disjoint weighted graphs G^+ and G^- (on the left and right of ℓ, respectively). Then*

$$\mathrm{M}(G) = 2^k\,\mathrm{M}(G^+)\,\mathrm{M}(G^-). \tag{3.2}$$

See Figure 3 for an example of the construction of weighted graphs G^+ and G^-.

The enumeration of the following regions will be employed in our proof. We start with a pentagonal region whose northern, northeastern, southeastern, southern sides

[2]It is easy to see that if G admits a perfect matching, then G has an even number of vertices on ℓ.

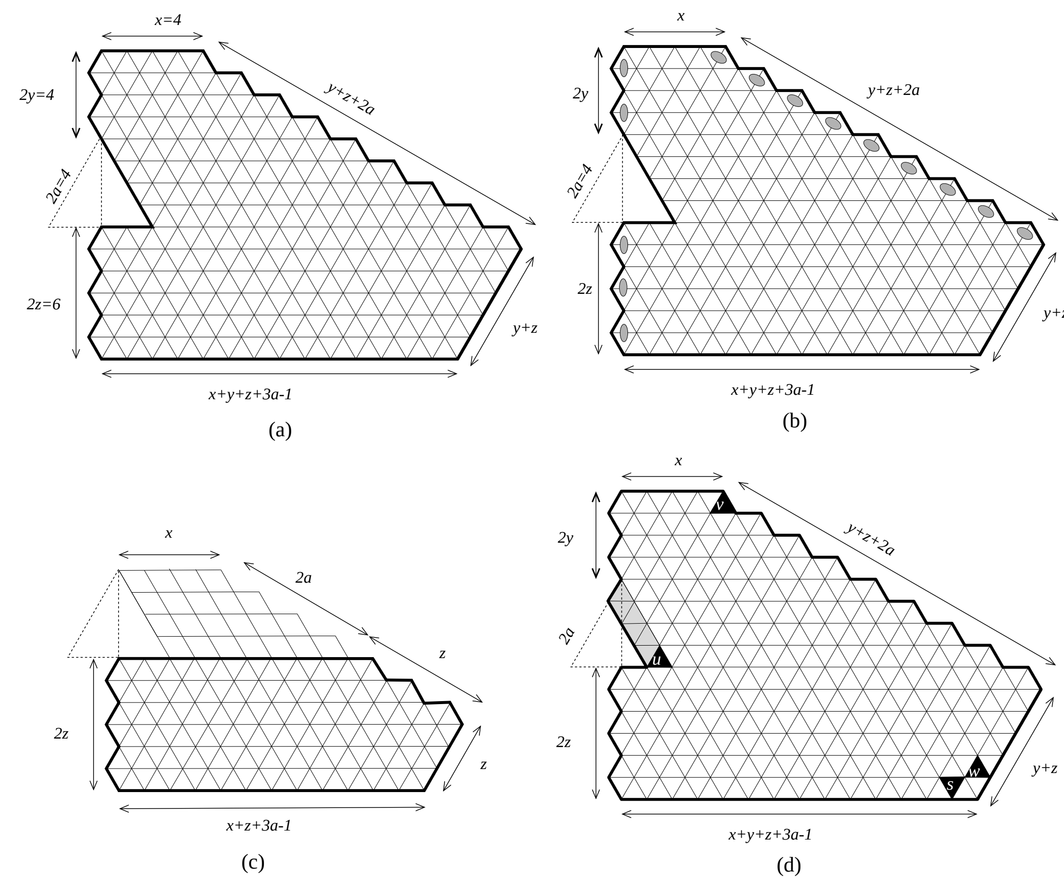


Figure 4: (a) The region $\mathcal{R}_{4,2,3}(2)$, (b) The region ${}^{*}_{*}\mathcal{R}_{4,2,3}(2)$; the lozenges with shaded cores have weight $1/2$. (c) How to apply Kuo condensation to an $\mathcal{R}$-type region. (d) The region $\mathcal{R}_{4,0,3}(2)$ with forced lozenges.

have lengths $x,\ y+z+2a,\ y+z,\ x+y+z+3a+1$, respectively, and the western side follows a zigzag lattice path with length $2y+2a+2z$. We remove a half triangle of side $2a$ at level $2z$ from the western side. Denote by $\mathcal{R}_{x,y,z}(a)$ the resulting region (see Figure 4(a)). We are also interested in a weighted variation ${}^{*}_{*}\mathcal{R}_{x,y,z}(a)$ of $\mathcal{R}_{x,y,z}(a)$, that are obtained by assigning weight $1/2$ to lozenges along the western and northeastern sides (see Figures 4(b)). Each tiling of ${}^{*}_{*}\mathcal{R}_{x,y,z}(a)$ is weighted by $1/2^n$, where n is the number of the latter weighted lozenges in the tiling.

Lemma 3.3. *For any non-negative integers x, y, z, a*

$$\mathrm{M}(\mathcal{R}_{x,y,z}(a)) = P_1(x,y,z,a) \tag{3.3}$$

and

$$\mathrm{M}({}^{*}_{*}\mathcal{R}_{x,y,z}(a)) = P_2(x,y,z,a), \tag{3.4}$$

where $P_1(x,y,z,a)$ and $P_2(x,y,z,a)$ are defined in (2.3) and (2.4), respectively.

Proof. We only proof here (3.3), as (3.4) can be obtained in an analogous manner.

We prove (3.3) by induction on $z+a$. The base cases are the situations when at least one of the parameters y, z, a is equal to 0.

The case $a=0$ was proved in [3, Proposition 3.1], and the case $z=0$ was enumerated in [4, Theorem 1.1]. Finally, if $y=0$, our region has several lozenges that are forced to be in any tilings. By removing these forced lozenges, we get back a region in the case $a=0$ (see the region restricted by the bold contour in Figure 4(c)).

For the induction step, we assume that $y,z,a>0$ and that equation (3.3) holds for any $\mathcal{R}$-type regions in which the sum of the z- and a-parameters is strictly less then $z+a$. We apply Kuo condensation (Lemma 3.1) to the dual graph G of the region $\mathcal{R}$ obtained from $\mathcal{R}_{x,y,z}(a)$ by adding a band of unit triangles along the side of the semi-triangular hole (see the shaded band in Figure 4(d)). The four vertices u,v,w,s in Lemma 3.1 correspond to the black triangles in the figure.

We consider the region corresponding to the graph $G-\{v\}$. The removal of the v-triangle yields several forced lozenges on the top of the region. By removing these forced lozenges, we obtain the region $\mathcal{R}_{x+3,y,z}(a-1)$ (see the region restricted by the bold contour in Figure 5(a)). Therefore,

$$\mathrm{M}(G-\{v\}) = \mathrm{M}(\mathcal{R}_{x+3,y,z}(a-1)). \tag{3.5}$$

Similarly, by considering forced lozenges as indicated respectively in Figures 5(b)–(f), we have: $\mathrm{M}(G-\{u,s,w\}) = \mathrm{M}(\mathcal{R}_{x,y,z-1}(a))$, $\mathrm{M}(G-\{u\}) = \mathrm{M}(\mathcal{R}_{x,y,z}(a))$, $\mathrm{M}(G-\{v,w,s\}) = \mathrm{M}(\mathcal{R}_{x+3,y,z-1}(a-1))$, $\mathrm{M}(G-\{w\}) = \mathrm{M}(\mathcal{R}_{x,y+1,z}(a-1))$, $\mathrm{M}(G-\{u,v,s\}) = \mathrm{M}(\mathcal{R}_{x+3,y-1,z-1}(a))$.

Plugging the above six equalities into equation (3.1) in Lemma 3.1, we obtain the following recurrence:

$$\begin{aligned}\mathrm{M}(\mathcal{R}_{x+3,y,z}(a-1))\,\mathrm{M}(\mathcal{R}_{x,y,z-1}(a)) =& \,\mathrm{M}(\mathcal{R}_{x,y,z}(a))\,\mathrm{M}(\mathcal{R}_{x+3,y,z-1}(a-1)) \\ &+\mathrm{M}(\mathcal{R}_{x,y+1,z}(a-1))\,\mathrm{M}(\mathcal{R}_{x+3,y-1,z-1}(a)). \end{aligned} \tag{3.6}$$

One readily sees that all the regions in (3.6), except for $\mathcal{R}_{x,y,z}(a)$, have the sum of their z- and a-parameters strictly less than $z+a$. Thus, their numbers of tilings are given by the formula (2.3). Plugging in these formulas into (3.6) and performing some straightforward algebraic simplification, one gets $\mathrm{M}(\mathcal{R}_{x,y,z}(a))$ equal to $P_1(x,y,z,a)$. □

Proof of Theorem 2.1. We only show here the proof of (2.7) and (2.8), the other formulas can be proved similarly.

As the lozenge tilings of the defected hexagon $\mathcal{H}_{2t+1,y}(2a,2x)$ can be naturally identified with the matchings of its dual graph G, the number of cyclically symmetric tilings

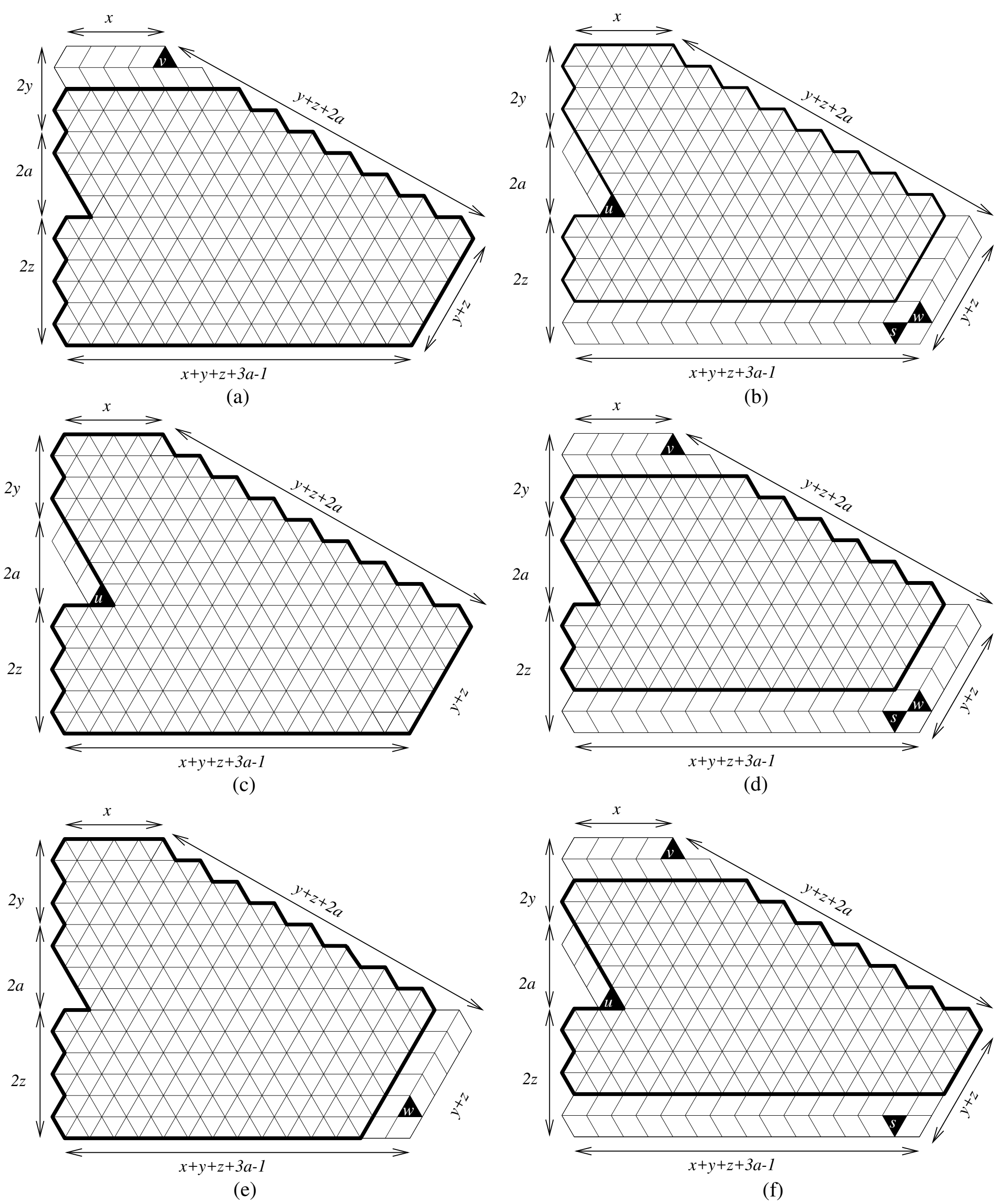


Figure 5: Obtaining a recurrence for the number of tilings of $\mathcal{R}$-type regions.

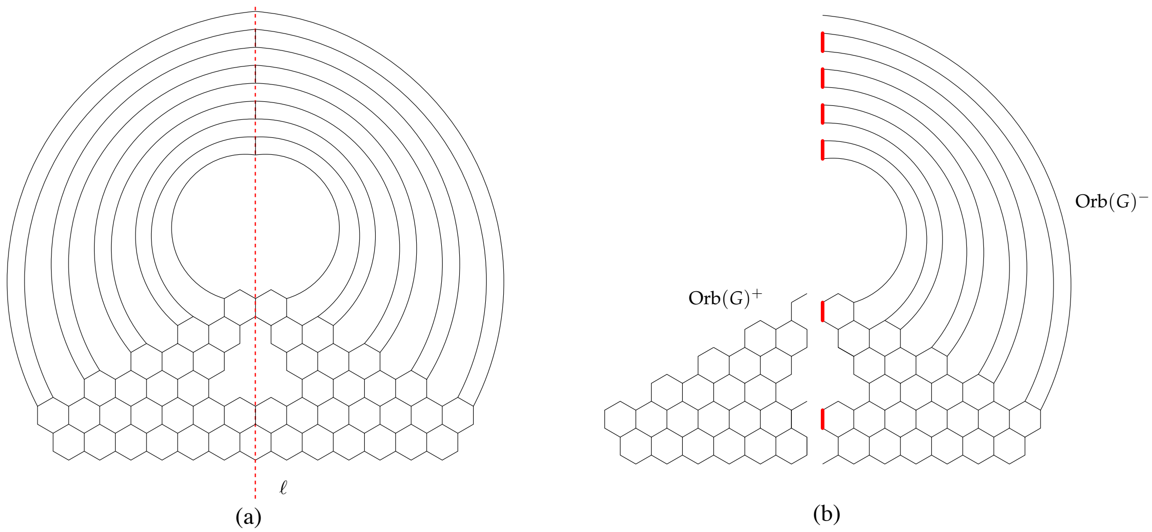


Figure 6: A symmetric embedding of $\mathrm{Orb}(G)$ in the plane and the subsequent application of Ciucu's Factorization Theorem.

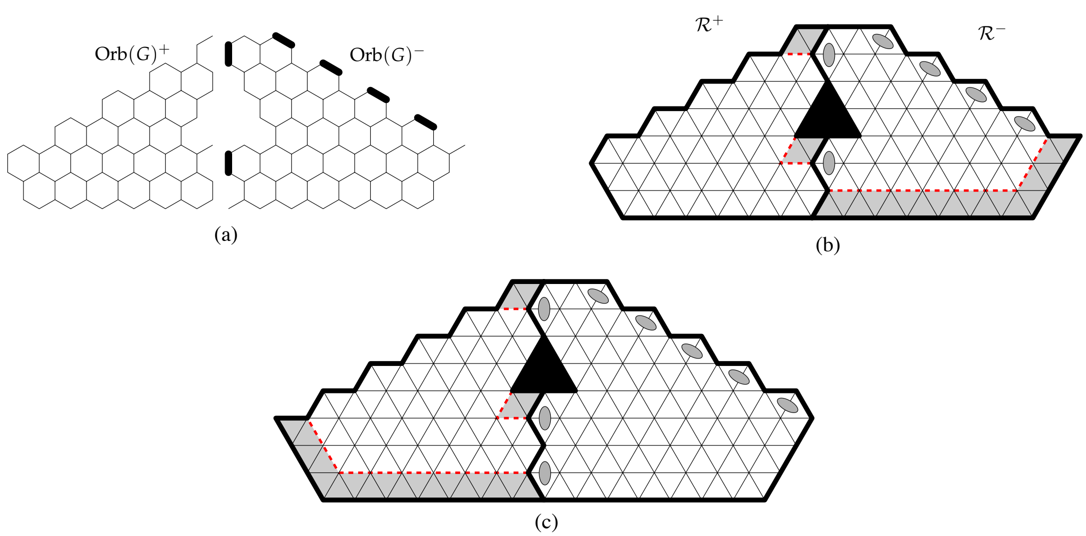


Figure 7: $\mathrm{Orb}(G)^+$ and $\mathrm{Orb}(G)^-$ are the dual graphs of certain (possibly weighted) $\mathcal{R}$-type regions.

of $\mathcal{H}_{2t+1,y}(2a,2x)$ is equal to the number of matchings of G, which are invariant under the rotation r by 120°.

Consider the action of the group generated by r on G, and let $\mathrm{Orb}(G)$ be the 'orbit graph'. The matchings of $\mathrm{Orb}(G)$ can be identified with the r-invariant matchings of G. Figure 6(a) shows that the graph $\mathrm{Orb}(G)$ in the case $t=2$, $a=1$, $b=1$, $y=1$; moreover, the orbit graph can be embedded in the plane so that it accepts a vertical symmetry axis ℓ. This allows us to apply Ciucu's Factorization Theorem (Lemma 3.2) to $\mathrm{Orb}(G)$ as in Figure 6(b) and obtain

$$\mathrm{CS}(\mathcal{H}_{2t+1,y}(2a,2x)) = \mathrm{M}(\mathrm{Orb}(G)) = 2^{2t+4a+1}\,\mathrm{M}(\mathrm{Orb}(G)^{+})\,\mathrm{M}(\mathrm{Orb}(G)^{-}), \tag{3.7}$$

where the component graphs $\mathrm{Orb}(G)^{+}$ and $\mathrm{Orb}(G)^{-}$ is obtained by applying the cutting procedure as in Lemma 3.2.

The component graphs $\mathrm{Orb}(G)^{+}$ and $\mathrm{Orb}(G)^{-}$ can be re-drawn as in Figure 7(a), where the bold edges have weight 1/2. One readily sees that $\mathrm{Orb}(G)^{+}$ and $\mathrm{Orb}(G)^{-}$ are the dual graphs of the regions $\mathcal{R}^{+}$ and $\mathcal{R}^{-}$ in Figure 7(b). The region $\mathcal{R}^{+}$, after removing forced lozenges on the top, is exactly the region $\mathcal{R}_{x+1,y,t-y}(a)$ (see Figure 7(b)); and the region $\mathcal{R}^{-}$ is the region ${}^{*}_{*}\mathcal{R}_{x+1,y,t-y}(a)$. Therefore, (2.7) follows from (3.7) and Lemma 3.3.

Next, we prove (2.8). Similar to the proof of (2.7), by applying Ciucu's Factorization Theorem to the dual graph G' of the region $\mathcal{H}_{2t,y}(2a,2x)$, we obtain

$$\mathrm{CS}(\mathcal{H}_{y,2t}(2a,2x)) = \mathrm{M}(\mathrm{Orb}(G')) = 2^{2t+4a}\,\mathrm{M}(\mathrm{Orb}(G')^{+})\,\mathrm{M}(\mathrm{Orb}(G')^{-}). \tag{3.8}$$

The components graphs $\mathrm{Orb}(G')^{+}$ and $\mathrm{Orb}(G')^{-}$ now correspond to the regions $\mathcal{R}_{x+1,y,t-y-1}(a)$ and ${}^{*}_{*}\mathcal{R}_{x+1,y,t-y}(a)$, respectively (illustrated in Figure 7(c)). Then (2.8) follows from (3.8) and Lemma 3.3.

□

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