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Generalized nil-Coxeter algebras

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Abstract. Motivated by work of Coxeter (1957), we study a class of algebras associated to Coxeter groups, which we term 'generalized nil-Coxeter algebras'. We construct the first finite-dimensional examples other than usual nil-Coxeter algebras; these form a 2-parameter type A family that we term $NC_A(n,d)$. We explore the combinatorial properties of these algebras, including the Coxeter word basis, length function, maximal words, and their connection to Khovanov's categorification of the Weyl algebra.

Our broader motivation arises from complex reflection groups and the Broué–Malle–Rouquier freeness conjecture (1998). With generic Hecke algebras over real and complex groups in mind, we show that the first 'non-usual' finite-dimensional examples $NC_A(n,d)$ are in fact the only ones, outside of the usual nil-Coxeter algebras. The proofs use a diagrammatic calculus akin to crystal theory.

Keywords: Coxeter group, generalized nil-Coxeter algebra, length function, Frobenius algebra, complex reflection group

1 Introduction and main results

We study a new class of finite-dimensional algebras arising out of Coxeter theory, with connections to old work by Coxeter and new work on generic Hecke algebras, combinatorics, and categorification. We work throughout over a ground field $\Bbbk$ for ease of exposition, although our results hold over any commutative unital ground ring.

We begin with background and notation. Real reflection groups W and their Iwahori–Hecke algebras $\mathcal{H}_W(q)$ are classical objects that have long been studied in algebraic combinatorics, representation theory, and mathematical physics. Recall that every Coxeter group is specified by a Coxeter matrix $M \in \mathbb{Z}^{I \times I}$, with finite index set I and entries $m_{ii} = 2 \leqslant m_{ij} \leqslant \infty \ \forall i \neq j$. The corresponding Artin monoid $\mathcal{B}_M^{\geqslant 0}$ has generators $\{T_i : i \in I\}$, and braid relations $T_i T_j T_i \cdots = T_j T_i T_j \cdots$ with m_{ij} factors on each side whenever $m_{ij} < \infty$. We will denote the corresponding Coxeter group by $W(M)$.

Three prominent algebras associated to $W(M)$ are its group algebra $\Bbbk W(M)$, its 0-Hecke algebra, and its nil-Coxeter algebra $NC(M)$ (also known in the literature as the nil Hecke ring, nil Coxeter algebra, and nilCoxeter algebra). All three algebras are quotients

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of the monoid algebra $\Bbbk\mathcal{B}_M^{\geqslant 0}$ by quadratic relations for the T_i, and are 'generic Hecke algebras' [10, Chapter 7]. Among such algebras, the T_i satisfy homogeneous relations only in the case of the nil-Coxeter algebra $NC(M)$. Using this, one shows that

$$NC(M) := \Bbbk\mathcal{B}_M^{\geqslant 0} / (T_i^2 : i \in I)$$

is the monoid algebra of a monoid with $|W(M)|+1$ elements, say $\{T_w : w \in W(M)\} \sqcup \{O_{W(M)}\}$ quotiented by the 'absorbing' central ideal $\Bbbk O_{W(M)}$. Nil-Coxeter algebras were introduced by Fomin and Stanley [8], and are related to flag varieties [13], symmetric function theory [2], and categorification [12].

We now present a larger family of algebras, which constitute the main object of study. Note that the algebras $NC(M)$ are the associated graded versions of the group algebra $\Bbbk W(M)$; indeed, taking the top-degree component of the non-homogeneous relations $T_i^2 = 1$ yields the nil-Coxeter relations $T_i^2 = 0$. The following construction is motivated by both real and complex reflection groups, and allows the nilpotence degree to vary.

Definition 1.1. *Define a* generalized Coxeter matrix *to be a symmetric matrix $M := (m_{ij})_{i,j \in I}$ with I finite, $m_{ii} < \infty\ \forall i \in I$, and $2 \leqslant m_{ij} \leqslant \infty\ \forall i \neq j$. Now fix such a matrix M.*

1. *Given an integer tuple $\mathbf{d} = (d_i)_{i \in I}$ with all $d_i \geqslant 2$, let $M(\mathbf{d})$ denote the matrix where the diagonal in M is replaced by the coordinates of $\mathbf{d}$. Let $M_2 := M((2, \dots, 2))$.*
2. *The* generalized Coxeter group *$W(M)$ is the group generated by $\{s_i : i \in I\}$ modulo the braid relations $s_i s_j s_i \cdots = s_j s_i s_j \cdots$ whenever $m_{ij} < \infty$, and the relations $s_i^{m_{ii}} = 1\ \forall i$.*
3. *Define the corresponding* generalized nil-Coxeter algebra *to be:*

$$NC(M) := \frac{\Bbbk\langle T_i, i \in I \rangle}{(\underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ times}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ times}},\ T_i^{m_{ii}} = 0,\ \forall i \neq j \in I)} = \frac{\Bbbk\mathcal{B}_{M_2}^{\geqslant 0}}{(T_i^{m_{ii}} = 0\ \forall i)}, \tag{1.1}$$

where we omit the braid relation $T_i T_j T_i \cdots = T_j T_i T_j \cdots$ if $m_{ij} = \infty$.

4. *As an important special case, we denote by M_{A_n} the usual type A Coxeter matrix with $|I| = n$, given by: $m_{ij} = 3$ if $|i-j| = 1$, and 2 otherwise.*

Working with generalized nil-Coxeter algebras $NC(M)$ yields a larger class of objects than the corresponding groups $W(M)$. For example, Marin [15] has shown that in rank 2 in type A, the algebra $NC(M_{A_2}((3,n)))$ is not finite-dimensional for $n \geqslant 3$, in particular for even n. However, the corresponding generalized Coxeter group $W(M_{A_2}((3,n)))$ is trivial for $3 \nmid n$, since in it the generators s_1, s_2 are conjugate, hence have equal orders.

In fact, this reasoning shows that for for all integers $d_1, \dots, d_n \geqslant 2$, we have

$$W(M_{A_n}(\mathbf{d})) = W(M_{A_n}((d, \dots, d))), \quad \text{where} \quad d = \gcd(d_1, \dots, d_n).$$

Now it is natural to ask for which integers $n, d \geqslant 2$ is the group $W(M_{A_n}((d, \dots, d)))$ finite – and what is its order. These questions were considered by Coxeter [5], and he proved that $W = W(M_{A_n}((d, \dots, d)))$ is finite if and only if $\frac{1}{n} + \frac{1}{d} > \frac{1}{2}$; moreover, in this case W has size $\left(\frac{1}{n} + \frac{1}{d} - \frac{1}{2}\right)^{1-n} \cdot n!/n^{n-1}$. In his thesis [14], Koster extended Coxeter's results to classify all finite generalized Coxeter groups; apart from the finite 'usual' Coxeter groups, one obtains precisely the Shephard groups.

In a parallel vein to these works, we explore for which matrices is the algebra $NC(M)$ finite-dimensional. In this we are also strongly motivated by the larger picture, which involves *complex* reflection groups and the *BMR freeness conjecture* [3]. We elaborate on these motivations presently; for now we remark that since complex reflections can have order $\geqslant 3$, working with them provides a natural reason to define and study generalized nil-Coxeter algebras.

Returning to real groups: recall that 'usual' nil-Coxeter algebras $NC(M((2, \dots, 2)))$ are finite-dimensional precisely for finite Coxeter groups, since for 'usual' Coxeter matrices $M_2 := M((2, \dots, 2))$ one has $\dim NC(M_2) = |W(M_2)|$. To our knowledge there are no other finite-dimensional examples $NC(M)$ known to date.

Our first main result parallels Coxeter's construction, and exhibits the first such 'non-usual' family of finite-dimensional algebras $NC(M)$ in type A:

Theorem 1.2. *For integers $n \geqslant 1$ and $d \geqslant 2$, define the $\Bbbk$-algebra*

$$NC_A(n, d) := NC(M_{A_n}((2, \dots, 2, d))). \tag{1.2}$$

Thus, $NC_A(n, d)$ has generators $T_1, \dots, T_n$, with relations:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad \forall\, 0 < i < n; \tag{1.3}$$

$$T_i T_j = T_j T_i, \qquad \forall\, |i - j| > 1; \tag{1.4}$$

$$T_1^2 = \cdots = T_{n-1}^2 = T_n^d = 0. \tag{1.5}$$

Then $NC_A(n, d)$ has a Coxeter word basis of $n!(1 + n(d-1))$ generators

$$\{T_w : w \in S_n\} \sqcup \{T_w T_n^k T_{n-1} T_{n-2} \cdots T_{m+1} T_m : w \in S_n,\ k \in [1, d-1],\ m \in [1, n]\}.$$

In particular, the subalgebra R_l generated by $T_1, \dots, T_l$ is isomorphic to the type A nil-Coxeter algebra $NC(M_{A_l}((2, \dots, 2)))$, for all $0 < l < n$.

Remark 1.3. *We adopt the following notation in the sequel without further reference: let*

$$w_\circ \in S_{n+1}, \quad w'_\circ \in S_n \quad \text{denote the respective longest elements,} \tag{1.6}$$

where the symmetric group S_{l+1} corresponds to the basis of the algebra R_l for $l = n-1, n$.

In a later section, we will discuss additional properties of the algebras $NC_A(n, d)$, including identifying the 'maximal' words, and exploring the Frobenius property.

Classification of finite-dimensional nil-Coxeter algebras

Our next main result classifies the matrices M for which the generalized nil-Coxeter algebra $NC(M)$ is finite-dimensional. In combinatorics and in algebra, classifying Coxeter-type objects of finite size, dimension, or type is a problem of significant classical as well as modern interest. Such settings include real and complex reflection groups [4, 6, 17] and associated Hecke algebras; finite type quivers, simple Lie algebras, the McKay–Slodowy correspondence, and Kleinian singularities (as well as the above results by Coxeter and Koster). The recent classification of finite-dimensional pointed Hopf algebras [1] reveals connections to small quantum groups. Even more recently, the classification of finite-dimensional Nichols algebras has been well-received (see [9] and the references therein); some ingredients used in proving those results show up in the present work as well.

We now classify the generalized Coxeter matrices M for which $NC(M)$ is finite-dimensional. Remarkably, outside of the usual nil-Coxeter algebras, our first family of examples $NC_A(n,d)$ turns out to be the only one:

Theorem 1.4. *Suppose W is a Coxeter group with connected Dynkin diagram. Fix an integer vector $\mathbf{d}$ with $d_i \geqslant 2\ \forall i$, i.e., a generalized Coxeter matrix $M(\mathbf{d})$. The following are equivalent:*

1. *The generalized nil-Coxeter algebra $NC(M(\mathbf{d}))$ is finite-dimensional.*
2. *Either W is a finite Coxeter group and $d_i = 2\ \forall i$, or W is of type A_n and $\mathbf{d} = (2, \dots, 2, d)$ or $(d, 2, \dots, 2)$ for some $d > 2$.*

Remark 1.5. *The above results are characteristic-free; in fact they hold over arbitrary ground rings $\mathbb{k}$, in which case Theorem 1.2 yields a $\mathbb{k}$-basis of the free $\mathbb{k}$-module $NC_A(n,d)$; and Theorem 1.4 classifies the finitely generated $\mathbb{k}$-algebras $NC(M)$. In sketching the proofs of these results below, we will continue to assume $\mathbb{k}$ is a field; for the general case over a ring $\mathbb{k}$, for full details, and for further ramifications, we refer the reader to [11], of which this note is an extended abstract.*

Before proceeding further, we mention another strong motivation for Theorem 1.4, arising from generic Hecke algebras over *complex* reflection groups. As mentioned above, the varying nilpotence degree of the T_i is natural in the setting of complex reflection groups W. A prominent area of research has been the study of the associated generic Hecke algebras $\mathcal{H}_W$ and the Broué–Malle–Rouquier freeness conjecture [3]. The conjecture says that $\mathcal{H}_W$ is a free R-module of rank $|W|$, where R is the ground ring. See also its recent resolution in characteristic zero [7], and the references therein.

In studying these topics, Marin [15] remarks that the lack of nil-Coxeter algebras of dimension $|W|$ is a striking difference between complex and real reflection groups W. This was verified in some cases in *loc. cit.*; and it motivated us to define generalized nil-Coxeter algebras over all complex reflection groups. We do so in [11], and then completely classify the finite-dimensional algebras over all such groups. Remarkably, Theorem 1.4 extends to all complex W as well, and the only finite-dimensional families are real (usual) nil-Coxeter algebras, and the family $NC_A(n,d)$. In particular, this shows the above statement of Marin.

Remark 1.6. *Our result holds even more generally: following the classification of finite complex reflection groups in the celebrated work [17], Popov classified in [16] the infinite discrete groups generated by unitary reflections. In [11] we extend Theorem 1.4 to also cover all of these groups; once again, we show there are no finite-dimensional nil-Coxeter analogues.*

The equidimensionality (or not) of $\mathcal{H}_W$ and its nil-Coxeter analogue amounts to whether the former – a filtered algebra – is a *flat* deformation of the latter, which is $\mathbb{Z}^{\geqslant 0}$-graded. The study of flat deformations goes back to classical work of Gerstenhaber, and also by Braverman–Gaitsgory, Drinfeld, Etingof–Ginzburg, and the recent program by Shepler and Witherspoon; see [18, 11] for more on this. In this formalism, Theorem 1.4 – or its extension to complex groups – says that over complex reflection groups, generic Hecke algebras are not flat deformations of their nil-Coxeter analogues. This is in stark contrast to the real case, where $\dim NC(M) = |W(M)|$.

2 A finite-dimensional generalized nil-Coxeter algebra

We now outline the proof of Theorem 1.2, using a diagrammatic calculus as well as braid monoid computations. Note that $NC_A(1,d) = \Bbbk[T_1]/(T_1^d)$, while $NC_A(n,2)$ is the usual type A nil-Coxeter algebra, for which the theorem is well-known (see e.g. [10]). Thus, in this section we will assume $d \geqslant 3$ and $n \geqslant 2$.

We begin by showing that the set from Theorem 1.2 spans $NC_A(n,d)$. As a first step:

Lemma 2.1. *A word in the generators T_i either vanishes in $NC_A(n,d)$, or can be equated with a word in which all occurrences of T_n are successive.*

Proof. Suppose a word $\mathcal{T}$ has a sub-word of the form $T_n^a T_{i_1} \cdots T_{i_k} T_n^b$ for some $a, b > 0$, with $0 < i_j < n \ \forall j$. Using the relations, we may assume the above representation of $\mathcal{T}$ is such that k is minimal. Thus $i_1 = i_k = n-1$, $i_2 = i_{k-1} = n-2$, and so on (else we may push some T_{i_j} outside of the sub-string). Hence the sub-string is of the form

$$T_{n-1}T_{n-2}\cdots T_{m+1}T_mT_{m+1}\cdots T_{n-2}T_{n-1}, \quad \text{for some } 1 \leqslant m \leqslant n-1.$$

Now one shows by descending induction on $m \leqslant n-1$ that in the Artin monoid $\mathcal{B}_{M_{A_n}}^{\geqslant 0}$,

$$T_{n-1}\cdots T_m \cdots T_{n-1} = T_mT_{m+1}\cdots T_{n-2}T_{n-1}T_{n-2}\cdots T_{m+1}T_m.$$

Hence,

$$\begin{aligned} T_n^a \cdot (T_{n-1}\cdots T_m\cdots T_{n-1}) \cdot T_n^b &= T_n^a \cdot (T_m \cdots T_{n-2}T_{n-1}T_{n-2}\cdots T_m) \cdot T_n^b \\ &= (T_m\cdots T_{n-2})T_n^{a-1}(T_nT_{n-1}T_n)T_n^{b-1}(T_{n-2}\cdots T_m) \\ &= (T_m\cdots T_{n-2})T_n^{a-1}(T_{n-1}T_nT_{n-1})T_n^{b-1}(T_{n-2}\cdots T_m). \end{aligned} \tag{2.1}$$

If $a, b \leqslant 1$ then the lemma follows. If instead $b > 1$ then this expression contains as a substring $T_{n-1}(T_n T_{n-1} T_n) = T_{n-1}^2 T_n T_{n-1} = 0$, so we are done. Similarly if $a > 1$. $\square$

Now the subalgebra R_{n-1} generated by $T_1, \dots, T_{n-1}$ satisfies the relations of the usual nil-Coxeter algebra $NC_A(n-1, 2)$, so the words $\{T_w : w \in S_n\}$ span it. By (2.1), every nonzero word not in R_{n-1} is of the form $T_w T_n^k T_{w'}$; writing $T_{w'}$ as a sub-string of minimal length, by above we may rewrite the word such that $T_{w'} = T_{n-1} \cdots T_m$. Hence,

$$NC_A(n,d) = R_{n-1} + \sum_{k=1}^{d-1} \sum_{m=1}^{n} R_{n-1} \cdot T_n^k \cdot (T_{n-1} \cdots T_m).$$

Since $\dim R_{n-1} \leqslant n!$, the upper bound on $\dim NC_A(n,d)$ follows.

The proof of the converse – i.e., linear independence of the claimed word basis – repeatedly uses some results about the permutation group S_n and its nil-Coxeter algebra:

Lemma 2.2. *Suppose $W = S_n$, with simple reflections $s_1, \dots, s_{n-1}$ labelled as usual. Let S_{n-1} be generated by $s_1, \dots, s_{n-2}$; then for all $w \in S_n \setminus S_{n-1}$, w has a reduced expression as $w = w' s_{n-1} \cdots s_{m'}$, where $w' \in S_{n-1}$ and $m' \in [1, n-1]$ are unique. Given such an element $w \in S_n$, we have in the usual nil-Coxeter algebra $NC(M_{A_n}((2, \dots, 2)))$:*

$$T_n \cdot T_w \cdot T_n \cdots T_m = \begin{cases} T_{w'} T_{n-1} \cdots T_{m-1} \cdot T_n \cdots T_{m'}, & \text{if } m' < m, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Now we introduce a diagrammatic calculus reminiscent of crystal theory from combinatorics and quantum groups. For simplicity, we begin by presenting the $n = 2$ case. Let $\mathscr{M}$ be a $\Bbbk$-vector space, with basis given by the nodes in Figure 1.

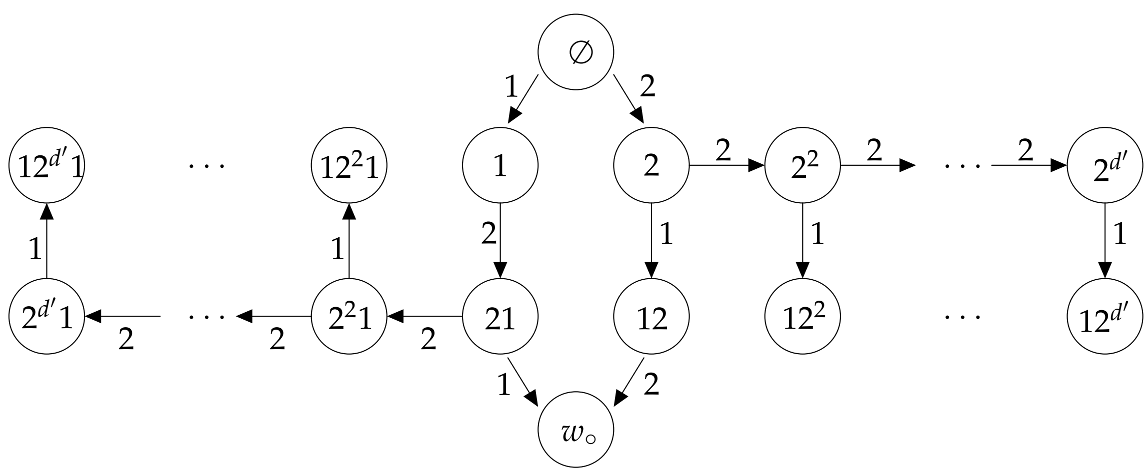

Figure 1: Regular representation for $NC_A(2, d)$, with $d' = d - 1$

In this figure, the node $2^2 1$ should be thought of as applying $T_2^2 T_1$ to the generating basis vector/node $\emptyset$; similarly for all other nodes. The arrows show the action of T_1, T_2

on the basis vectors (i.e., nodes), and the lack of an arrow labeled i with source $v \in \mathscr{M}$ means $T_i v = 0$. Now verify by inspection that the relations in $NC_A(2,d)$ are satisfied in $\mathrm{End}_{\Bbbk}(\mathscr{M})$, whence $\mathscr{M}$ is a cyclic $NC_A(2,d)$-module generated by the vector $\emptyset$ – in fact, the regular representation. This gives the desired result for $NC_A(2,d)$.

For general $n \geqslant 2$, the strategy is similar but with more involved notation. For $w \in S_n$, let T_w denote the (well-defined) word in $T_1, \dots, T_{n-1} \in NC_A(n,d)$. Now define a vector space $\mathscr{M}$ with basis given by (2.3) and $NC_A(n,d)$-action as in Figure 2 below:

$$\mathcal{B} := \{B(w,k,m) : w \in S_n,\ k \in [1,d-1],\ m \in [1,n]\} \sqcup \{B(w) : w \in S_n\}. \tag{2.3}$$

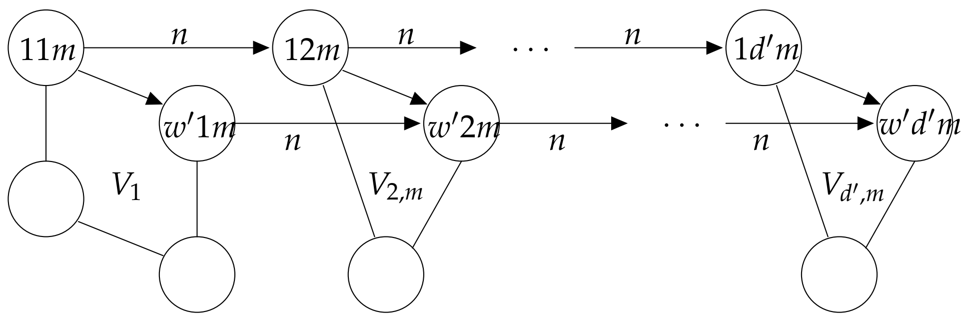

Figure 2: Regular representation for $NC_A(n,d)$, with $d' = d-1$

Note that $\dim_{\Bbbk} \mathscr{M} = n!(1 + n(d-1))$; that the basis vectors in (2.3) are to be thought of as akin to $T_w T_n^k T_{n-1} \cdots T_m$ and T_w respectively; and the nodes $(wkm), (w)$ precisely denote the basis vectors $B(w,k,m), B(w)$ respectively.

Now let V_1 denote the span of the vectors $\{B(w) : w \in S_{n-1}\} \sqcup \{B(w,1,m) : w \in S_{n-1}, m \in [1,n]\}$. These vectors are in bijection with the word basis of the usual nil-Coxeter algebra $NC_A(n,2)$. Similarly for $k \in [1,d-1]$ and $m \in [1,n]$, define $V_{k,m}$ to be the span of the vectors $B(w,k,m), w \in S_n$.

Define the $NC_A(n,d)$-action on $\mathscr{M}$ as follows. First for the action of $T_1, \dots, T_{n-1}$, write $V_{1,n+1} := \mathrm{span}\{B(w) : w \in S_n\}$; and equip each space $V(k,m)$ for $k \in [1,d-1], m \in [1,n]$ and also $V(1,n+1)$ with the structure of the regular representation of R_{n-1}. Next, if $w \in S_{n-1}$, then we define

$$T_n \cdot B(w,k,m) := \mathbf{1}(k \leqslant d-2) B(w,k+1,m), \qquad T_n \cdot B(w) := B(w,1,n).$$

Now suppose $w \in S_n \setminus S_{n-1}$. Using Lemma 2.2, write $w = w' s_{n-1} \cdots s_m$; then $m \leqslant n-1$. Define $T_n \cdot B(w,k,m) := 0$ if $k > 1$; also set $T_n \cdot B(w) := B(w',1,m')$; finally,

$$T_n \cdot B(w,1,m) := \begin{cases} B(w' s_{n-1} \cdots s_{m-1}, 1, m'), & \text{if } m' < m, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Then some involved computations using the identities mentioned above show that the proposed action indeed equips $\mathscr{M}$ with the structure of a cyclic $NC_A(n,d)$-module, generated by $B(1)$. This allows us to complete the proof of Theorem 1.2. □

3 Further properties

We now discuss several additional properties of the algebras $NC_A(n,d)$. For proofs of results in this section, we refer the reader to [11]. The first set of properties shows how these algebras resemble usual nil-Coxeter algebras.

Theorem 3.1 (see [11]). *Fix integers $n \geqslant 1$ and $d \geqslant 2$.*

1. *The algebra $NC_A(n,d)$ has a length function that restricts to the usual length function $\ell_{A_{n-1}}$ on $R_{n-1} \simeq NC_{A_{n-1}}((2,\dots,2))$ (from Theorem 1.2), and*
$$\ell(T_w T_n^k T_{n-1} \cdots T_m) = \ell_{A_{n-1}}(w) + k + n - m, \tag{3.1}$$
for all $w \in S_n$, $k \in [1, d-1]$, and $m \in [1, n]$.
2. *There is a unique longest word $T_{w'_\circ} T_n^{d-1} T_{n-1} \cdots T_1$ of length*
$$l_{n,d} := \ell_{A_{n-1}}(w'_\circ) + d + n - 2.$$
3. *The algebra $NC_A(n,d)$ is local, with unique maximal (augmentation) ideal $\mathfrak{m}$ generated by $T_1, \dots, T_n$. The ideal $\mathfrak{m}$ is nilpotent with $\mathfrak{m}^{1+l_{n,d}} = 0$.*

Thus there is a variant of the Coxeter word length, as well as a unique longest word and nilpotent augmentation ideal. As an immediate consequence, one can compute the Hilbert polynomial of the graded algebra $NC_A(n,d)$:

Corollary 3.2. *If $T_1, \dots, T_n$ all have degree 1, then $NC_A(n,d)$ has Hilbert–Poincaré series*
$$[n]_q!\ (1 + [n]_q\ [d-1]_q), \qquad \text{where } [n]_q := \frac{q^n - 1}{q - 1},\ [n]_q! := \prod_{j=1}^n [j]_q.$$

Here, we also use the standard result that the usual nil-Coxeter algebra $NC_A(n,2)$ has Hilbert–Poincaré series $[n]_q!$ (see e.g. [10, Sections 3.12, 3.15]).

Having discussed similarities with usual nil-Coxeter algebras, we next present certain differences in structure. For any generalized Coxeter matrix M, define $x \in NC(M)$ to be *left-primitive* if $T_i x = 0\ \forall i \in I$. Similarly define *right-primitive* elements; and an element that is both is said to be *primitive*. We denote these subspaces of $NC(M)$ by
$$\operatorname{Prim}_L(NC(M)), \quad \operatorname{Prim}_R(NC(M)), \quad \operatorname{Prim}(NC(M)).$$

Proposition 3.3 (see [11]). *Every generalized nil-Coxeter algebra $NC(M)$ is equipped with an anti-involution θ that fixes each generator T_i. Now θ is an isomorphism : $\operatorname{Prim}_L(NC(M)) \longleftrightarrow \operatorname{Prim}_R(NC(M))$. Moreover, the following hold.*

1. *If* $NC(M) = NC_A(1,d)$, *then*
$$\mathrm{Prim}_L(NC(M)) = \mathrm{Prim}_R(NC(M)) = \mathrm{Prim}(NC(M)) = \Bbbk \cdot T_1^{d-1}.$$
2. *If* $NC(M) = NC_A(n,d)$ *with* $n \geqslant 2$ *and* $d \geqslant 2$, *then:*
   (a) $\mathrm{Prim}_L(NC(M))$ *is spanned by* $T_{w_\circ} := T_{w'_\circ} T_n T_{n-1} \cdots T_1$ *and the* $n(d-2)$ *words*
$$\{T_{w'_\circ} T_n^k T_{n-1} \cdots T_m : \ k \in [2, d-1], \ m \in [1,n]\}.$$
   (b) $\mathrm{Prim}(NC(M))$ *is spanned by the words* $T_{w'_\circ} T_n^k T_{n-1} \cdots T_1$, *where* $1 \leqslant k \leqslant d-1$.

In all cases, the map θ *fixes both* $\mathrm{Prim}(NC(M))$ *as well as the lengths of all nonzero words.*

(Thus there are multiple primitive words for $d > 2$.) Using Proposition 3.3, we address another difference with usual nil-Coxeter algebras: the latter are always Frobenius [12]. It is natural to ask when the finite-dimensional algebras $NC_A(n,d)$ share this property.

Proposition 3.4. *The algebra* $NC_A(n,d)$ *is Frobenius if and only if* $n = 1$ *or* $d = 2$.

In fact this happens if and only if the group algebra $\Bbbk W(M_{A_n}(\mathbf{d}))$ is a flat deformation of $NC_A(n,d)$.

Proof. If $W(M)$ is a finite Coxeter group, [12, Section 2.2] shows that $NC(M)$ is Frobenius. Next, one easily verifies $NC_A(1,d) = \Bbbk[T_1]/(T_1^d)$ is Frobenius, via the symmetric bilinear form given by: $\sigma(T_1^i, T_1^j) = \mathbf{1}(i + j = d-1)$. Now suppose for some n,d that $NC_A(n,d)$ is Frobenius, with nondegenerate invariant bilinear form σ. For each primitive $p \neq 0$, there exists a_p such that $0 \neq \sigma(p, a_p) = \sigma(pa_p, 1)$. Thus, we can take $a_p = 1$, $\forall p$. Since the functional $\sigma(-,1) : \mathrm{Prim}(NC_A(n,d)) \to \Bbbk$ is nonsingular, we obtain $\dim_{\Bbbk} \mathrm{Prim}(NC_A(n,d)) = 1$. Applying Proposition 3.3, we get $n = 1$ or $d = 2$. □

Finally, recall the famous result by Khovanov [12] that the Weyl algebra $W_n := \mathbb{Z}\langle x, \partial \rangle / (\partial x = 1 + x\partial)$ can be represented by functors on bimodule categories over usual nil-Coxeter algebras. (Here we use that the nil-Coxeter algebra $\mathcal{A}_n := NC_A(n,2)$ is a bimodule over $\mathcal{A}_{n-1}$.) We now explain how $NC_A(n,d)$ fits into Khovanov's framework for $d \geqslant 2$, noting that for $d = 2$ it was proved in [12]:

Proposition 3.5 (see [11])**.** *For* $n \geqslant 1$ *and* $d \geqslant 2$, *there is an isomorphism of* $\mathcal{A}_{n-1}$*-bimodules:*
$$NC_A(n,d) \simeq \mathcal{A}_{n-1} \oplus \bigoplus_{k=1}^{d-1} \left(\mathcal{A}_{n-1} \otimes_{\mathcal{A}_{n-2}} \mathcal{A}_{n-1} \right).$$

For $d \geqslant 2$, in the notation of [12] this result implies that over $\mathcal{A}_{n-1}$-bimodules, the algebra $NC_A(n,d)$ corresponds to $1 + (d-1)x\partial$. Thus, Proposition 3.5 strengthens Theorems 1.2 and 3.1, which discussed a left $\mathcal{A}_{n-1}$-module structure on $NC_A(n,d)$ (namely, $NC_A(n,d)$ is free of rank $1 + n(d-1)$).

4 All finite-dimensional generalized nil-Coxeter algebras

We conclude by proving Theorem 1.4. Clearly (2) $\implies$ (1) by Theorem 1.2 and [10, Chapter 7]. Now suppose (1) holds and $\mathbf{d} \neq (2, \dots, 2)$. We again use the diagrammatic calculus above, now for the diagrams in Figure 3.

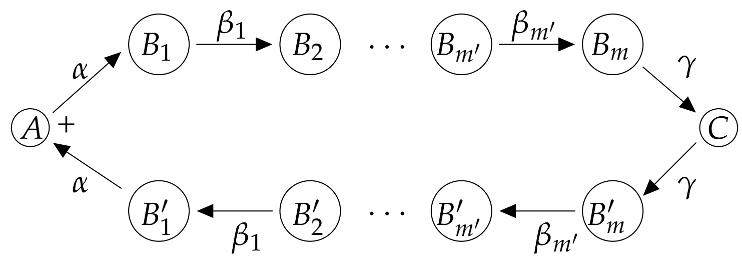

Fig. 3.1 ($m' = m - 1$)

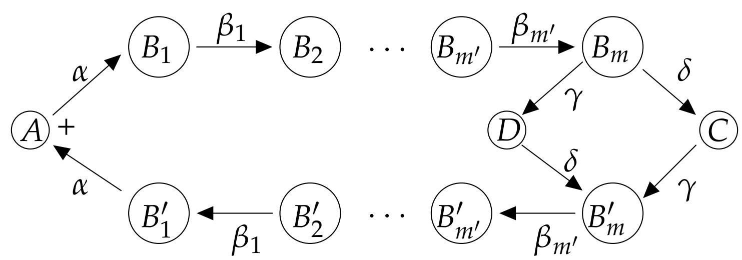

Fig. 3.3 ($m' = m - 1$)

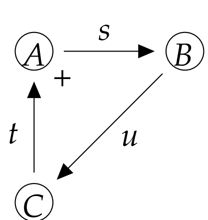

Fig. 3.2

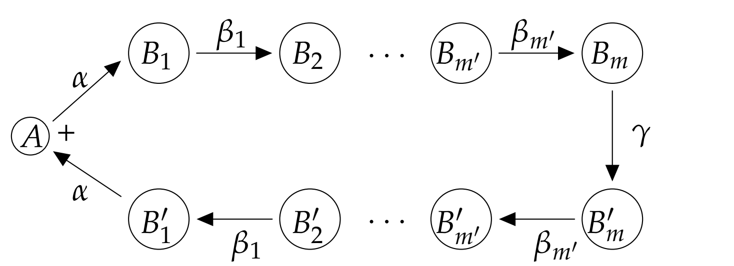

Fig. 3.4 ($m' = m - 1$)

Figure 3: Modules for the infinite-dimensional generalized nil-Coxeter algebras

We consider the possible cases, showing in each case that the algebra $NC(M)$ is infinite-dimensional, until we are left with only $NC_A(n, d)$. First suppose there exist two nodes $\alpha, \gamma \in I$ with $m_{\alpha\alpha}, m_{\gamma\gamma} \geqslant 3$. Since the Dynkin diagram of I is connected by assumption, there exist $\beta_1, \dots, \beta_{m-1} \in I$ such that

$$\alpha \quad \longleftrightarrow \quad \beta_1 \quad \longleftrightarrow \quad \cdots \quad \longleftrightarrow \quad \beta_{m-1} \quad \longleftrightarrow \quad \gamma$$

are all connected in I, i.e., a path. Now define an $NC(M)$-module $\mathscr{M}$ with basis

$$A_r, B_{1r}, \dots, B_{mr}, C_r, B'_{1r}, \dots, B'_{mr}, \quad r \geqslant 1,$$

and where every T_i kills all basis vectors, *except* for the actions described in Figure 3.1, namely $T_\alpha(A_r) := B_{1r}, T_{\beta_1}(B'_{2r}) := B'_{1r}$, and so on for all $r \geqslant 1$. The '+' indicates that $T_\alpha(B'_{1r}) := A_{r+1}\ \forall r \geqslant 1$. One verifies that the T_i satisfy the $NC(M)$-relations on every basis vector, whence on $\mathscr{M}$. Now as $\mathscr{M}$ is cyclic and infinite-dimensional, so is $NC(M)$.

The strategy is similar for the remainder of the proof. Henceforth we fix the unique node $\alpha \in I$ such that $m_{\alpha\alpha} \geqslant 3$. If α is connected in I to γ with $m_{\alpha\gamma} \geqslant 4$, then we work with Figure 3.2, setting $(s,t,u) \rightsquigarrow (\alpha, \alpha, \gamma)$, and define $\mathscr{M} := \operatorname{span}_{\Bbbk}\{A_r, B_r, C_r : r \geqslant 1\}$. Now check that $\mathscr{M}$ is an infinite-dimensional cyclic $NC(M)$-module. Next, if α is adjacent to two nodes $\gamma, \delta \in I$, work with Figure 3.3 for $m = 1$. This shows α must be extremal.

Note that if $NC(M)$ is finite-dimensional then so is its quotient $NC(M((2,\dots,2)))$, which is a nil-Coxeter algebra. Hence $W(M((2,\dots,2)))$ is a finite Coxeter group, and these are known [4, 6]. We now sketch how to eliminate all cases not of type A, whence from above, $NC(M) \cong NC_A(n,d)$, where we set $n := |I|$.

The dihedral types G_2, H_2, I do not hold from above. Suppose I is of type B, C, H:

$$\alpha \quad \longleftrightarrow \quad \beta_1 \quad \longleftrightarrow \quad \cdots \quad \longleftrightarrow \quad \beta_{m-1} \quad \longleftrightarrow \quad \gamma,$$

with $m_{\alpha\alpha} \geqslant 3, m_{\gamma\gamma} = 2, m_{\beta_{m-1}\gamma} \geqslant 4$. Then work with Figure 3.4. Note this also rules out the F_4 case, since $NC(M_{F_4}) \twoheadrightarrow NC(M_{B_3})$ or $NC(M_{C_3})$ by killing the extremal generator T_δ with $m_{\delta\delta} = 2$, and we showed that the latter two algebras are infinite-dimensional.

Next suppose I is of type D. If α is a (extremal) node on the 'long arm', work with Figure 3.3 with $m = n-2$, with the other extremal nodes γ, δ. Else if α is extremal on a short arm, we work as above with the quotient algebra $NC(M_{D_4})$ of Dynkin type D_4, by killing all T_j with node j on the long arm having degree $\leqslant 2$. Working with Figure 3.3 for $m = 2$, it follows that $NC(M_{D_4})$, and hence $NC(M)$, is infinite-dimensional.

Finally, in all remaining cases, the Coxeter graph of I is of type E. Akin to above, these cases are ruled out by quotienting to reduce to type D. This concludes the proof. □

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