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Proof of Chapoton's conjecture on Newton polygons of q-Ehrhart polynomials

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Abstract. Recently, Chapoton found a q-analog of Ehrhart polynomials, which are polynomials in x whose coefficients are rational functions in q. Chapoton conjectured the shape of the Newton polygon of the numerator of the q-Ehrhart polynomial of an order polytope. In this paper, we prove Chapoton's conjecture.

Keywords: q-Ehrhart polynomial, Newton polygon, order polytope, P-partition

1 Introduction

In 1962, Ehrhart [5] discovered certain polynomials associated to lattice polytopes. These polynomials are now widely known and called Ehrhart polynomials. They contain important information of lattice polytopes such as the number of lattice points in the polytope, the number of lattice points in the relative interior and the relative volume of the polytope.

Recently, Chapoton [4] found a q-analog of Ehrhart polynomials and generalized some properties of them. A q-Ehrhart polynomial is a polynomial in variable x whose coefficients are rational functions in q. Thus we can write a q-Ehrhart polynomial as a rational function in q and x whose numerator is a polynomial in q and x, and whose denominator is a polynomial in q. In the same paper, Chapoton conjectured the shape of the Newton polygon of the numerator of the q-Ehrhart polynomial associated to an order polytope. The goal of this paper is to prove Chapoton's conjecture.

First, we briefly review basic properties of Ehrhart polynomials and their q-analogs. See [1, 3, 2] for more details in Ehrhart polynomials.

A point in $\mathbb{R}^m$ is called a *lattice point* if all the coordinates are integers. A *lattice polytope* is a polytope whose vertices are lattice points. All polytopes considered in this paper are lattice polytopes.

For a polytope M and an integer n, we denote by nM the dilation of M by a scale factor of n, i.e.,

$$nM = \{n\mathbf{x} : \mathbf{x} \in M\}.$$

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For a lattice polytope M in $\mathbb{R}^m$, there exists a polynomial $E(x)$, called the *Ehrhart polynomial of M*, satisfying the following interesting properties:

- $E(n) = |nM \cap \mathbb{Z}^m|$ for all integers $n \geq 0$.
- $(-1)^{\dim M} E(-n) = |nM^\circ \cap \mathbb{Z}^m|$ for all integers $n \geq 0$, where $\dim M$ is the dimension of M and M° is the relative interior of M.
- The degree of $E(x)$ is equal to the dimension of M.
- The leading coefficient of $E(x)$ is equal to the relative volume of M.

For a polytope M in $\mathbb{R}^m$, let

$$W(M,q) = \sum_{\mathbf{x} \in M \cap \mathbb{Z}^m} q^{|\mathbf{x}|},$$

where for $\mathbf{x} = (x_1, \dots, x_m)$ we denote

$$|\mathbf{x}| = x_1 + \cdots + x_m.$$

We use the standard notation for q-integers: for $n \in \mathbb{Z}$,

$$[n]_q := \frac{1-q^n}{1-q},$$

and, for integers $n \geq k \geq 0$,

$$[n]_q! := [1]_q [2]_q \dots [n]_q, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Note that for $n \geq 0$ and $a, b \in \mathbb{Z}$, we have

$$\begin{aligned} [n]_q &= 1 + q + q^2 + \cdots + q^{n-1}, \\ [-n]_q &= -q^{-n} [n]_q, \\ [a+b]_q &= [a]_q + q^a [b]_q. \end{aligned}$$

Chapoton [4, Theorem 3.1] found a q-analog of Ehrhart polynomials as follows.

Theorem 1.1 (Chapoton)**.** *Let M be a polytope satisfying the following conditions:*

- *For every vertex $\mathbf{x}$ of M, we have $|\mathbf{x}| \geq 0$.*
- *For every edge between two vertices $\mathbf{x}$ and $\mathbf{y}$ of M, we have $|\mathbf{x}| \neq |\mathbf{y}|$.*

Then there is a polynomial $E(x) \in \mathbb{Q}(q)[x]$ such that for every integer $n \geq 0$,

$$E([n]_q) = W(nM, q).$$

The polynomial $E(x)$ in Theorem 1.1 is called the *q-Ehrhart polynomial* of the polytope M. We note that in [4], more generally, Chapoton considers a linear form $\lambda(\mathbf{x})$ on $\mathbb{R}^m$ in place of $|\mathbf{x}|$. In this setting with a linear form, Chapoton [4, Theorem 3.5] also shows a nice q-analog of the Ehrhart–Macdonald reciprocity:

$$E([-n]_q) = (-1)^{\dim M} W(nM^\circ, 1/q).$$

We note that Kim and Stanton [7, Theorem 9.3] showed that the leading coefficient of the q-Ehrhart polynomial of an order polytope is equal to the q-volume of the order polytope, which is defined as a Jackson's q-integral over the order polytope.

In order to state Chapoton's conjecture we need some notation and terminology.

For a polynomial $f(x_1,\dots,x_k)$ in $x_1,\dots,x_k$, we denote by $[x_1^{i_1}\dots x_k^{i_k}]f(x_1,\dots,x_k)$ the coefficient of $x_1^{i_1}\dots x_k^{i_k}$ in $f(x_1,\dots,x_k)$. For a polynomial $f(x_1,\dots,x_k)$ in $x_1,\dots,x_k$, the *Newton polytope* of $f(x_1,\dots,x_k)$, denoted by $\mathrm{Newton}(f(x_1,\dots,x_k))$, is the convex hull of the points $(i_1,\dots,i_k)$ such that $[x_1^{i_1}\dots x_k^{i_k}]f(x_1,\dots,x_k) \neq 0$. In this paper, we consider *Newton polygons*, which are Newton polytopes of two-variable functions.

For a poset P on $\{1,2,\dots,m\}$, the *order polytope* $\mathcal{O}(P)$ of P is defined by

$$\mathcal{O}(P) = \{(x_1,\dots,x_m) \in [0,1]^m : x_i \le x_j \text{ if } i \le_P j\}.$$

As mentioned in [4], using the properties of vertices and edges of an order polytope in [9] one can check that every order polytope satisfies the conditions in Theorem 1.1. Therefore, we can consider the q-Ehrhart polynomial of an order polytope.

Let $E_P(x)$ be the q-Ehrhart polynomial of $\mathcal{O}(P)$. We denote by $N_P(q,x)$ be the numerator of $E_P(x)$. More precisely, $N_P(q,x)$ is the unique polynomial in $\mathbb{Z}[q,x]$ with positive leading coefficient such that

$$E_P(x) = \frac{N_P(q,x)}{D(q)},$$

for some polynomial $D(q) \in \mathbb{Z}[q]$ with $\gcd(N_P(q,x), D(q)) = 1$.

For integers $1 \le a_1 \le a_2 \le \cdots \le a_m$ and $h \ge a_1 + \cdots + a_m$, we define $C(a_1,\dots,a_m;h)$ to be the convex hull of the points $(0,0)$, $(a_1+\cdots+a_i, i)$ for $1 \le i \le m$, (h,m) and $(h-m,0)$. See Figure 1 for an example.

Let P be a poset and $x \in P$. A *chain ending at* x (resp. *starting at* x) is a subset $\{t_1 <_P \cdots <_P t_k\}$ of P with $t_k = x$ (resp. $t_1 = x$). The *size* of a chain is the number of elements in the chain. We denote by $\mathrm{mc}_P(x)$ the maximum size of a chain ending at x. We also denote by $\overline{\mathrm{mc}}_P(x)$ the maximum size of a chain starting at x. When there is no possible confusion, we will simply write as $\mathrm{mc}(x)$ and $\overline{\mathrm{mc}}(x)$ instead of $\mathrm{mc}_P(x)$ and $\overline{\mathrm{mc}}_P(x)$.

In [4, Conjecture 5.3], Chapoton proposed the following conjecture on the shape of the Newton polygon of $N_P(q,x)$.

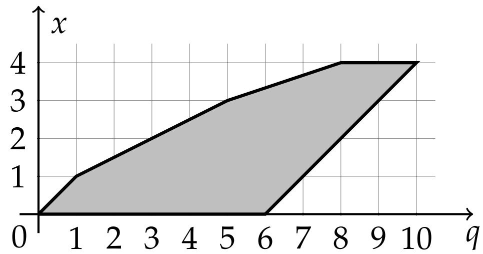

Figure 1: The polygon $C(1, 2, 2, 3; 10)$ in the (q, x)-coordinate system.

Conjecture 1.2. Let P be a poset on $\{1, 2, \dots, m\}$. Suppose that $a_1 \le a_2 \le \cdots \le a_m$ is the increasing rearrangement of $\overline{\mathrm{mc}}(1), \dots, \overline{\mathrm{mc}}(m)$. Then the Newton polygon of the numerator of the q-Ehrhart polynomial of $\mathcal{O}(P)$ is given by

$$\mathrm{Newton}(N_P(q, x)) = C(a_1, \dots, a_m; h),$$

for some integer $h \ge a_1 + \cdots + a_m$.

The goal of this paper is to prove Conjecture 1.2. As Chapoton points out in [4], the q-Ehrhart polynomial $E_P(x)$ of $\mathcal{O}(P)$ can be understood as a generating function for P-partitions of $\overline{P}$, the dual poset of P. It is well-known that the generating function for P-partitions can be expressed in terms of linear extensions of the poset. One of the main ingredients of our proof of Conjecture 1.2 is Corollary 3.4, which gives a description of the minimum of $\mathrm{maj}(\pi) - k\,\mathrm{des}(\pi)$ over all linear extensions π of P.

The rest of this paper is organized as follows. In Section 2, we recall necessary definitions and state our main result (Theorem 2.5), which describes the precise shape of the Newton polygon of $[m]_q!E_P(x)$. Then we show that Theorem 2.5 implies Conjecture 1.2. In Section 3 we find some property of the linear extensions of a poset. In Section 4 we prove Theorem 2.5.

This is an extended abstract of [6].

2 The main result

In this section we state our main theorem, which implies Conjecture 1.2.

We first recall some definitions on permutations and posets. We refer the reader to [8] for more details.

The set of nonnegative integers is denoted by $\mathbb{N}$.

Let $\mathfrak{S}_m$ be the set of permutations of $\{1, 2, \dots, m\}$. For $\pi = \pi_1 \dots \pi_m \in \mathfrak{S}_m$, a *descent* of π is an integer $1 \le i \le m-1$ such that $\pi_i > \pi_{i+1}$. We denote by $\mathrm{Des}(\pi)$ the set of descents of π. We define $\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i$ and $\mathrm{des}(\pi) = |\,\mathrm{Des}(\pi)|$.

Let P be a poset on $\{1, 2, \dots, m\}$. A *P-partition* is an order-reversing map $\sigma : P \to \mathbb{N}$, i.e., $\sigma(x) \ge \sigma(y)$ if $x \le_P y$. For a P-partition σ, let $|\sigma| = \sigma(1) + \cdots + \sigma(m)$. We denote by

$\mathcal{P}(P)$ the set of P-partitions. For an integer n, we denote by $\mathcal{P}(P,n)$ the set of P-partitions σ satisfying $\sigma(x) \le n$ for all $x \in P$.

We say that P is *naturally labeled* if $x \le_P y$ implies $x \le y$. A *linear extension* of P is a permutation $\pi = \pi_1 \dots \pi_m \in \mathfrak{S}_m$ such that $\pi_i \le_P \pi_j$ implies $i \le j$. We denote by $\mathcal{L}(P)$ the set of linear extensions of P. Note that if P is naturally labeled, $\mathcal{L}(P)$ always contains the identity permutation.

We need the following lemma, which gives a connection between certain generating functions for $\mathcal{P}(P,n)$ and $\mathcal{L}(P)$.

Lemma 2.1. *For a naturally labeled poset P on $\{1,2,\dots,m\}$, we have*

$$\sum_{\sigma\in\mathcal{P}(P,n)} q^{|\sigma|} = \sum_{\pi\in\mathcal{L}(P)} q^{\operatorname{maj}(\pi)} \begin{bmatrix} n-\operatorname{des}(\pi)+m \\ m \end{bmatrix}_q.$$

Proof. For a permutation $w \in \mathfrak{S}_m$, let S_w denote the set of all functions $f : P \to \mathbb{N}$ satisfying the following conditions:

- $f(w_1) \ge f(w_2) \ge \cdots \ge f(w_m)$ and
- $f(w_i) > f(w_{i+1})$ if $i \in \operatorname{Des}(w)$.

It is well known [8, Lemma 3.15.3] that

$$\mathcal{P}(P) = \biguplus_{\pi\in\mathcal{L}(P)} S_\pi.$$

Let $S_\pi(n) = S_\pi \cap \mathcal{P}(P,n)$. Then we have

$$\mathcal{P}(P,n) = \biguplus_{\pi\in\mathcal{L}(P)} S_\pi(n).$$

Thus,

$$\sum_{\sigma\in\mathcal{P}(P,n)} q^{|\sigma|} = \sum_{\pi\in\mathcal{L}(P)} \sum_{\sigma\in S_\pi(n)} q^{|\sigma|} = \sum_{\pi\in\mathcal{L}(P)} \sum_{\substack{n\ge i_1\ge\cdots\ge i_m\ge 0\\ i_j>i_{j+1} \text{ if } j\in\operatorname{Des}(\pi)}} q^{i_1+\cdots+i_m}.$$

It is shown in [7, Lemma 4.5] that

$$\sum_{\substack{n\ge i_1\ge\cdots\ge i_m\ge 0\\ i_j>i_{j+1} \text{ if } j\in\operatorname{Des}(\pi)}} q^{i_1+\cdots+i_m} = q^{\operatorname{maj}(\pi)} \begin{bmatrix} n-\operatorname{des}(\pi)+m \\ m \end{bmatrix}_q,$$

which completes the proof. $\square$

For a poset P, we denote its dual by $\overline{P}$, that is, $x \le_P y$ if and only if $y \le_{\overline{P}} x$. By definition, for a poset P and an integer $n \in \mathbb{N}$, we have

$$W(n\mathcal{O}(P), q) = \sum_{\sigma \in \mathcal{P}(\overline{P}, n)} q^{|\sigma|}. \tag{2.1}$$

Therefore, the q-Ehrhart polynomial $E_P(x)$ of $\mathcal{O}(P)$ is closely related to P-partitions of $\overline{P}$. The next proposition shows that $E_P(x)$ can be written as a generating function for linear extensions of $\overline{P}$.

Proposition 2.2. *Let P be a poset on $\{1, 2, \dots, m\}$. Suppose that $\overline{P}$ is naturally labeled. Then the q-Ehrhart polynomial of $\mathcal{O}(P)$ is*

$$E_P(x) = \frac{1}{[m]_q!} \sum_{\pi \in \mathcal{L}(\overline{P})} q^{\mathrm{maj}(\pi)} \prod_{i=1}^{m} ([i - \mathrm{des}(\pi)]_q + q^{i-\mathrm{des}(\pi)} x).$$

Proof. Let $f(x)$ be the right hand side. Then

$$f([n]_q) = \sum_{\pi \in \mathcal{L}(\overline{P})} q^{\mathrm{maj}(\pi)} \frac{\prod_{i=1}^{m} [i - \mathrm{des}(\pi) + n]_q}{[m]_q!} = \sum_{\pi \in \mathcal{L}(\overline{P})} q^{\mathrm{maj}(\pi)} \begin{bmatrix} n - \mathrm{des}(\pi) + m \\ m \end{bmatrix}_q .$$

On the other hand, by Lemma (2.1) and (2.1), we have

$$W(n\mathcal{O}(P), q) = \sum_{\pi \in \mathcal{L}(\overline{P})} q^{\mathrm{maj}(\pi)} \begin{bmatrix} n - \mathrm{des}(\pi) + m \\ m \end{bmatrix}_q .$$

Thus $f([n]_q) = W(n\mathcal{O}(P), q)$ for all $n \in \mathbb{N}$ and we obtain $E_P(x) = f(x)$. $\square$

Now we define a polynomial $F_P(q, x)$ in q and x, which will be used throughout this paper.

Definition 2.3. For a poset P on $\{1, 2, \dots, m\}$, we define

$$F_P(q, x) = \sum_{\pi \in \mathcal{L}(P)} q^{\mathrm{maj}(\pi)} \prod_{i=1}^{m} ([i - \mathrm{des}(\pi)]_q + q^{i-\mathrm{des}(\pi)} x).$$

Note that we always have $F_P(q, x) \in \mathbb{Z}[q, x]$ because for every $\pi \in \mathcal{L}(P)$, the power of q in each summand is at least

$$\mathrm{maj}(\pi) + \sum_{i=1}^{\mathrm{des}(\pi)} (i - \mathrm{des}(\pi)) \ge \binom{\mathrm{des}(\pi) + 1}{2} + \binom{\mathrm{des}(\pi) + 1}{2} - \mathrm{des}(\pi)^2 \ge 0.$$

Proposition 2.2 implies that for a naturally labeled poset P on $\{1, 2, \dots, m\}$, we have

$$F_P(q, x) = [m]_q! E_{\overline{P}}(x). \tag{2.2}$$

Proposition 2.4. *Let P be a poset on $\{1,2,\dots,m\}$ such that $\overline{P}$ is naturally labeled. Suppose that $a_1 \le a_2 \le \cdots \le a_m$ is the increasing rearrangement of $\overline{\mathrm{mc}}(1),\dots,\overline{\mathrm{mc}}(m)$. Then we have*

$$\mathrm{Newton}(N_P(q,x)) = C(a_1,\dots,a_m;h),$$

for some $h \ge a_1+\cdots+a_m$ if and only if

$$\mathrm{Newton}(F_{\overline{P}}(q,x)) = C(a_1,\dots,a_m;h'),$$

for some $h' \ge a_1+\cdots+a_m$. Moreover, in this case we always have $h' = h+r$, where $r = \deg\phi(q)$ and $\phi(q) = \gcd(F_{\overline{P}}(q,x),[m]_q!)$.

Proof. By (2.2), we have

$$F_{\overline{P}}(q,x) = N_P(q,x)\phi(q).$$

Since $\phi(q)$ divides $[m]_q!$, the leading coefficient and the constant term of $\phi(q)$ are both 1. Thus, we have

$$\phi(q) = q^r + c_{r-1}q^{r-1}+\cdots+c_1q^1+1,$$

for some $c_1,\dots,c_{r-1}\in\mathbb{Z}$. Hence, for each $1\le k\le m$, we have

$$\max\{i : [q^ix^k]N_P(q,x)\neq 0\} = \max\{i : [q^{i+r}x^k]F_{\overline{P}}(q,x)\neq 0\},$$

$$\min\{i : [q^ix^k]N_P(q,x)\neq 0\} = \min\{i : [q^ix^k]F_{\overline{P}}(q,x)\neq 0\},$$

which imply the statement. □

Now we state our main theorem.

Theorem 2.5. *Let P be a naturally labeled poset on $\{1,2,\dots,m\}$. Let $b_1\le b_2\le\cdots\le b_m$ be the increasing rearrangement of $\mathrm{mc}(1),\mathrm{mc}(2),\dots,\mathrm{mc}(m)$. Then the Newton polygon of*

$$F_P(q,x) = [m]_q!E_{\overline{P}}(x) = \sum_{\pi\in\mathcal{L}(P)} q^{\mathrm{maj}(\pi)}\prod_{i=1}^{m}([i-\mathrm{des}(\pi)]_q + q^{i-\mathrm{des}(\pi)}x)$$

is given by

$$\mathrm{Newton}(F_P(q,x)) = C\left(b_1,\dots,b_m;\binom{m+1}{2}\right).$$

We prove Theorem 2.5 in Section 4. Note that in Theorem 2.5 we have

$$b_1+\cdots+b_m \le \binom{m+1}{2},$$

which follows from the fact that $b_1 = 1$ and $b_{i+1}\le b_i+1$ for all i.

We finish this section by showing that Theorem 2.5 implies Conjecture 1.2.

Proof of Conjecture 1.2. Note that relabeling of P does not affect $E_P(q,x)$. Hence, we can assume that $\overline{P}$ is naturally labeled. Observe that $\overline{\mathrm{mc}}_P(x) = \mathrm{mc}_{\overline{P}}(x)$ for all $x \in \{1,2,\dots,m\}$. By Theorem 2.5,

$$\mathrm{Newton}(F_{\overline{P}}(q,x)) = C\left(a_1,\dots,a_m;\binom{m+1}{2}\right).$$

By Proposition 2.4, we obtain that

$$\mathrm{Newton}(N_P(q,x)) = C(a_1,\dots,a_m;h),$$

for some integer $h \ge a_1 + \cdots + a_m$. This completes the proof. $\square$

3 Some properties of linear extensions

In this section we prove some properties of posets which will be used in the next section.

Lemma 3.1. *Let P be a naturally labeled poset on $\{1,2,\dots,m\}$ and $\pi \in \mathcal{L}(P)$. Suppose that $\mathrm{Des}(\pi) \ne \emptyset$ and c is the largest descent of π. Then there is a permutation $\sigma \in \mathcal{L}(P)$ such that $\mathrm{Des}(\sigma) = \mathrm{Des}(\pi) \setminus \{c\}$.*

Definition 3.2. Let $\pi = \pi_1 \dots \pi_m \in \mathfrak{S}_m$. A *descent block* of π is a maximal consecutive subsequence of π which is in decreasing order. We denote by $\mathrm{DB}_i(\pi)$ the set of elements in the ith descent block of π.

For example, if $\pi = 384196725$, then the descent blocks of π are 3, 841, 96, 72, 5, and $\mathrm{DB}_1(\pi) = \{3\}$, $\mathrm{DB}_2(\pi) = \{1,4,8\}$, $\mathrm{DB}_3(\pi) = \{6,9\}$, $\mathrm{DB}_4(\pi) = \{2,7\}$, $\mathrm{DB}_5(\pi) = \{5\}$.

The following proposition is the key ingredient for proving Chapoton's conjecture.

Proposition 3.3. *Let P be a naturally labeled poset on $\{1,2,\dots,m\}$. Suppose that $b_1 \le b_2 \le \cdots \le b_m$ is the increasing rearrangement of $\mathrm{mc}(1),\mathrm{mc}(2),\dots,\mathrm{mc}(m)$ and*

$$C_i = \{x \in P : \mathrm{mc}(x) = i\}.$$

Then, for $\pi \in \mathcal{L}(P)$ and $0 \le k \le m$, we have

$$\mathrm{maj}(\pi) - k\,\mathrm{des}(\pi) + \binom{k+1}{2} \ge b_1 + \cdots + b_k.$$

The equality holds if and only if all of the following conditions hold:

- $\mathrm{Des}(\pi) \subseteq \{1,2,\dots,k\}$,
- $\mathrm{DB}_i(\pi) = C_i$, *for* $1 \le i \le p-1$,

- $\mathrm{DB}_p(\pi) \subseteq C_p$,

where p is the integer satisfying

$$|C_1| + \cdots + |C_{p-1}| < k \le |C_1| + \cdots + |C_p|.$$

Furthermore, for every $0 \le k \le m$, there is a permutation in $\mathcal{L}(P)$ satisfying these conditions.

The following corollary is an immediate consequence of Proposition 3.3.

Corollary 3.4. *Let P be a naturally labeled poset on $\{1, 2, \ldots, m\}$. Suppose that $b_1 \le b_2 \le \cdots \le b_m$ is the increasing rearrangement of $\mathrm{mc}(1), \mathrm{mc}(2), \ldots, \mathrm{mc}(m)$. Then, for $0 \le k \le m$, we have*

$$\min \left\{ \mathrm{maj}(\pi) - k\,\mathrm{des}(\pi) + \binom{k+1}{2} : \pi \in \mathcal{L}(P) \right\} = b_1 + \cdots + b_k.$$

Moreover, for $1 \le k \le m$, if P is not a chain, we have

$$\min \left\{ \mathrm{maj}(\pi) - k\,\mathrm{des}(\pi) + \binom{k+1}{2} : \pi \in \mathcal{L}(P), 1 \le \mathrm{des}(\pi) \le k \right\} = b_1 + \cdots + b_k.$$

Note that Corollary 3.4 allows us to find the minimum of $\mathrm{maj}(\pi) - k\,\mathrm{des}(\pi)$ over $\pi \in \mathcal{L}(P)$. The second part of Corollary 3.4 means that if $1 \le k \le m$ and P is not a chain, the minimum of $\mathrm{maj}(\pi) - k\,\mathrm{des}(\pi) + \binom{k+1}{2}$ for all $\pi \in \mathcal{L}(P)$ is attained for $\pi \in \mathcal{L}(P)$ satisfying $1 \le \mathrm{des}(\pi) \le k$. This will be used in the next section.

4 Proof of Theorem 2.5

In this section we assume that P is a naturally labeled poset on $\{1, 2, \ldots, m\}$ and $b_1 \le b_2 \le \cdots \le b_m$ is the increasing rearrangement of $\mathrm{mc}(1), \ldots, \mathrm{mc}(m)$.

For a polynomial $f(q)$ in q, define

$$q_{\max}(f(q)) = \max\{i : [q^i]f(q) \ne 0\},$$
$$q_{\min}(f(q)) = \min\{i : [q^i]f(q) \ne 0\}.$$

When $f(q) = 0$, we use the following convention:

$$q_{\max}(0) = -\infty, \qquad q_{\min}(0) = \infty.$$

Recall that

$$\begin{aligned} F_P(q,x) &= \sum_{\pi \in \mathcal{L}(P)} q^{\mathrm{maj}(\pi)} \prod_{i=1}^{m} ([i - \mathrm{des}(\pi)]_q + q^{i-\mathrm{des}(\pi)} x) \\ &= \sum_{s=0}^{m-1} \sum_{\pi \in \mathcal{L}(P), \mathrm{des}(\pi)=s} q^{\mathrm{maj}(\pi)} \prod_{i=1}^{m} (q^{i-s} x + [i-s]_q). \end{aligned}$$

Since P is naturally labeled, $\mathcal{L}(P)$ contains the identity permutation. Therefore,

$$F_P(q,x) = A + B, \tag{4.1}$$

where

$$A = \prod_{i=1}^{m}(q^i x + [i]_q),$$

$$B = x\sum_{s=1}^{m-1}\sum_{\pi\in\mathcal{L}(P),\mathrm{des}(\pi)=s} q^{\mathrm{maj}(\pi)-\binom{s}{2}}\prod_{i=1}^{s-1}(x-[i]_q)\prod_{i=1}^{m-s}(q^i x + [i]_q).$$

Since $[x^0]F_P(q,x) = [x^0]A = [m]_q!$, we have

$$q_{\max}([x^0]F_P(q,x)) = \binom{m}{2}, \qquad q_{\min}([x^0]F_P(q,x)) = 0.$$

Therefore, in order to prove Theorem 2.5, it suffices to show the following two propositions.

Proposition 4.1. *For $1 \le k \le m$, we have*

$$q_{\max}([x^k]F_P(q,x)) = \binom{m}{2} + k.$$

Proposition 4.2. *For $1 \le k \le m$, we have*

$$q_{\min}([x^k]F_P(q,x)) = b_1 + \cdots + b_k.$$

Proof of Proposition 4.1. By (4.1), it is enough to show that

$$q_{\max}([x^k]A) = \binom{m}{2} + k, \tag{4.2}$$

$$q_{\max}([x^k]B) < \binom{m}{2} + k. \tag{4.3}$$

In order to get the largest power of q, when we expand the product in A, we must select $q^i x$ or q^{i-1}. This implies (4.2).

To prove (4.3), consider $\pi \in \mathfrak{S}_m$ with $\mathrm{des}(\pi) = s \ge 1$. Then

$$\begin{aligned}
&q_{\max}\left([x^{k-1}]q^{\mathrm{maj}(\pi)-\binom{s}{2}}\prod_{i=1}^{s-1}(x-[i]_q)\prod_{i=1}^{m-s}(q^i x+[i]_q)\right)\\
&\le \sum_{i=1}^{s}(m-i) - \binom{s}{2} + \sum_{i=1}^{s-1}(i-1) + \sum_{i=1}^{m-s}(i-1) + (k-1)\\
&= \binom{m}{2} - (s-1) + (k-1) < \binom{m}{2} + k.
\end{aligned}$$

Therefore, we obtain (4.3). □

The rest of this section is devoted to proving Proposition 4.2.

For $\pi \in \mathfrak{S}_m$ with $\operatorname{des}(\pi) = s \geq 1$ and an integer $1 \leq k \leq m$, let

$$t(\pi, k) = [x^{k-1}] q^{\operatorname{maj}(\pi) - \binom{s}{2}} \prod_{i=1}^{s-1} (x - [i]_q) \prod_{i=1}^{m-s} (q^i x + [i]_q). \tag{4.4}$$

Then we always have

$$q_{\min}(t(\pi, k)) \geq \operatorname{maj}(\pi) - \binom{\operatorname{des}(\pi)}{2}. \tag{4.5}$$

We need the following two lemmas.

Lemma 4.3. *Let $\pi \in \mathfrak{S}_m$ with $\operatorname{des}(\pi) = s \geq 1$. Then, for $s \leq k \leq m$, we have*

$$q_{\min}(t(\pi, k)) = \operatorname{maj}(\pi) - ks + \binom{k+1}{2}.$$

Lemma 4.4. *Let P be a naturally labeled poset on $\{1, 2, \dots, m\}$. Suppose that P is not a chain. Then, for $1 \leq k \leq m$, we have*

$$q_{\min}([x^k]B) = \min \left\{ \operatorname{maj}(\pi) - k \operatorname{des}(\pi) + \binom{k+1}{2} : \pi \in \mathcal{L}(P), 1 \leq \operatorname{des}(\pi) \leq k \right\}.$$

Now we give a proof of Proposition 4.2.

Proof of Proposition 4.2. First, observe that

$$q_{\min}([x^k]A) = \binom{k+1}{2}.$$

If P is a chain, then the identity permutation is the only linear extension of P. In this case $B = 0$ and $b_i = i$. Thus

$$q_{\min}([x^k]F_P(q, x)) = q_{\min}([x^k]A) = \binom{k+1}{2} = b_1 + \cdots + b_k.$$

Now suppose that P is not a chain. By Lemma 4.4 and Corollary 3.4 we have

$$q_{\min}([x^k]B) = b_1 + \cdots + b_k \leq \binom{k+1}{2}.$$

Therefore we also obtain

$$q_{\min}([x^k]F_P(q, x)) = b_1 + \cdots + b_k. \qquad \square$$

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