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Description of crystals for generalized Kac–Moody algebras using rigged configurations

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Abstract. We give a model for $B(\infty)$ and explicitly describe the $*$-involution using rigged configurations when $\mathfrak{g}$ is a generalized Kac–Moody algebra.

Keywords: rigged configuration, crystal, star-involution, tableau, quantum group

1 Introduction

Generalized Kac–Moody algebras, also known as Borcherds algebras, are infinite-dimensional Lie algebras introduced by Borcherds [1, 2] as a result of his study of the "Monstrous Moonshine" conjectures of Conway and Norton [4]. See, *e.g.*, [10].

With respect to a symmetrizable Kac–Moody algebra $\mathfrak{g}$, crystal bases are combinatorial analogues of representations of the quantized universal enveloping algebra of $\mathfrak{g}$. Defined by Kashiwara in the early 1990s [14, 15], crystals have become an integral part of combinatorial representation theory and have seen application to algebraic combinatorics, mathematical physics, the theory of automorphic forms, and more. In [6], Kashiwara's construction of the crystal basis was extended to the symmetrizable generalized Kac–Moody algebra setting. In particular, the crystal basis for the negative half of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ was introduced, denoted $B(\infty)$, and the crystal basis for the irreducible highest weight module $V(\lambda)$ was also introduced, denoted $B(\lambda)$. The general combinatorial properties of these crystals were then abstracted in [8], much in the same way that Kashiwara had done in [16] for the classical case. There, theorems characterizing the crystals $B(\infty)$ and $B(\lambda)$ were also proved. More recently, other combinatorial models for crystals over generalized Kac–Moody algebras are known: Nakajima monomials [8], Littelmann's path model [9], the polyhedral model [23, 24], and irreducible components of quiver varieties [12, 13].

This extended abstract aims to achieve analogous results to [19, 20, 21] for the case in which $\mathfrak{g}$ is a generalized Kac–Moody algebra; that is, to develop a rigged configuration

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model for the infinity crystal $B(\infty)$, including the $*$-crystal operators, and the irreducible highest weight crystals $B(\lambda)$ when the underlying algebra is a generalized Kac–Moody algebra. In order to do this, a new recognition theorem (see Theorem 3.3) for $B(\infty)$, mimicking the recognition theorem in the classical Kac–Moody cases by Tingley–Webster [26, Proposition 1.4] (which is a reformulation of [17, Proposition 3.2.3]), is presented. The major difference in this new recognition theorem is the existence of imaginary simple roots; the crystal operators associated with imaginary simple roots behave inherently different than that of the case of only real simple roots. Once the new recognition theorem is established, we state new crystal operators (see Definition 4.1) and the $*$-crystal operators (see Definition 4.6) on rigged configurations. We then appeal to the fact that $B(\lambda)$ naturally injects into $B(\infty)$ by [7, Theorem 5.2].

There are currently no known non-recursive characterizations of $\mathrm{RC}(\infty)$ in any Kac–Moody or generalized Kac–Moody type. However, there is a recursive description of $\mathrm{RC}(\infty)$ in type A_n given in [25].

We note that our results give the first model for crystals of generalized Kac–Moody algebras that has a direct combinatorial description of the $*$-involution on $B(\infty)$; *i.e.*, by not recursively using the crystal and $*$-crystal operators. Moreover, the rigged configuration model for $B(\lambda)$ does not require knowledge other than the combinatorial description of the element, in contrast to the Littelmann path or Nakajima monomial models.

2 Quantum generalized Kac–Moody algebras and crystals

Let I be a countable set. An *(even, integral, symmetrizable) Borcherds–Cartan matrix* $A = (A_{ab})_{a,b\in I}$ is a matrix with integer entries such that

1. $A_{aa} = 2$ or $A_{aa} \in -2\mathbf{Z}_{>0}$ for all $a \in I$;
2. $A_{ab} \leq 0$ if $a \neq b$;
3. $A_{ab} = 0$ if and only if $A_{ba} = 0$;
4. there exists a diagonal matrix D such that DA is symmetric.

An index $a \in I$ is called *real* if $A_{aa} = 2$ and is called *imaginary* if $A_{aa} \leq 0$. The subset of I of all real (resp. imaginary) indices is denoted I^{re} (resp. I^{im}).

Example 2.1. Let $I = \{(i,t) : i \in \mathbf{Z}_{\geq -1},\ 1 \leq t \leq c(i)\}$, where $c(i)$ is the i-th coefficient of the elliptic modular function

$$j(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots = \sum_{i\geq -1} c(i)q^i.$$

Define $A = (A_{(i,t),(j,s)})$, where each entry is defined by $A_{(i,t),(j,s)} = -(i+j)$. This is a Borcherds–Cartan matrix, and it is associated to the Monster Lie algebra used by Borcherds in [2].

A *Borcherds–Cartan datum* is a tuple $(A, P^\vee, P, \Pi^\vee, \Pi)$, where

1. A is an even, integral, symmetrizable Borcherds–Cartan matrix;
2. $P^\vee = (\bigoplus_{a\in I} \mathbf{Z}h_a) \oplus (\bigoplus_{a\in I} \mathbf{Z}d_a)$, called the *dual weight lattice*;
3. $P = \{\lambda \in \mathfrak{h}^* : \lambda(P^\vee) \subset \mathbf{Z}\}$, where $\mathfrak{h}^* = \mathbf{Q} \otimes_{\mathbf{Z}} P^\vee$, called the *weight lattice*;
4. $\Pi^\vee = \{h_a : a \in I\}$, called the set of *simple coroots*;
5. $\Pi = \{\alpha_a : a \in I\}$, called the set of *simple roots*.

Define the canonical pairing $\langle\ ,\ \rangle\colon P^\vee \times P \longrightarrow \mathbf{Z}$ by $\langle h_a, \alpha_b\rangle = A_{ab}$ for all $a, b \in I$.

The set of *dominant integral weights* is $P^+ = \{\lambda \in P : \lambda(h_a) \geq 0 \text{ for all } a \in I\}$. The *fundamental weights*, denoted $\Lambda_a \in P^+$ for $a \in I$, are defined by $\Lambda_a(h_b) = \delta_{ab}$ and $\Lambda_a(d_b) = 0$ for all $a, b \in I$. Finally, set $Q = \bigoplus_{a\in I} \mathbf{Z}\alpha_a$ and $Q^+ = \sum_{a\in I} \mathbf{Z}_{\geq 0}\alpha_a$.

Let $U_q(\mathfrak{g})$ be the *quantum generalized Kac–Moody algebra* associated with the Borcherds–Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$. (For more detailed information on $U_q(\mathfrak{g})$, see, *e.g.*, [6].)

Definition 2.2 (See [6])**.** An *abstract $U_q(\mathfrak{g})$-crystal* is a set B together with maps

$$e_a, f_a\colon B \longrightarrow B \sqcup \{\mathbf{0}\}, \qquad \varepsilon_a, \varphi_a\colon B \longrightarrow \mathbf{Z} \sqcup \{-\infty\}, \qquad \mathrm{wt}\colon B \longrightarrow P,$$

subject to the following conditions:

1. $\mathrm{wt}(e_a v) = \mathrm{wt}(v) + \alpha_a$ if $e_a v \neq \mathbf{0}$;
2. $\mathrm{wt}(f_a v) = \mathrm{wt}(v) - \alpha_a$ if $f_a v \neq \mathbf{0}$;
3. for any $a \in I$ and $v \in B$, $\varphi_a(v) = \varepsilon_a(v) + \langle h_a, \mathrm{wt}(v)\rangle$;
4. for any $a \in I$ and $v, v' \in B$, $f_a v = v'$ if and only if $v = e_a v'$;
5. for any $a \in I$ and $v \in B$ such that $e_a v \neq \mathbf{0}$, we have
   (a) $\varepsilon_a(e_a v) = \varepsilon_a(v) - 1$ and $\varphi_a(e_a v) = \varphi_a(v) + 1$ if $a \in I^{\mathrm{re}}$,
   (b) $\varepsilon_a(e_a v) = \varepsilon_a(v)$ and $\varphi_a(e_a v) = \varphi_a(v) + A_{aa}$ if $a \in I^{\mathrm{im}}$;
6. for any $a \in I$ and $v \in B$ such that $f_a v \neq \mathbf{0}$, we have
   (a) $\varepsilon_a(f_a v) = \varepsilon_a(v) + 1$ and $\varphi_a(f_a v) = \varphi_a(v) - 1$ if $a \in I^{\mathrm{re}}$,
   (b) $\varepsilon_a(f_a v) = \varepsilon_a(v)$ and $\varphi_a(f_a v) = \varphi_a(v) - A_{aa}$ if $a \in I^{\mathrm{im}}$;
7. for any $a \in I$ and $v \in B$ such that $\varphi_a(v) = -\infty$, we have $e_a v = f_a v = \mathbf{0}$.

Here, $\mathbf{0}$ is considered to be a formal object; *i.e.*, it is not an element of a crystal.

Example 2.3. For each $\lambda \in P^+$, by [6, Section 3], there exists a unique irreducible highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ in the category $\mathcal{O}_{\mathrm{int}}$. (See [6] for the details and explanation of the notation.) Associated to each $V(\lambda)$ is a crystal basis $\big(L(\lambda), B(\lambda)\big)$, in the sense of [6]. Then $B(\lambda)$ is an abstract $U_q(\mathfrak{g})$-crystal. In this case, for all $a \in I$ and $v \in B(\lambda)$, we have

$$\varepsilon_a(v) = \begin{cases} \max\{k \geq 0 : e_a^k v \neq \mathbf{0}\} & \text{if } a \in I^{\mathrm{re}}, \\ 0 & \text{if } a \in I^{\mathrm{im}}, \end{cases} \quad \varphi_a(v) = \begin{cases} \max\{k \geq 0 : f_a^k v \neq \mathbf{0}\} & \text{if } a \in I^{\mathrm{re}}, \\ \langle h_a, \mathrm{wt}(v) \rangle & \text{if } a \in I^{\mathrm{im}}. \end{cases}$$

Moreover, there exists a unique $v_\lambda \in B(\lambda)$ such that $\mathrm{wt}(v_\lambda) = \lambda$ and

$$B(\lambda) = \{f_{a_1} \cdots f_{a_r} v_\lambda : r \geq 0,\ a_1, \dots, a_r \in I\} \setminus \{\mathbf{0}\}.$$

Example 2.4. The negative half of the generalized quantum algebra $U_q^-(\mathfrak{g})$ has a crystal basis $\big(L(\infty), B(\infty)\big)$ in the sense of [6]. Then $B(\infty)$ is an abstract $U_q(\mathfrak{g})$-crystal. In this case, there exists a unique element $\mathbf{1} \in B(\infty)$ such that $\mathrm{wt}(\mathbf{1}) = 0$ and

$$B(\infty) = \{f_{a_1} \cdots f_{a_r} \mathbf{1} : r \geq 0,\ a_1, \dots, a_r \in I\}.$$

Moreover, for all $v \in B(\infty)$ and $a, a_1, \dots, a_r \in I$, we have

$$\varepsilon_a(v) = \begin{cases} \max\{k \geq 0 : e_a^k v \neq \mathbf{0}\} & \text{if } a \in I^{\mathrm{re}}, \\ 0 & \text{if } a \in I^{\mathrm{im}}, \end{cases} \tag{2.1}$$

$$\varphi_a(v) = \varepsilon_a(v) + \langle h_a, \mathrm{wt}(v) \rangle, \tag{2.2}$$

$$\mathrm{wt}(v) = -\alpha_{a_1} - \cdots - \alpha_{a_r} \quad \text{if } v = f_{a_1} \cdots f_{a_r} \mathbf{1}. \tag{2.3}$$

Definition 2.5 (See [7])**.** Let B_1 and B_2 be abstract $U_q(\mathfrak{g})$-crystals. A *crystal morphism* $\psi\colon B_1 \longrightarrow B_2$ is a map $B_1 \sqcup \{\mathbf{0}\} \longrightarrow B_2 \sqcup \{\mathbf{0}\}$ such that

1. for $v \in B_1$ and $a \in I$, $\varepsilon_a\big(\psi(v)\big) = \varepsilon_a(v)$, $\varphi_a\big(\psi(v)\big) = \varphi_a(v)$, and $\mathrm{wt}\big(\psi(v)\big) = \mathrm{wt}(v)$,
2. if $v \in B_1$ and $f_a v \in B_1$, then $\psi(f_a v) = f_a \psi(v)$.

Let $\psi\colon B_1 \longrightarrow B_2$ be a crystal morphism. Then ψ is called *strict* if $\psi(e_a v) = e_a \psi(v)$ and $\psi(f_a v) = f_a \psi(v)$ for all $a \in I$. The morphism ψ is an *embedding* if the underlying map is injective. An *isomorphism* of crystals is a bijective, strict crystal morphism.

Definition 2.6 (See [7])**.** Let B_1 and B_2 be abstract $U_q(\mathfrak{g})$-crystals. The *tensor product*

$B_1 \otimes B_2$ is a crystal with underlying set $B_1 \times B_2$ and operations defined, for $a \in I$, by

$$e_a(v_1 \otimes v_2) = \begin{cases} e_a v_1 \otimes v_2 & \text{if } a \in I^{\mathrm{re}} \text{ and } \varphi_a(v_1) \geq \varepsilon_a(v_2), \\ e_a v_1 \otimes v_2 & \text{if } a \in I^{\mathrm{im}} \text{ and } \varphi_a(v_1) > \varepsilon_a(v_2) - A_{aa}, \\ \mathbf{0} & \text{if } a \in I^{\mathrm{im}} \text{ and } \varepsilon_a(v_2) < \varphi_a(v_1) \leq \varepsilon_a(v_2) - A_{aa}, \\ v_1 \otimes e_a v_2 & \text{if } a \in I^{\mathrm{re}} \text{ and } \varphi_a(v_1) < \varepsilon_a(v_2), \\ v_1 \otimes e_a v_2 & \text{if } a \in I^{\mathrm{im}} \text{ and } \varphi_a(v_1) \leq \varepsilon_a(v_2), \end{cases}$$

$$f_a(v_1 \otimes v_2) = \begin{cases} f_a v_1 \otimes v_2 & \text{if } \varphi_a(v_1) > \varepsilon_a(v_2), \\ v_1 \otimes f_a v_2 & \text{if } \varphi_a(v_1) \leq \varepsilon_a(v_2), \end{cases}$$

$$\varepsilon_a(v_1 \otimes v_2) = \max\{\varepsilon_a(v_1), \varepsilon_a(v_2) - \langle h_a, \mathrm{wt}(v_1)\rangle\},$$

$$\varphi_a(v_1 \otimes v_2) = \max\{\varphi_a(v_1) + \langle h_a, \mathrm{wt}(v_2)\rangle, \varphi_a(v_2)\},$$

$$\mathrm{wt}(v_1 \otimes v_2) = \mathrm{wt}(v_1) + \mathrm{wt}(v_2).$$

Example 2.7. Let $\lambda \in P$ and set $T_\lambda = \{t_\lambda\}$. For all $a \in I$, define crystal operations

$$e_a t_\lambda = f_a t_\lambda = \mathbf{0}, \qquad \varepsilon_a(t_\lambda) = \varphi_a(t_\lambda) = -\infty, \qquad \mathrm{wt}(t_\lambda) = \lambda.$$

Note that $T_\lambda \otimes T_\mu \cong T_{\lambda+\mu}$, for $\lambda, \mu \in P$. Moreover, by [7, Proposition 3.9], for every $\lambda \in P^+$, there exists a crystal embedding $\iota_\lambda \colon B(\lambda) \longrightarrow B(\infty) \otimes T_\lambda$.

Example 2.8. Let $C = \{c\}$. Then C is a crystal with operations defined, for $a \in I$, by

$$e_a c = f_a c = \mathbf{0}, \qquad \varepsilon_a(c) = \varphi_a(c) = 0, \qquad \mathrm{wt}(c) = 0.$$

Theorem 2.9 (See [7, Theorem 5.2])**.** *Let $\lambda \in P^+$. Then $B(\lambda)$ is isomorphic to the connected component of $B(\infty) \otimes T_\lambda \otimes C$ containing $\mathbf{1} \otimes t_\lambda \otimes c$.*

Example 2.10. For each $a \in I$, set $\mathbf{N}_{(a)} = \{z_a(-n) : n \geq 0\}$. Then $\mathbf{N}_{(a)}$ is a crystal with maps defined, for $b \in I$, by

$$e_b z_a(-n) = \begin{cases} z_a(-n+1) & \text{if } b = a, \\ \mathbf{0} & \text{otherwise}, \end{cases} \qquad f_b z_a(-n) = \begin{cases} z_a(-n-1) & \text{if } b = a, \\ \mathbf{0} & \text{otherwise}, \end{cases}$$

$$\varepsilon_b(z_a(-n)) = \begin{cases} n & \text{if } b = a \in I^{\mathrm{re}}, \\ 0 & \text{if } b = a \in I^{\mathrm{im}}, \\ -\infty & \text{otherwise}, \end{cases} \qquad \varphi_b(z_a(-n)) = \begin{cases} -n & \text{if } b = a \in I^{\mathrm{re}}, \\ -nA_{aa} & \text{if } b = a \in I^{\mathrm{im}}, \\ -\infty & \text{otherwise}, \end{cases}$$

$$\mathrm{wt}(z_a(-n)) = -n\alpha_a.$$

By convention, $z_a(-n) = \mathbf{0}$ for $n < 0$.

Theorem 2.11 (See [7, Theorem 4.1])**.** *For any $a \in I$, there exists a unique strict crystal embedding $\Psi_a \colon B(\infty) \longrightarrow B(\infty) \otimes \mathbf{N}_{(a)}$.*

3 Recognition theorem for $B(\infty)$

Theorem 3.1 (See [7, Theorem 5.1]). *Let B be an abstract $U_q(\mathfrak{g})$-crystal such that*

1. $\mathrm{wt}(B) \subset -Q^+$;
2. *there exists an element $v_0 \in B$ such that $\mathrm{wt}(v_0) = 0$;*
3. *for any $v \in B$ such that $v \neq v_0$, there exists some $a \in I$ such that $e_a v \neq \mathbf{0}$;*
4. *for all $a \in I$, there exists a strict embedding $\Psi_a \colon B \longhookrightarrow B \otimes \mathbf{N}_{(a)}$.*

Then there exists a crystal isomorphism $B \cong B(\infty)$ such that $v_0 \mapsto \mathbf{1}$.

In [6], it was shown there is a antiautomorphism $* \colon U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g})$ defined by

$$E_a^* = E_a, \qquad F_a^* = F_a, \qquad (q^h)^* = q^{-h},$$

where E_a, F_a, q^h, $a \in I$ and $h \in P^\vee$, are the generators of $U_q(\mathfrak{g})$. By [6, Corollary 7.40], this admits a combinatorial construction on $B(\infty)$ that is called the $*$-*involution* or *star involution* (sometimes known as Kashiwara's involution [3, 5, 11, 18, 22, 26]), which we describe here. For $a \in I$, define

$$\mathbf{N}^{(a)} = \{v \in B(\infty) : v \otimes z_a(-n) \in \mathrm{Im}(\Psi_a) \text{ for some } n \geq 0\}.$$

This forms a subcrystal of $B(\infty)$. By [18, Lemma 3.12], we have $B(\infty) = \mathbf{N}^{(a)} \otimes \mathbf{N}_{(a)}$ as a $U_q(\mathfrak{g})$-crystal. Using this expression for $B(\infty)$, the Kashiwara operators e_b and f_b act on the left-hand factor, for all $b \in I$. Define new operators e_b^* and f_b^*, respectively, on $B(\infty)$ to be the operators that act by e_b and f_b, respectively, on the right-hand factor. Additionally, for $b \in I$ and $v \in B(\infty)$, set $\mathrm{wt}^*(v) = \mathrm{wt}(v)$ and

$$\varepsilon_b^*(v) = \begin{cases} \max\{k \geq 0 : e_b^* v \neq 0\} & \text{if } b \in I^{\mathrm{re}}, \\ 0 & \text{if } b \in I^{\mathrm{im}}, \end{cases} \qquad \varphi_b^*(v) = \varepsilon_b^*(v) + \langle h_b, \mathrm{wt}(v) \rangle.$$

The set $B(\infty)$ equipped with the $*$-crystal operators will be denoted by $B(\infty)^*$.

Theorem 3.2 (See [18, Theorem 4.7]). *We have $B(\infty) \cong B(\infty)^*$.*

Define $B(\infty) \longrightarrow B(\infty)^*$ by asserting $v^* = f_{a_k}^* \cdots f_{a_2}^* f_{a_1}^* \mathbf{1}$ if $v = f_{a_k} \cdots f_{a_2} f_{a_1} \mathbf{1}$. Then

$$e_a^* = * \circ e_a \circ *, \qquad f_a^* = * \circ f_a \circ *, \qquad \varepsilon_a^* = \varepsilon_a \circ *, \qquad \varphi_a^* = \varphi_a \circ *, \qquad \mathrm{wt}^* = \mathrm{wt}.$$

In the sequel, we will require the following modified statistics:

$$\widetilde{\varepsilon}_a(v) := \max\{k' \geq 0 : e_a^{k'} v \neq \mathbf{0}\}, \qquad \widetilde{\varphi}_a(v) := \max\{k' \geq 0 : f_a^{k'} v \neq \mathbf{0}\},$$

and similarly for $\widetilde{\varepsilon}_a^*$ and $\widetilde{\varphi}_a^*$ using e_a^* and f_a^* respectively. Note that $\widetilde{\varepsilon}_a(v) = \varepsilon_a(v)$ and $\widetilde{\varphi}_a(v) = \varphi_a(v)$, as well as for the $*$-versions, when $a \in I^{\mathrm{re}}$. Additionally, for $v \in B(\infty)$ and $a \in I$, define

$$\kappa_a(v) := \begin{cases} \varepsilon_a(v) + \varepsilon_a^*(v) + \langle h_a, \mathrm{wt}(v) \rangle & \text{if } a \in I^{\mathrm{re}}, \\ \varepsilon_a(v) + \widetilde{\varepsilon}_a^*(v) A_{aa} + \langle h_a, \mathrm{wt}(v) \rangle & \text{if } a \in I^{\mathrm{im}}. \end{cases} \tag{3.1}$$

We will appeal to the following statement, which is a generalized Kac–Moody analog of the result used in [20] coming from [26] (but based on Kashiwara and Saito's classification theorem for $B(\infty)$ in the Kac–Moody setting from [17]). First, a *bicrystal* is a set B with two abstract $U_q(\mathfrak{g})$-crystal structures $(B, e_a, f_a, \varepsilon_a, \varphi_a, \mathrm{wt})$ and $(B, e_a^\circ, f_a^\circ, \varepsilon_a^\circ, \varphi_a^\circ, \mathrm{wt})$ with the same weight function. In such a bicrystal B, we say $v \in B$ is a *highest weight element* if $e_a v = e_a^\circ v = 0$ for all $a \in I$.

Theorem 3.3. *Let $(B, e_a, f_a, \varepsilon_a, \varphi_a, \mathrm{wt})$ and $(B^\circ, e_a^\circ, f_a^\circ, \varepsilon_a^\circ, \varphi_a^\circ, \mathrm{wt})$ be connected abstract $U_q(\mathfrak{g})$-crystals with the same highest weight element $v_0 \in B \cap B^\circ$ that is the unique element of weight 0, where the remaining crystal data is determined by setting $\mathrm{wt}(v_0) = 0$ and $\varepsilon_a(v)$ by Equation (2.1). Assume further that, for all $a \neq b$ in I and all $v \in B$,*

1. *$f_a v$, $f_a^\circ v \neq \mathbf{0}$;*
2. *$f_a^\circ f_b v = f_b f_a^\circ v$ and $\widetilde{\varepsilon}_a^\circ(f_b v) = \widetilde{\varepsilon}_a^\circ(v)$ and $\widetilde{\varepsilon}_b(f_a^\circ v) = \widetilde{\varepsilon}_b(v)$;*
3. *$\kappa_a(v) = 0$ implies $f_a v = f_a^\circ v$;*
4. *for $a \in I^{\mathrm{re}}$:*
   - *(a) $\kappa_a(v) \geq 0$;*
   - *(b) $\kappa_a(v) \geq 1$ implies $\varepsilon_a^\circ(f_a v) = \varepsilon_a^\circ(v)$ and $\varepsilon_a(f_a^\circ v) = \varepsilon_a(v)$;*
   - *(c) $\kappa_a(v) \geq 2$ implies $f_a f_a^\circ v = f_a^\circ f_a v$;*
5. *for $a \in I^{\mathrm{im}}$: $\kappa_a(v) > 0$ implies $\widetilde{\varepsilon}_a^\circ(f_a v) = \widetilde{\varepsilon}_a(v)$ and $f_a f_a^\circ v = f_a^\circ f_a v$.*

Then $(B, e_a, f_a, \varepsilon_a, \varphi_a, \mathrm{wt}) \cong B(\infty)$. Moreover, suppose $\kappa_a(v) = 0$ if and only if

$$\kappa_a^\circ(v) := \varepsilon_a^\circ(v) + \widetilde{\varepsilon}_a(v) A_{aa} + \langle h_a, \mathrm{wt}(v) \rangle = 0$$

for all $a \in I^{\mathrm{im}}$ and $v \in B$. Then $(B^\circ, e_a^\circ, f_a^\circ, \varepsilon_a^\circ, \varphi_a^\circ, \mathrm{wt}) \cong B(\infty)$ with $e_a^\circ = e_a^*$ and $f_a^\circ = f_a^*$.

Theorem 3.3 is logically equivalent to Theorem 3.1.

4 Rigged configurations

Let $\mathcal{H} = I \times \mathbf{Z}_{>0}$. A rigged configuration is a sequence of partitions $\nu = (\nu^{(a)} : a \in I)$ such that each row $\nu_i^{(a)}$ has an integer called a *rigging*, and we let $J = \big(J_i^{(a)} : (a, i) \in \mathcal{H}\big)$, where $J_i^{(a)}$ is the multiset of riggings of rows of length i in $\nu^{(a)}$. We consider there to be an infinite number of rows of length 0 with rigging 0; *i.e.*, $J_0^{(a)} = \{0, 0, \dots\}$ for all $a \in I$. The term rigging will be interchanged freely with the term *label*. We identify two rigged configurations (ν, J) and $(\widetilde{\nu}, \widetilde{J})$ if $\nu = \widetilde{\nu}$ and $J_i^{(a)} = \widetilde{J}_i^{(a)}$ for any fixed $(a,i) \in \mathcal{H}$. Let $(\nu, J)^{(a)}$ denote the rigged partition $(\nu^{(a)}, J^{(a)})$.

Define the *vacancy numbers* of ν to be

$$p_i^{(a)}(\nu) = p_i^{(a)} = -\sum_{(b,j)\in\mathcal{H}} A_{ab} \min(i,j) m_j^{(b)}, \tag{4.1}$$

where $m_i^{(a)}$ is the number of parts of length i in $\nu^{(a)}$ and $(A_{ab})_{a,b\in I}$ is the underlying Cartan matrix. The *corigging*, or *colabel*, of a row in $(\nu, J)^{(a)}$ with rigging x is $p_i^{(a)} - x$. In addition, we can extend the vacancy numbers to

$$p_\infty^{(a)} = \lim_{i\to\infty} p_i^{(a)} = -\sum_{b\in I} A_{ab} |\nu^{(b)}|$$

since $\sum_{j=1}^{\infty} \min(i,j) m_j^{(b)} = |\nu^{(b)}|$ for $i \gg 1$. Note this is consistent with letting $i = \infty$ in Equation (4.1).

Let $\mathrm{RC}(\infty)$ denote the set of rigged configurations generated by $(\nu_\emptyset, J_\emptyset)$, where $\nu_\emptyset^{(a)} = 0$ for all $a \in I$, and closed under the operators e_a and f_a $(a \in I)$ defined next. Recall that, in our convention, $x \leq 0$ since there the string $(0,0)$ is in each $(\nu, J)^{(a)}$.

Definition 4.1. Fix some $a \in I$.

e_a: We initially split this into two cases:

- $a \in I^{\mathrm{re}}$: Let x be the smallest rigging in $(\nu, J)^{(a)}$. If $x = 0$, then $e_a(\nu, J) = \mathbf{0}$. Otherwise, let r be a row in $(\nu, J)^{(a)}$ of minimal length ℓ with rigging x.
- $a \in I^{\mathrm{im}}$: If $\nu^{(a)} = \emptyset$ or the smallest rigging of $(\nu, J)^{(a)}$ is not equal to $-\frac{1}{2}A_{aa}$, then $e_a(\nu, J) = \mathbf{0}$. Otherwise let r be a row with rigging $x = -\frac{1}{2}A_{aa}$.

If $e_a(\nu, J) \neq \mathbf{0}$, then $e_a(\nu, J)$ is the rigged configuration that removes a box from row r, sets the new rigging of r to be $x + \frac{1}{2}A_{aa}$, and changes all other riggings such that the coriggings remain fixed.

f_a: Let x be the smallest rigging in $(\nu, J)^{(a)}$. Let r be a row in $(\nu, J)^{(a)}$ of maximal length ℓ with rigging x. Then $f_a(\nu, J)$ is the rigged configuration that adds a box to row r, sets the new rigging of r to be $x - \frac{1}{2}A_{aa}$, and changes all other riggings such that the coriggings remain fixed.

Define the following additional maps on $\mathrm{RC}(\infty)$ by

$$\begin{aligned}
\varepsilon_a(\nu, J) &= \begin{cases} \max\{k \in \mathbf{Z} : e_a^k(\nu, J) \neq 0\} & \text{if } a \in I^{\mathrm{re}}, \\ 0 & \text{if } a \in I^{\mathrm{im}}, \end{cases} \\
\varphi_a(\nu, J) &= \langle h_a, \mathrm{wt}(\nu, J)\rangle + \varepsilon_a(\nu, J), \\
\mathrm{wt}(\nu, J) &= -\sum_{a \in I} |\nu^{(a)}| \alpha_a.
\end{aligned}$$

Remark 4.2. The reasoning for the cases in the definition of e_a amounts to the definition of $B(\infty)$ for generalized Kac–Moody algebras, which is outlined in Example 2.4. It is also a manifestation of the tensor product rule in Definition 2.6.

Unlike in the case when $a \in I^{\mathrm{re}}$, there is a lot more structure for $(\nu, J)^{(a)}$ when $a \in I^{\mathrm{im}}$. This is a key observation in our proofs.

Lemma 4.3. *Suppose $a \in I^{\mathrm{im}}$ and $(\nu, J) \in \mathrm{RC}(\infty)$. Then, $\nu^{(a)} = (1^k)$ and, for any string (i, x) such that $i > 0$, we have $x \geq -\frac{1}{2}A_{aa}$.*

Proposition 4.4. *With the operations above, $\mathrm{RC}(\infty)$ is an abstract $U_q(\mathfrak{g})$-crystal.*

Proposition 4.5. *Let $(\nu, J) \in \mathrm{RC}(\infty)$ and fix some $a \in I$. Let $x \leq 0$ denote the smallest label in $(\nu, J)^{(a)}$. Then we have $\varepsilon_a(\nu, J) = -x$ and $\varphi_a(\nu, J) = p_\infty^{(a)} - x$.*

Definition 4.6. Fix some $a \in I$.

- e_a^*: We initially split this into two cases:
  - $a \in I^{\mathrm{re}}$: Let x be the smallest corigging in $(\nu, J)^{(a)}$. If $x = 0$, then $e_a^*(\nu, J) = \mathbf{0}$. Otherwise, let r be a row in $(\nu, J)^{(a)}$ of minimal length ℓ with corigging x.
  - $a \in I^{\mathrm{im}}$: If $\nu^{(a)} = \emptyset$ or the smallest corigging of $(\nu, J)^{(a)}$ is not equal to $-\frac{1}{2}A_{aa}$, then $e_a^*(\nu, J) = \mathbf{0}$. Otherwise let r be a row with corigging $x = -\frac{1}{2}A_{aa}$.

  If $e_a^*(\nu, J) \neq \mathbf{0}$, then $e_a^*(\nu, J)$ is the rigged configuration that removes a box from row r, sets the rigging of r so that the corigging is $x - \frac{1}{2}A_{aa}$, and keeps all other riggings fixed.
- f_a^*: Let x be the smallest corigging in $(\nu, J)^{(a)}$. Let r be a row in $(\nu, J)^{(a)}$ of maximal length ℓ with corigging x. Then $f_a^*(\nu, J)$ is the rigged configuration that adds a box to row r, sets the rigging of r so that the corigging is $x - \frac{1}{2}A_{aa}$, and keeps all other riggings fixed.

If e_a^* removes a box from a row of length ℓ in (ν, J), then the vacancy numbers change by the formula

$$\widetilde{p}_i^{(b)} = \begin{cases} p_i^{(b)} & \text{if } i < \ell, \\ p_i^{(b)} + A_{ab} & \text{if } i \geq \ell. \end{cases} \tag{4.2}$$

On the other hand, if f_a^* adds a box to a row of length ℓ, then the vacancy numbers change by

$$\widetilde{p}_i^{(b)} = \begin{cases} p_i^{(b)} & \text{if } i \leq \ell, \\ p_i^{(b)} - A_{ab} & \text{if } i > \ell. \end{cases} \tag{4.3}$$

Similar equations hold for e_a and f_a, respectively. So the riggings of unchanged rows are changed according to Equation (4.2) and Equation (4.3) under e_a and f_a, respectively.

Remark 4.7. By Equation (4.2) and Equation (4.3), the crystal operators e_a and f_a preserve all colabels of (ν, J) other than the row changed in $(\nu, J)^{(a)}$.

Let $\mathrm{RC}(\infty)^*$ denote the closure of $(\nu_\emptyset, J_\emptyset)$ under f_a^* and e_a^*. We define the remaining crystal structure by

$$\begin{aligned} \varepsilon_a^*(\nu, J) &= \begin{cases} \max\{k \in \mathbf{Z} : (e_a^*)^k(\nu, J) \neq 0\} & \text{if } a \in I^{\mathrm{re}}, \\ 0 & \text{if } a \in I^{\mathrm{im}}, \end{cases} \\ \varphi_a^*(\nu, J) &= \langle h_a, \mathrm{wt}(\nu, J) \rangle + \varepsilon_a^*(\nu, J), \\ \mathrm{wt}(\nu, J) &= -\sum_{a \in I} |\nu^{(a)}| \alpha_a. \end{aligned}$$

Proposition 4.8. *With the operations above,* $\mathrm{RC}(\infty)^*$ *is an abstract* $U_q(\mathfrak{g})$*-crystal.*

Theorem 4.9. *As* $U_q(\mathfrak{g})$*-crystals,* $\mathrm{RC}(\infty) \cong \mathrm{RC}(\infty)^* \cong B(\infty)$*.*

Example 4.10. Let $I = \{1, 2\}$ and

$$A = \begin{pmatrix} -2\alpha & -\beta \\ -\gamma & -2\delta \end{pmatrix},$$

such that $\beta, \gamma \in \mathbf{Z}_{\geq 0}$ and $\alpha, \delta \in \mathbf{Z}_{>0}$. Then $I = I^{\mathrm{im}}$. The top part of the crystal graph $\mathrm{RC}(\infty)$ is pictured in Figure 1. For example,

$$f_2 f_1^3 f_2(\nu_\emptyset, J_\emptyset) = \begin{array}{r|c|l} 6\alpha+2\beta & & 3\alpha+\gamma \\ 6\alpha+2\beta & & 2\alpha+\gamma \\ 6\alpha+2\beta & & \alpha+\gamma \end{array} \quad \begin{array}{r|c|l} 4\delta+3\gamma & & 2\delta+3\beta \\ 4\delta+3\gamma & & \delta \end{array}$$

As previously mentioned, the proof of Theorem 4.9 is given by showing the conditions of Theorem 3.3 hold. Thus, we obtain the following corollary as in [20].

Corollary 4.11. *The $*$-involution on* $\mathrm{RC}(\infty)$ *is given by replacing every rigging x of a row of length i in $(\nu, J)^{(a)}$ by the corresponding corigging $p_i^{(a)} - x$ for all $(a, i) \in \mathcal{H}$.*

We can describe highest weight crystals $B(\lambda)$ by utilizing Theorem 2.9. We give this explicitly on rigged configurations by defining new crystal operators $f_a'(\nu, J)$ as $f_a(\nu, J)$ unless $p_i^{(a)} < x + \langle h_a, \lambda \rangle$ for some $(a, i) \in \mathcal{H}$ and $x \in J_i^{(a)}$ or $\varphi_a(\nu, J) = 0$ for $a \in I^{\mathrm{im}}$, in which case $f_a'(\nu, J) = 0$. Let $\mathrm{RC}(\lambda)$ denote the closure of $(\nu_\emptyset, J_\emptyset)$ under f_a'.

Theorem 4.12. *Let $\lambda \in P^+$. Then* $\mathrm{RC}(\lambda) \cong B(\lambda)$*.*

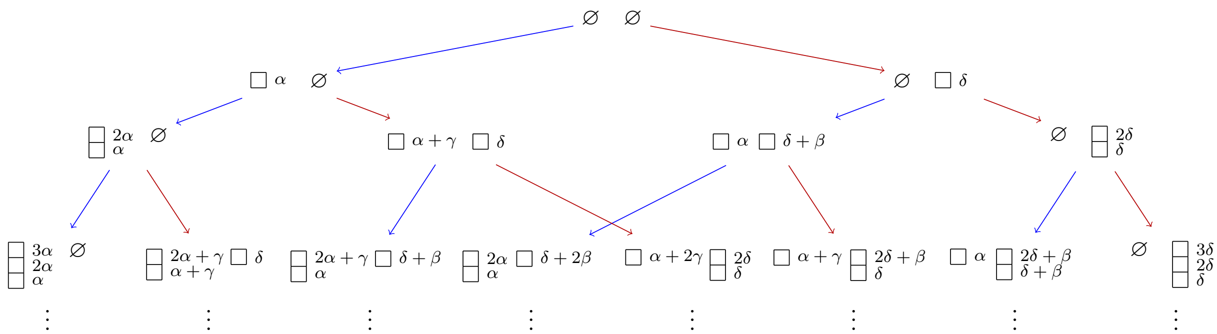

Figure 1: Top of the crystal graph for a purely imaginary Borcherds–Cartan matrix in terms of rigged configurations. Here, the blue arrows correspond to f_1 and the red arrows correspond to f_2.

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