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Quasisymmetric Power Sums

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Abstract. In the 1995 paper entitled "Noncommutative symmetric functions," Gelfand, et. al. defined several noncommutative symmetric function analogues for well-known symmetric function bases, including two distinct types of power sum bases. This paper explores the combinatorial properties of their duals, two distinct quasisymmetric power sum bases. In particular, we show that they refine the classical symmetric power sum basis, and give transition matrices to other well-understood bases, as well as explicit formulas for products of quasisymmetric power sums.

Keywords: quasisymmetric functions, noncommutative symmetric functions, power sums

1 Introduction

The ring of symmetric functions Sym has two important generalizations: the ring QSym of quasisymmetric functions, and the ring NSym of noncommutative symmetric functions. Most well-known bases of Sym generalize nicely to bases of QSym and NSym. Moreover, a duality between QSym and NSym as Hopf algebras interconnects their structure in powerful ways, lifting the traditional Hall inner product on Sym [1, 3, 5, 6, 7, 10].

Here, we explore analogues to the symmetric power sum bases in QSym. With respect to the pairing on Sym, the power sum basis is (up to a constant) self-dual, a relationship expected to hold between analogues in QSym and NSym. Two types of noncommutative power sum bases, $\overline{\mathbf{\Psi}}$ and $\overline{\mathbf{\Phi}}$, were already defined by Gelfand, et. al. [5]. The quasisymmetric duals to the noncommutative power sums have briefly appeared (one type in [4]

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and the other in [11]) but very little has been said about their structure or their relationship to other bases. The main objective of this work is to fill this gap in the field. To that end, we define two types of quasisymmetric power sum bases as scaled duals to $\overline{\mathbf{\Psi}}$ and $\overline{\mathbf{\Phi}}$. We give combinatorial interpretations of their coefficients, and use those to show that they both refine the symmetric power sums (Theorems 3.6 and 3.10). We also give transition matrices to other well-understood bases, and explore algebraic properties, giving explicit formulas for products of quasisymmetric power sums.

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2 Preliminaries

We generally use lower case letters (e.g. e, m, h, s, and p) to indicate *symmetric functions*, bold barred letters (e.g. $\overline{\boldsymbol{e}}$, $\overline{\boldsymbol{h}}$, and $\overline{\boldsymbol{r}}$) to indicate *noncommutative symmetric functions*, and capital letters (e.g. M and F) to indicate *quasisymmetric functions*. When there is a single clear analogue of a symmetric function basis, we use the same letter for the symmetric functions and their analogue (following [10] rather than [5]). For the two different analogues to the power sums, we echo [5] in using $\overline{\mathbf{\Psi}}$ and $\overline{\mathbf{\Phi}}$ for the noncommutative symmetric power sums, and then use Ψ and Φ as quasisymmetric analogues.

2.1 Quasisymmetric functions

The ring of *quasisymmetric functions*, denoted QSym, is defined as the set of formal power series $f \in \mathbb{C}[\![x_1, x_2, \ldots]\!]$ of bounded degree, where the coefficient of $x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$ in f is the same as the coefficient for $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ for any $i_1 < i_2 < \cdots < i_k$ (see [10, 11, 12]). There are a number of common bases for QSym_n as a vector space over $\mathbb{C}$. These bases are indexed by (strong) integer compositions.

Recall $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vDash n$ is a *composition* of n if $\alpha_i > 0$ for each i and $\sum_i \alpha_i = n$. The *size* of a composition α is $|\alpha| = \sum \alpha_i$ and the *length* is $\ell(\alpha) = k$. We denote by $\widetilde{\alpha}$ the partition obtained by placing the parts of α in weakly decreasing order.

For $\alpha, \beta \vDash n$, we say that β *refines* α (equivalently, α is a *coarsening* of β), written $\beta \preccurlyeq \alpha$ (following the convention of [10]), if there are $i_1 < \cdots < i_{k-1}$ such that

$$\alpha = (\beta_1 + \cdots + \beta_{i_1}, \beta_{i_1+1} + \cdots + \beta_{i_1+i_2}, \ldots, \beta_{i_1+\cdots+i_{k-1}+1} + \cdots + \beta_{i_1+\cdots+i_k}).$$

We denote by $\beta^{(j)}$ the composition made up of the parts of β (in order) that sum to α_j.

The *quasisymmetric monomial function* indexed by α is

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}, \qquad \text{where } k = \ell(\alpha),$$

and the *fundamental quasisymmetric function* is

$$F_\alpha = \sum_{\beta \preccurlyeq \alpha} M_\beta, \qquad \text{so that} \qquad M_\alpha = \sum_{\beta \preccurlyeq \alpha} (-1)^{\ell(\beta)-\ell(\alpha)} F_\beta. \tag{2.1}$$

2.2 Noncommutative symmetric functions

The ring of *noncommutative symmetric functions*, denoted NSym, is formally defined as a free associative algebra $\mathbb{C}\langle \overline{\boldsymbol{e}}_1, \overline{\boldsymbol{e}}_2, \ldots \rangle$, where the $\overline{\boldsymbol{e}}_i$ are regarded as *noncommutative elementary functions* and $\overline{\boldsymbol{e}}_\alpha = \overline{\boldsymbol{e}}_{\alpha_1} \overline{\boldsymbol{e}}_{\alpha_2} \cdots \overline{\boldsymbol{e}}_{\alpha_k}$, for a composition α. Define the *noncommutative complete homogeneous symmetric functions* as in [5, Section 4.1] by

$$\overline{\boldsymbol{h}}_n = \sum_{\alpha \vDash n} (-1)^{n-\ell(\alpha)} \overline{\boldsymbol{e}}_\alpha, \quad \text{and} \quad \overline{\boldsymbol{h}}_\alpha = \overline{\boldsymbol{h}}_{\alpha_1} \cdots \overline{\boldsymbol{h}}_{\alpha_k} = \sum_{\beta \preccurlyeq \alpha} (-1)^{|\alpha|-\ell(\beta)} \overline{\boldsymbol{e}}_\beta. \tag{2.2}$$

Via the pairing between QSym and NSym, we have $\langle M_\alpha, \overline{\boldsymbol{h}}_\beta \rangle = \delta_{\alpha,\beta}$.

In [5, Section 3] the *noncommutative power sums of the first kind* (or *type*) $\overline{\boldsymbol{\Psi}}_\alpha$ and the *second kind* $\overline{\boldsymbol{\Phi}}_\alpha$ are defined by considering two natural reformulations of logarithmic differentiation on the generating functions of $\overline{\boldsymbol{h}}$ and $\overline{\boldsymbol{e}}$ in a manner analogous to the commuting power sums. Each can be expressed in terms of the $\overline{\boldsymbol{h}}$ basis [5, Section 4]. The first type is given by

$$\overline{\boldsymbol{\Psi}}_n = \sum_{\beta \vDash n} (-1)^{\ell(\beta)-1} \beta_{\ell(\beta)} \overline{\boldsymbol{h}}_\beta. \tag{2.3}$$

Given compositions $\beta \preccurlyeq \alpha$, let $\mathrm{lp}(\beta) = \beta_{\ell(\beta)}$ (last part) and $\mathrm{lp}(\beta, \alpha) = \prod_{i=1}^{\ell(\alpha)} \mathrm{lp}(\beta^{(i)})$. Then

$$\overline{\boldsymbol{\Psi}}_\alpha = \overline{\boldsymbol{\Psi}}_{\alpha_1} \cdots \overline{\boldsymbol{\Psi}}_{\alpha_m} = \sum_{\beta \preccurlyeq \alpha} (-1)^{\ell(\beta)-\ell(\alpha)} \mathrm{lp}(\beta, \alpha) \overline{\boldsymbol{h}}_\beta. \tag{2.4}$$

Letting $\ell(\beta, \alpha) = \prod_{j=1}^{\ell(\alpha)} \ell(\beta^{(j)})$, the noncommutative power sums of the second kind are

$$\overline{\boldsymbol{\Phi}}_n = \sum_{\alpha \vDash n} (-1)^{\ell(\alpha)-1} \frac{n}{\ell(\alpha)} \overline{\boldsymbol{h}}_\alpha, \quad \text{and} \quad \overline{\boldsymbol{\Phi}}_\alpha = \sum_{\beta \preccurlyeq \alpha} (-1)^{\ell(\beta)-\ell(\alpha)} \frac{\prod_i \alpha_i}{\ell(\beta, \alpha)} \overline{\boldsymbol{h}}_\beta. \tag{2.5}$$

3 Quasisymmetric power sum bases

We describe two quasisymmetric analogues of the power sums. The symmetric power sums satisfy $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}$, where z_λ is the number of permutations of cycle type λ. Specifically, $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots k^{m_k} m_k!$, where m_i is the multiplicity of i in λ. For a composition α, we set $z_\alpha = z_{\widetilde{\alpha}}$.

3.1 Type 1 quasisymmetric power sums

We define the type 1 quasisymmetric power sums to be the basis Ψ of QSym satisfying

$$\langle \Psi_\alpha, \overline{\mathbf{\Psi}}_\beta \rangle = z_\alpha \delta_{\alpha,\beta}.$$

Duality makes most of this definition straightforward; our choice of scalar makes Ψ a refinement of the symmetric power sums as shown in Theorem 3.6. In [5, Section 4.5], the authors give both the transition matrix from the $\overline{\boldsymbol{h}}$ basis to the $\overline{\mathbf{\Psi}}$ basis (above in (2.3)), and its inverse. Using the latter and duality, we compute a quasisymmetric monomial function expansion of Ψ_α. Given $\alpha \preccurlyeq \beta$, define

$$\pi(\alpha) = \prod_{i=1}^{\ell(\alpha)} \sum_{j=1}^{i} \alpha_j \qquad \text{and} \qquad \pi(\alpha,\beta) = \prod_{i=1}^{\ell(\beta)} \pi(\alpha^{(i)}).$$

Then

$$\overline{\boldsymbol{h}}_\alpha = \sum_{\beta \preccurlyeq \alpha} \frac{1}{\pi(\beta,\alpha)} \overline{\mathbf{\Psi}}_\beta, \qquad \text{so that} \qquad \psi_\alpha = \sum_{\beta \succcurlyeq \alpha} \frac{1}{\pi(\alpha,\beta)} M_\beta$$

has the property that $\langle \psi_\alpha, \overline{\mathbf{\Psi}}_\beta \rangle = \delta_{\alpha,\beta}$. Then the type 1 quasisymmetric power sums satisfy

$$\Psi_\alpha = z_\alpha \psi_\alpha = z_\alpha \sum_{\beta \succcurlyeq \alpha} \frac{1}{\pi(\alpha,\beta)} M_\beta. \tag{3.1}$$

For example $\Psi_{(2,3,2)} = (2^2 \cdot 2! \cdot 3) \left(\frac{1}{2\cdot 3\cdot 2} M_{(2,3,2)} + \frac{1}{2\cdot 5\cdot 2} M_{(5,2)} + \frac{1}{2\cdot 3\cdot 5} M_{(2,5)} + \frac{1}{2\cdot 5\cdot 7} M_{(7)} \right)$.

The remainder of this section is devoted to computing an explicit formula for Ψ and a combinatorial proof of the refinement of the symmetric power sums. We consider permutations of $[n] = \{1, 2, \ldots, n\}$ both in one-line notation and in cycle notation. We work with two canonical forms for writing a permutation according to its cycle type.

Definition 3.1. A permutation in cycle notation is said to be in *standard form* if each cycle is written with the largest element last and the cycles are listed in increasing order according to their largest element. It is said to be in *partition form* if each cycle is written with the largest element last, the cycles are listed in descending length order, and cycles of equal length are listed in increasing order according to their largest element. Given a fixed order of the cycles, the *(ordered) cycle type* is the composition $\alpha \vDash n$ with α_i equal to the length of the ith cycle.

For example, the standard form of $(26)(397)(54)(1)(8)$ is $(1)(45)(26)(8)(739)$ (with cycle type $(1,2,2,1,3)$) while the partition form is $(739)(45)(26)(1)(8)$ (with cycle type $(3,2,2,1,1)$).

Next, we define a set of permutations that is a key combinatorial ingredient in our proofs. Let S_n be the set of permutations of $[n]$. Fix $\beta \vDash n$ and $\sigma \in S_n$, written in one-line

permutation σ	$\text{split}_\beta(\sigma)$	add () by α	consistent?
571423689	$57 \parallel 14 \parallel 23689$	$(5)(7) \parallel (14) \parallel (2)(368)(9)$	yes
571428369	$57 \parallel 14 \parallel 28369$	$(5)(7) \parallel (14) \parallel (2)(\mathbf{8}36)(9)$	**no**
571493682	$\underbrace{57}_{\sigma^1} \parallel \underbrace{14}_{\sigma^2} \parallel \underbrace{93682}_{\sigma^3}$	$\underbrace{(5)(7)}_{\bar{\sigma}^1} \parallel \underbrace{(14)}_{\bar{\sigma}^2} \parallel \underbrace{(\mathbf{9})(36\mathbf{8})(\mathbf{2})}_{\bar{\sigma}^3}$	**no**

Table 1: Determining if $\sigma \in \text{Cons}_{\alpha \preccurlyeq \beta}$ where $\beta = (2,2,5)$ and $\alpha = (1,1,2,1,3,1)$.

notation. Partition σ according to β (which we draw using $\parallel$), and consider the (disjoint) words $\text{split}_\beta(\sigma) = [\sigma^1, \dots, \sigma^\ell]$, where $\ell = \ell(\beta)$. Let $[\text{split}_\beta(\sigma)]_j = \sigma^j$. See the second column of Table 1 for examples.

Definition 3.2. Fix $\alpha \preccurlyeq \beta$ compositions of n. Given $\sigma \in S_n$ written in one-line notation, let $\sigma^j = [\text{split}_\beta(\sigma)]_j$. Then, for each $i = 1, \dots, \ell$, add parentheses to σ^i according to $\alpha^{(i)}$, yielding disjoint permutations $\bar{\sigma}^i$ (of subalphabets of $[n]$) of cycle type $\alpha^{(i)}$. If the resulting subpermutations $\bar{\sigma}^i$ are all in standard form, we say σ is *consistent* with $\alpha \preccurlyeq \beta$. In other words, we look at subsequences of σ (split according to β) separately to see if each subsequence is in standard form upon adding parentheses according to $\alpha^{(j)}$. Define

$$\text{Cons}_{\alpha \preccurlyeq \beta} = \{\sigma \in S_n : \sigma \text{ is consistent with } \alpha \preccurlyeq \beta\}.$$

For $\alpha = (1,1,2,1,3,1)$ and $\beta = (2,2,5)$, Table 1 shows several examples of permutations and the partitioning process. Note how β subtly influences consistency in the example of the last row.

To compute coefficients in the quasisymmetric monomial function expansion of Ψ_α, we fix α and let β range over all coarsenings of α. Notice that if σ is consistent with $\alpha \preccurlyeq \beta$ for some choice of β, then σ is consistent with $\alpha \preccurlyeq \gamma$ for all $\alpha \preccurlyeq \gamma \preccurlyeq \beta$. This implies $\text{Cons}_{\alpha \preccurlyeq \beta} \subseteq \text{Cons}_{\alpha \preccurlyeq \alpha}$ for all $\alpha \preccurlyeq \beta$. The next lemma is used to establish a combinatorial interpretation of the coefficients of Ψ_α.

Lemma 3.3. *Let $\alpha, \beta \vDash n$. If $\alpha \preccurlyeq \beta$, we have $|\text{Cons}_{\alpha \preccurlyeq \beta}| \cdot \pi(\alpha, \beta) = n!$.*

Proof. Consider the set A_α of size $\pi(\alpha, \beta)$ defined by

$$A_\alpha = \bigotimes_{i=1}^{\ell(\beta)} \left(\bigotimes_{j=1}^{\ell(\alpha^{(i)})} \mathbb{Z}/a_j^{(i)}\mathbb{Z} \right), \qquad \text{where } a_j^{(i)} = \sum_{r=1}^{j} \alpha_r^{(i)}.$$

Define a map $\text{Sh} : \text{Cons}_{\alpha \preccurlyeq \beta} \times A_\alpha \to S_n$ with $(\sigma, s) \mapsto \sigma_s$ as follows (see also Example 3.4).

Start with $s = [s_j^{(i)}]_{i=1}^{\ell(\beta)}{}_{j=1}^{\ell(\alpha^{(i)})} \in A_\alpha$ and $\sigma \in \mathrm{Cons}_{\alpha\preccurlyeq\beta}$. First partition σ into words $\sigma^1, \ldots, \sigma^\ell$ according to β so that $\sigma^i = [\mathrm{split}_\beta(\sigma)]_i$. Then for each $i = 1, \ldots, \ell(\beta)$, modify σ^i by cycling the first $a_j^{(i)}$ values right by $s_j^{(i)}$ for $j = 1, \ldots, \ell(\alpha^{(i)})$. Call the resulting word σ_s^i. Let $\sigma_s = \sigma_s^1 \cdots \sigma_s^\ell$. (We omit the straightforward check that the process is invertible. See [2] for details.) □

Example 3.4. Let $\beta = (5,4) \vDash 9$, and let $\alpha = (2,3,2,2) \preccurlyeq \beta$. Then $\alpha^{(1)} = (2,3)$, so that $a_1^{(1)} = 2$ and $a_2^{(1)} = 2+3 = 5$; and $\alpha^{(2)} = (2,2)$, so that $a_1^{(2)} = 2$ and $a_2^{(2)} = 2+2 = 4$. Fix $\sigma = 267394518 \in \mathrm{Cons}_{\alpha\preccurlyeq\beta}$, and $s = (s^{(1)}, s^{(2)}) = ((1,3),(0,1)) \in A_\alpha$.
To determine σ_s, first partition σ according to β: $\sigma^1 = 26739$ and $\sigma^2 = 4518$.
Next, cycle σ^i according to $\alpha^{(i)}$:

$$\sigma^1 = 26739 \to \underline{26}739 \to \underline{62}739 \to \underline{62739} \to \underline{73962} = \sigma_s^1;$$

$$\sigma^2 = 4518 \to \underline{45}18 \to \underline{45}18 \to \underline{4518} \to \underline{8451} = \sigma_s^2.$$

Finally, combine to get $\sigma_s = \sigma_s^1\sigma_s^2 = 739628451$.

We now give an explicit formula for Ψ_α. The proof (omitted for this abstract) is a direct computation via careful use of Lemma 3.3 [2].

Theorem 3.5. *Let m_i is the multiplicity of i in α. Then*

$$\Psi_{(\alpha_1,\cdots,\alpha_k)}(x_1,\cdots,x_m) = \frac{\prod_{i=1}^n m_i!}{n!} \sum_{\sigma\in S_n} \sum_{\mathbf{i}\in I_\sigma} x_{i_1}^{\alpha_1}\cdots x_{i_k}^{\alpha_k},$$

where

$$I_\sigma = \{1 \le i_1 \le \cdots \le i_k \le m \ : \ \textit{if } i_j = i_{j+1} \textit{ then } \max([\mathrm{split}_\alpha(\sigma)]_j) < \max([\mathrm{split}_\alpha(\sigma)]_{j+1})\}.$$

We next turn our attention to a proof that the type 1 quasisymmetric power sums refine the symmetric power sums in a natural way. For $\alpha, \beta \vDash n$, let $R_{\alpha\beta} = |\mathcal{O}_{\alpha\beta}|$, where

$$\mathcal{O}_{\alpha\beta} = \left\{ \begin{array}{c} \text{ordered set partitions} \\ (B_1,\cdots,B_{\ell(\beta)}) \text{ of } \{1,\cdots,\ell(\alpha)\} \end{array} \middle| \ \beta_j = \sum_{i\in B_j} \alpha_i \text{ for } 1 \le j \le \ell(\beta) \right\},$$

i.e., $R_{\alpha\beta}$ is the number of ways to group the parts of α so that the parts in the jth (unordered) group sum to β_j. In particular, for a partition λ, from [12, p.297] we have

$$p_\lambda = \sum_{\alpha\vDash n} R_{\lambda\alpha} M_\alpha.$$

The following theorem can be established either by exploiting duality or through a bijective proof. We sketch the bijective proof with details provided in full in [2].

Theorem 3.6. *Let $\lambda \vdash n$. Then*

$$p_\lambda = \sum_{\widetilde{\alpha}=\lambda} \Psi_\alpha.$$

In particular, $\Psi_\alpha = z_\alpha \psi_\alpha = z_{\widetilde{\alpha}} \psi_\alpha$ is the unique rescaling of the ψ basis that refines the symmetric power sums with unit coefficients.

Recall from (3.1) that, for a composition α, we have $\Psi_\alpha = \sum_{\beta \succcurlyeq \alpha} \frac{z_\alpha}{\pi(\alpha,\beta)} M_\beta$. Summing over α rearranging to λ and multiplying on both sides by $n!/z_\lambda$, we see that to prove Theorem 3.6 it is sufficient to establish the following for a fixed β.

Proposition 3.7. *For $\lambda \vdash n$ and $\beta \vDash n$,*

$$R_{\lambda\beta} \frac{n!}{z_\lambda} = \sum_{\substack{\alpha \preccurlyeq \beta \\ \widetilde{\alpha}=\lambda}} \frac{n!}{\pi(\alpha,\beta)}.$$

Proof. Let $\lambda \vdash n$ and $\beta \vDash n$. Using Lemma 3.3, we only need show that

$$R_{\lambda\beta} \frac{n!}{z_\lambda} = \sum_{\substack{\alpha \preccurlyeq \beta \\ \widetilde{\alpha}=\lambda}} |\mathrm{Cons}_{\alpha \preccurlyeq \beta}|. \tag{3.2}$$

For each refinement $\alpha \preccurlyeq \beta$, let $C_\alpha = \{(\alpha,\sigma) : \sigma \in \mathrm{Cons}_{\alpha \preccurlyeq \beta}\}$ and define $C = \bigcup_{\substack{\alpha \preccurlyeq \beta \\ \widetilde{\alpha}=\lambda}} C_\alpha$. Let S_n^λ be the set of permutations of n of cycle type λ. We prove (3.2) by defining the map

$$\mathrm{Br} : C \to \mathcal{O}_{\lambda\beta} \times S_n^\lambda$$

as follows (see also Example 3.8). Start with $(\alpha,\sigma) \in C$, with σ written in one-line notation. Add parentheses to σ according to α, and denote the corresponding permutation (now in cycle notation) by $\bar{\sigma}$. Next, sort the cycles of $\bar{\sigma}$ into partition form (as in Definition 3.1), and let c_i be the ith cycle in this ordering. Then with $\bar{\sigma}^1, \ldots, \bar{\sigma}^\ell$ as in Definition 3.2 (see Table 1), define $B = (B_1, \ldots, B_k)$ by $j \in B_i$ when c_j belongs to $\bar{\sigma}^i$, i.e. $\bar{\sigma}^i = \prod_{j \in B_i} c_j$.

Define $\mathrm{Br}(\alpha,\sigma) = (B,\bar{\sigma})$. Since α rearranges to λ, $\bar{\sigma}$ has (unordered) cycle type λ. And since $\prod_{j \in B_i} c_j = \bar{\sigma}^i$, we have $\sum_{j \in B_i} \lambda_j = \beta_i$. Thus $\mathrm{Br} : (\alpha,\sigma) \mapsto (B,\bar{\sigma})$ is well-defined. (We omit the straightforward check that this process is invertible. See [2] for details.) $\square$

Example 3.8. Let $\beta = (5,4)$, $\alpha = (2,3,2,2)$ and $\sigma = 267394518$. To determine $\mathrm{Br}(\alpha,\sigma)$, first add parentheses to σ according to α: $\bar{\sigma} = (26)(739)(45)(18)$.
Next, partition-sort the cycles of $\bar{\sigma}$: $\bar{\sigma} = \underbrace{(739)}_{c_1}\underbrace{(45)}_{c_2}\underbrace{(26)}_{c_3}\underbrace{(18)}_{c_4}$.

Finally, compare to the β-partitioning: $\underbrace{(26)(739)}_{\bar{\sigma}^1} \parallel \underbrace{(45)(18)}_{\bar{\sigma}^2}$.

So $B = (\{1,3\},\{2,4\})$, since $\bar{\sigma}^1 = c_1 c_3$ and $\bar{\sigma}^2 = c_2 c_4$.

3.2 Type 2 quasisymmetric power sums

We define the type 2 quasisymmetric power sums to be the basis Φ of QSym satisfying

$$\langle \Phi_\alpha, \overline{\boldsymbol{\Phi}}_\beta \rangle = z_\alpha \delta_{\alpha,\beta}.$$

As in [5], define $\operatorname{sp}(\gamma) = \ell(\gamma)! \prod_j \gamma_j$ and $\operatorname{sp}(\beta,\alpha) = \prod_i \operatorname{sp}(\beta^{(i)})$. Then

$$\overline{\boldsymbol{h}}_\alpha = \sum_{\beta \preccurlyeq \alpha} \frac{1}{\operatorname{sp}(\beta,\alpha)} \overline{\boldsymbol{\Phi}}_\beta, \qquad \text{so that} \quad \phi_\alpha = \sum_{\beta \succcurlyeq \alpha} \frac{1}{\operatorname{sp}(\alpha,\beta)} M_\beta$$

has the property that $\langle \phi_\alpha, \overline{\boldsymbol{\Phi}}_\beta \rangle = \delta_{\alpha,\beta}$. Then the type 2 quasisymmetric power sums[1] satisfy

$$\Phi_\alpha = z_\alpha \phi_\alpha = z_\alpha \sum_{\beta \succcurlyeq \alpha} \frac{1}{\operatorname{sp}(\alpha,\beta)} M_\beta.$$

For example $\Phi_{(2,3,2)} = \frac{2^2 \cdot 2! \cdot 3}{2 \cdot 3 \cdot 2} \left(M_{(2,3,2)} + \frac{1}{2} M_{(5,2)} + \frac{1}{2} M_{(2,5)} + \frac{1}{3!} M_{(7)} \right)$.

We can obtain the following expansion for Φ_α in monomial functions, whose proof (omitted here) is given by rewriting the coefficients and interpreting them in terms of ordered set partitions. For $\alpha \preccurlyeq \beta$ let $\operatorname{OSP}(\alpha,\beta)$ denote the ordered set partitions of $\{1,\ldots,\ell(\alpha)\}$ with block size $|B_i| = \ell(\alpha^{(i)})$. If $\alpha \not\preccurlyeq \beta$, we set $\operatorname{OSP}(\alpha,\beta) = \emptyset$.

Theorem 3.9. *Let $\alpha \vDash n$ and let m_i denote the number of parts of α of size i. Then*

$$\Phi_\alpha = \binom{\ell(\alpha)}{m_1, m_2, \ldots, m_k}^{-1} \sum_{\beta \succcurlyeq \alpha} |\operatorname{OSP}(\alpha,\beta)| M_\beta.$$

As with the type 1 case above, applying Theorem 3.9 gives the following.

Theorem 3.10. *The type 2 quasisymmetric power sums refine the symmetric power sums by*

$$p_\lambda = \sum_{\widetilde{\alpha} = \lambda} \Phi_\alpha.$$

The proof relies on the following identity, which we prove bijectively in [2].

Lemma 3.11. *Let $\lambda \vdash n$ and $\beta \vDash n$. Let m_i denote the number of parts of λ of size i. Then*

$$\binom{\ell(\lambda)}{m_1, m_2, \ldots, m_k} R_{\lambda\beta} = \sum_{\substack{\alpha \preccurlyeq \beta \\ \widetilde{\alpha} = \lambda}} |\operatorname{OSP}(\alpha,\beta)|.$$

[1] Note that a similar polynomial is defined in [11], but is not dual to $\overline{\boldsymbol{\Phi}}$ nor does it refine p_λ.

4 Relationships between bases

Type 1 and type 2 quasisymmetric power sums. To determine the relationship between the two different types of quasisymmetric power sums, we first use duality to expand the monomial quasisymmetric functions in terms of the type 2 quasisymmetric power sums. Thus, from (2.5) and duality we obtain

$$M_\beta = \sum_{\alpha \succcurlyeq \beta} (-1)^{\ell(\beta)-\ell(\alpha)} \frac{\Pi_i \alpha_i}{\ell(\beta,\alpha)} \Phi_\alpha.$$

Combining this with the expansion of Ψ_α in terms of the quasisymmetric monomial function (3.1), we obtain

$$\Psi_\alpha = \sum_{\alpha \preccurlyeq \beta \preccurlyeq \gamma} (-1)^{\ell(\beta)-\ell(\gamma)} \frac{z_\alpha \Pi_i \gamma_i}{\pi(\alpha,\beta)\ell(\beta,\gamma)} \Phi_\gamma.$$

Similarly,

$$\Phi_\alpha = \sum_{\alpha \preccurlyeq \beta \preccurlyeq \gamma} (-1)^{\ell(\beta)-\ell(\gamma)} \frac{z_\alpha \operatorname{lp}(\beta,\gamma)}{\operatorname{sp}(\alpha,\beta)} \Psi_\gamma.$$

Quasisymmetric power sums and the fundamental functions. To describe the type 1 quasisymmetric power sums in terms of the fundamental quasisymmetric functions, we first need to compute the sum of quasisymmetric monomial functions over an interval in the refinement partial order. Using a natural bijection between compositions of n and subsets of $[n-1]$ given by partial sums, we write

$$\operatorname{Set}(\alpha) = \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\cdots+\alpha_{k-1}\} \qquad \text{for } \alpha = (\alpha_1,\ldots,\alpha_k) \vDash n, \text{ and}$$

$$\operatorname{comp}(A) = (a_1, a_2-a_1, \ldots, a_j-a_{j-1}, n-a_j), \qquad \text{for } A = \{a_1,\ldots,a_j\} \subseteq [n-1]$$

with $a_1 < a_2 < \cdots < a_j$. Then let $\alpha^c = \operatorname{comp}((\operatorname{Set}(\alpha))^c)$. Given a second composition β, let $\alpha \wedge \beta$ (respectively $\alpha \vee \beta$) denote the finest (respectively coarsest) composition γ such that $\gamma \succcurlyeq \alpha$ and $\gamma \succcurlyeq \beta$ (respectively $\gamma \preccurlyeq \alpha$ and $\gamma \preccurlyeq \beta$). Note that if $\alpha \wedge \beta = \gamma$, then $\operatorname{Set}(\alpha) \wedge \operatorname{Set}(\beta) = \operatorname{Set}(\gamma)$ in the boolean lattice. For example, if $\alpha = (2,3,1)$ and $\beta = (1,2,2,1)$, then $\alpha^c = (1,2,1,2)$, $\alpha \wedge \beta = (5,1)$, and $\alpha \vee \beta = (1,1,1,2,1)$. The next lemma, which is a straightforward exercise in Möbius inversion, gives a relationship between monomial and fundamental quasisymmetric functions.

Lemma 4.1. *Let $\alpha, \beta \vDash n$ with $\alpha \preccurlyeq \beta$. Then*

$$\sum_{\alpha \preccurlyeq \delta \preccurlyeq \beta} M_\delta = \sum_{\beta \vee \alpha^c \preccurlyeq \delta \preccurlyeq \beta} (-1)^{\ell(\beta)-\ell(\delta)} F_\delta.$$

Given a permutation σ and a composition α, let $\widehat{\alpha(\sigma)}$ denote the coarsest composition β with $\beta \succcurlyeq \alpha$ and $\sigma \in \operatorname{Cons}_{\alpha \preccurlyeq \beta}$. For example, if $\alpha = (3,2,2)$ and $\sigma = 1352467$, then $\widehat{\alpha(\sigma)} = (3,4)$. In addition, we write $\sigma \in \operatorname{Cons}_\alpha$ if we are considering $\beta = \alpha$.

Theorem 4.2. *Let $\alpha \vDash n$. Then*

$$\Psi_\alpha = \frac{z_\alpha}{n!} \sum_{\gamma \succcurlyeq \alpha} |\{\sigma \in \text{Cons}_\alpha : \widehat{\alpha(\sigma)} = \gamma\}| \sum_{\eta \succcurlyeq \alpha^c} (-1)^{\ell(\eta)-1} F_{\gamma \vee \eta}. \tag{4.1}$$

Proof. Let $\alpha \vDash n$. We use $\mathbb{1}_{\mathcal{R}}$ to denote the characteristic function of a relation $\mathcal{R}$. Combining the quasisymmetric monomial function expansion of Ψ_α given in (3.1), Lemma 3.3, and Lemma 4.1, we have

$$\begin{aligned}
\Psi_\alpha &= \frac{z_\alpha}{n!} \sum_{\alpha \preccurlyeq \beta} |\text{Cons}_{\alpha \preccurlyeq \beta}| M_\beta = \frac{z_\alpha}{n!} \sum_{\sigma \in \text{Cons}_\alpha} \sum_{\alpha \preccurlyeq \delta \preccurlyeq \widehat{\alpha(\sigma)}} M_\delta \\
&= \frac{z_\alpha}{n!} \sum_{\sigma \in \text{Cons}_\alpha} \sum_{\alpha^c \vee \widehat{\alpha(\sigma)} \preccurlyeq \delta \preccurlyeq \widehat{\alpha(\sigma)}} (-1)^{\ell(\widehat{\alpha(\sigma)}) - \ell(\delta)} F_\delta \\
&= \frac{z_\alpha}{n!} \sum_{\delta \vDash n} (-1)^{\ell(\delta)} F_\delta \sum_{\sigma \in \text{Cons}_\alpha} (-1)^{\ell(\widehat{\alpha(\sigma)})} \mathbb{1}_{\alpha^c \vee \widehat{\alpha(\sigma)} \preccurlyeq \delta \preccurlyeq \widehat{\alpha(\sigma)}} \\
&= \frac{z_\alpha}{n!} \sum_{\gamma \succcurlyeq \alpha} |\{\sigma \in \text{Cons}_\alpha : \widehat{\alpha(\sigma)} = \gamma\}| \sum_{\delta \vDash n} (-1)^{\ell(\gamma) - \ell(\delta)} F_\delta \mathbb{1}_{\alpha^c \vee \gamma \preccurlyeq \delta \preccurlyeq \gamma},
\end{aligned}$$

with the last equality holding since the compositions $\widehat{\alpha(\sigma)}$ are coarsenings of α. Given $\gamma \succcurlyeq \alpha$ and $\delta \vDash n$, there exists $\eta \succcurlyeq \alpha^c$ such that $\delta = \gamma \vee \eta$ if and only if $\delta \succcurlyeq \alpha^c \vee \gamma$. So

$$\begin{aligned}
\Psi_\alpha &= \frac{z_\alpha}{n!} \sum_{\gamma \succcurlyeq \alpha} |\{\sigma \in \text{Cons}_\alpha : \widehat{\alpha(\sigma)} = \gamma\}| (-1)^{\ell(\gamma)} \sum_{\eta \succcurlyeq \alpha^c} (-1)^{\ell(\gamma \vee \eta)} F_{\gamma \vee \eta} \\
&= \frac{z_\alpha}{n!} \sum_{\gamma \succcurlyeq \alpha} |\{\sigma \in \text{Cons}_\alpha : \widehat{\alpha(\sigma)} = \gamma\}| \sum_{\eta \succcurlyeq \alpha^c} (-1)^{\ell(\eta)-1} F_{\gamma \vee \eta}.
\end{aligned}$$ □

Note that in (4.1) each composition occurs at most once in the indexing set for F.

Theorem 4.3. *Let $\alpha \vDash n$. Then*

$$\Phi_\alpha = \binom{m_1 + \cdots + m_n}{m_1, \ldots, m_n}^{-1} \sum_{\gamma \vDash n} \left(\sum_{\beta \succcurlyeq (\gamma \wedge \alpha)} (-1)^{\ell(\gamma) - \ell(\beta)} |\text{OSP}(\alpha, \beta)| \right) F_\gamma.$$

This is proved by direct substitution of monomial and fundamental expansions [2].

5 Products of quasisymmetric power sums

Unlike symmetric power sums, the quasisymmetric power sums are not multiplicative. Comultiplication for the noncommutative symmetric power sums (type 1) (dual to multiplication in QSym) is given in [5] by $\Delta(\overline{\mathbf{\Psi}}_k) = 1 \oplus \overline{\mathbf{\Psi}}_k + \overline{\mathbf{\Psi}}_k \oplus 1$. Thus

$$\Delta(\overline{\mathbf{\Psi}}_\alpha) = \prod_i \Delta(\overline{\mathbf{\Psi}}_{\alpha_i}) = \prod_i (1 \oplus \overline{\mathbf{\Psi}}_{\alpha_i} + \overline{\mathbf{\Psi}}_{\alpha_i} \oplus 1) = \sum_{\substack{\gamma, \beta \\ \alpha \in \gamma \sqcup\!\sqcup \beta}} \overline{\mathbf{\Psi}}_\gamma \oplus \overline{\mathbf{\Psi}}_\beta,$$

where $\gamma \sqcup\!\sqcup \beta$ is the set of shuffles of $\gamma = (\gamma_1, \cdots, \gamma_n)$ and $\beta = (\beta_1, \cdots, \beta_m)$. Let a_j denote the number of parts of size j in α and b_j denote the number of parts of size j in β, and $\alpha \cdot \beta$ their concatenation. Define $C(\alpha, \beta) = \prod_j \binom{a_j + b_j}{a_j}$. It is straightforward to check that $C(\alpha, \beta) = z_{\alpha \cdot \beta}/(z_\alpha z_\beta)$.

Theorem 5.1. *Let α and β be compositions. Then*

$$\Psi_\alpha \Psi_\beta = \frac{1}{C(\alpha, \beta)} \sum_{\gamma \in \alpha \sqcup\!\sqcup \beta} \Psi_\gamma \qquad \text{and} \qquad \Phi_\alpha \Phi_\beta = \frac{1}{C(\alpha, \beta)} \sum_{\gamma \in \alpha \sqcup\!\sqcup \beta} \Phi_\gamma.$$

Proof. (sketch of type 1 proof.) This can be proved easily using duality, or directly using the quasisymmetric monomial function expansion of the quasisymmetric power sums. For the latter, one can show that the coefficients in the quasisymmetric monomial function expansions of both sides of the product formulas in Theorem 5.1 are the same. In particular, if $\alpha \vDash m$, $\beta \vDash n$, and ξ is a fixed coarsening of a shuffle of α and β, we have

$$\binom{m+n}{m} \sum_{\substack{\delta \succcurlyeq \alpha, \eta \succcurlyeq \beta \\ \xi \in \delta \overline{\sqcup\!\sqcup} \eta}} \frac{m!}{\pi(\alpha, \delta)} \frac{n!}{\pi(\beta, \eta)} = \sum_{\substack{\gamma \in \alpha \sqcup\!\sqcup \beta \\ \gamma \preccurlyeq \xi}} \frac{(m+n)!}{\pi(\gamma, \xi)},$$

where $\delta \,\overline{\sqcup\!\sqcup}\, \eta$ is the set of *overlapping shuffles* of δ and η, that is, shuffles where a part of δ and a part of η can be added to form a single part. □

6 Comments and future directions

Two immediate questions of interest are as follows. First, do the quasisymmetric power sums play a role in the representation theory of the 0-Hecke algebra? Namely, the Frobenius map sends class functions C_λ to $\frac{p_\lambda}{z_\lambda}$ in Sym, and maps irreducible characters to their corresponding Schur functions. Krob and Thibon [9] define quasisymmetric and noncommutative symmetric characteristic maps; one takes irreducible representations of the 0-Hecke algebra to the fundamental quasisymmetric basis, the other takes indecomposable representations of the same algebra to the ribbon basis. It would be interesting to explore whether these maps can now be defined in terms of the corresponding power sums, just as in the symmetric case.

Second, what is the product of a quasisymmetric power sum with a quasisymmetric Schur function? For power sums indexed by a single part (as in Murnaghan–Nakayama rules), Tewari [13] gives a noncommutative symmetric functions analogue and Tiefenbruck [14] a quasisymmetric analogue. But since the quasisymmetric power sums are not multiplicative, this is only a partial solution.

It is also natural to consider the role of these power sums in plethysm. However, while plethysm has been defined in QSym and NSym (see [8]), our functions do not play the expected role (as Adams operators in the language of λ-rings) in this case.

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