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Cylindric Reverse Plane Partitions and 2D TQFT

Christian Korff*[1] and David Palazzo†[1]

[1] *School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QQ*

Abstract. The ring of symmetric functions carries the structure of a Hopf algebra. When computing the coproduct of complete symmetric functions h_λ one arrives at weighted sums over reverse plane partitions (RPP) involving binomial coefficients. Employing the action of the extended affine symmetric group at fixed level n we generalise these weighted sums to cylindric RPP and define cylindric complete symmetric functions. The latter are shown to be h-positive, that is, their expansions coefficients in the basis of complete symmetric functions are non-negative integers. We state an explicit formula in terms of tensor multiplicities for irreducible representations of the generalised symmetric group. Moreover, we relate the complete symmetric functions to a 2D topological quantum field theory (TQFT) that is a generalisation of the celebrated $\widehat{\mathfrak{sl}}_n$-Verlinde algebra or Wess–Zumino–Witten fusion ring, which plays a prominent role in the context of vertex operator algebras and algebraic geometry.

Keywords: reverse plane partitions, symmetric functions, topological quantum field theory, Verlinde algebra, generalised symmetric group

1 Symmetric functions and reverse plane partitions

This section recalls some known results about the ring of symmetric functions and reverse plane partitions. It serves as a motivation for our definition of weighted sums over cylindric reverse plane partitions in Section 2 and the definition of a certain class of 2D TQFTs in Section 3, which are the new results. Detailed proofs and a full discussion will be published elsewhere [9], here we simply present a summary of our results.

1.1 Pieri type rules and coproduct

Let $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ denote the ring of symmetric functions equipped with the Hall inner product $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$, where m_λ denotes the monomial symmetric function and $h_\mu = h_{\mu_1} h_{\mu_2} \cdots$ the complete symmetric function with $h_r = \sum_{1 \le i_1 \le \cdots \le i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$ and $\lambda, \mu \in \mathcal{P}$ partitions. In particular, each of the sets $\{m_\lambda\}_{\lambda \in \mathcal{P}}$ and $\{h_\lambda\}_{\lambda \in \mathcal{P}}$ forms a $\mathbb{Z}$-basis of the ring Λ.

*christian.korff@glasgow.ac.uk

†d.palazzo1@research.gla.ac.uk. David Palazzo is supported by a College of Science and Engineering Scholarship of the University of Glasgow.

Introduce a coproduct $\Delta : \Lambda \to \Lambda \otimes \Lambda$ via the relation $\langle \Delta f, g \otimes h \rangle = \langle f, gh \rangle$ for $f, g, h \in \Lambda$; compare with [12, Chapter I.5, Ex. 25]. The latter allows one to use the product formulae

$$m_\lambda m_\mu = \sum_\nu f^\nu_{\lambda\mu} m_\nu \qquad \text{and} \qquad h_\lambda m_\mu = \sum_\nu \chi_{\nu/\mu}(\lambda) m_\nu \tag{1.1}$$

to define the 'skew complete symmetric function' $h_{\lambda/\mu}$ via

$$\Delta h_\lambda = \sum_\mu h_{\lambda/\mu} \otimes h_\mu, \qquad h_{\lambda/\mu} = \sum_\nu f^\lambda_{\mu\nu} h_\nu = \sum_\nu \chi_{\lambda/\mu}(\nu) m_\nu \,. \tag{1.2}$$

The expansion coefficients in both formulae are non-negative integers and have combinatorial expressions, which we know recall.

For the product of monomial symmetric functions the expansion coefficients $f^\nu_{\lambda\mu}$ in (1.1) are given by the cardinality of the set $\{(\alpha, \beta) \mid \alpha + \beta = \nu\}$ where respectively α and β are distinct permutations of λ and μ; see e.g. [3]. This also fixes the expansion coefficients for the second product in (1.1). Recall the known identity $h_\lambda = \sum_\mu L_{\lambda\mu} m_\mu$, where $L_{\lambda\mu}$ is the number of $\mathbb{N}$-matrices $A = (a_{ij})_{i,j\geq 1}$ which have only a finite number of nonzero entries and row sums $\text{row}(A) = \lambda$ and column sums $\text{col}(A) = \mu$; see e.g. [16, Chapter 7.5, Proposition 7.5.1]. Then it follows at once that

$$\chi_{\nu/\mu}(\lambda) = \sum_\alpha L_{\lambda\alpha} f^\nu_{\alpha\mu} \,. \tag{1.3}$$

Here we are interested in an alternative expression using reverse plane partitions as the latter suggests a natural generalisation to cylindric reverse plane partitions.

1.2 Weighted sums over reverse plane partitions

Given two partitions λ, μ with $\mu \subset \lambda$ recall that a *reverse plane partition* (RPP) π of skew shape λ/μ is a sequence $\{\lambda^{(i)}\}_{i=0}^l$ of partitions $\lambda^{(i)}$ with $\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(l)} = \lambda$. As usual we refer to the vector $\theta(\pi) = (|\lambda^{(1)}/\lambda^{(0)}|, |\lambda^{(2)}/\lambda^{(1)}|, \ldots)$ as the *weight* of π and denote by x^π the monomial $x^\pi = x_1^{\theta_1(\pi)} x_2^{\theta_2(\pi)} \cdots$ in the indeterminates x_i. Alternatively, we can think of a RPP as a map $\pi : \lambda/\mu \to \mathbb{N}$ which assigns to the squares $s = (x, y) \in \lambda^{(i)}/\lambda^{(i-1)} \subset \mathbb{Z} \times \mathbb{Z}$ the integer i. The result is a skew tableau whose entries are non-decreasing along each row from left to right and down each column.

We now introduce weighted sums over RPP in terms of binomial coefficients. Given any pair of partitions $\lambda, \mu \in \mathcal{P}$ denote by $\chi_{\lambda/\mu}$ the number of distinct permutations α of μ such that $\alpha \subset \lambda$. The latter has the following explicit expression in terms of the conjugate partitions λ', μ',

$$\chi_{\lambda/\mu} = \begin{cases} \prod_{i\geq 1} \binom{\lambda'_i - \mu'_{i+1}}{\mu'_i - \mu'_{i+1}}, & \mu \subset \lambda \\ 0, & \text{otherwise} \end{cases} \,. \tag{1.4}$$

The following result is probably known to experts but we were unable to find it in the literature.

Lemma 1.1. *The skew complete symmetric function defined in* (1.2) *is the weighted sum*

$$h_{\lambda/\mu}(x) = \sum_{\pi} \chi_\pi \, x^\pi, \qquad \chi_\pi = \prod_{i \geq 1} \chi_{\lambda^{(i)}/\lambda^{(i-1)}} \tag{1.5}$$

over all reverse plane partitions π of shape λ/μ. In particular, the coefficient (1.3) *has the alternative expression $\chi_{\lambda/\mu}(\nu) = \sum_\pi \chi_\pi$, where the sum is over all reverse plane partitions π of shape λ/μ and weight $\theta(\pi) = \nu$.*

1.3 The expansion coefficients in terms of the symmetric group

We now project onto the ring $\Lambda_k = \mathbb{Z}[x_1, \ldots, x_k]^{S_k}$ of symmetric functions in k-variables by setting $x_{k+1} = x_{k+2} = \cdots = 0$. Here S_k is the symmetric group in k letters. Let $\mathcal{P}_k = \bigoplus_{i=1}^k \mathbb{Z}\epsilon_i$ be the $\mathfrak{gl}_k$ weight lattice with standard basis $\epsilon_1, \ldots, \epsilon_k$ and inner product $(\epsilon_i, \epsilon_j) = \delta_{ij}$. Denote by $\mathcal{P}_k^+ \subset \mathcal{P}_k$ the *positive dominant weights*.

Each $\lambda \in \mathcal{P}_k$ defines a map $\lambda : [k] \to \mathbb{Z}$ in the obvious manner and we shall consider the right action $\mathcal{P}_k \times S_k \to \mathcal{P}_k$ given by $(\lambda, w) \mapsto \lambda \circ w = (\lambda_{w(1)}, \ldots, \lambda_{w(k)})$. For a fixed weight μ denote by $S_\mu \subset S_k$ its stabiliser group. The latter has cardinality $|S_\mu| = \prod_{i \in \mathbb{Z}} m_i(\mu)!$ with $m_i(\mu)$ being the number of elements in $[k]$ which are mapped to i under μ. Given any permutation $w \in S_k$ there exists a unique decomposition $w = w_\mu w^\mu$ with $w_\mu \in S_\mu$ and w^μ a minimal length representative of the right coset $S_\mu w$. Denote by $S^\mu \subset S_k$ the set of all minimal length coset representatives in $S_\mu \backslash S_k$.

Lemma 1.2. *Let $\lambda, \mu, \nu \in \mathcal{P}_k^+$. (i) The expansion coefficient $f_{\lambda\mu}^\nu$ in* (1.1) *can be expressed as the cardinality of the set*

$$\{(w, w') \in S^\lambda \times S^\mu \mid \lambda \circ w + \mu \circ w' = \nu\} \, . \tag{1.6}$$

(ii) The specialisation of the weight factor $\chi_{\lambda/\mu}$ defined in (1.4) *to elements in $\mathcal{P}_k^+$ equals the cardinality of the set*

$$\{w \in S^\mu \mid \mu \circ w \leq \lambda\} \, , \tag{1.7}$$

where $\mu \circ w \leq \lambda$ is shorthand notation for $\mu_{w(i)} \leq \lambda_i$ for all $i \in [k]$.

2 Cylindric Reverse Plane Partitions

Cylindric plane partitions were first considered by Gessel and Krattenthaler in [4]. There has been a growing interest in cylindric Schur functions [13] and their connections with other areas in mathematics. Here we introduce *cylindric complete symmetric functions* as weighted sums over cylindric RPP by generalising the definition of the sets (1.6) and (1.7) to the extended affine symmetric group $\hat{S}_k = \mathcal{P}_k \rtimes S_k$. In the last section we will relate the expansions of these cylindric complete symmetric functions in the h_λ-basis to the (non-negative) structure constants of generalised Verlinde algebras.

2.1 Lusztig's realisation of the affine symmetric group

Recall the realisation of the affine symmetric group $\tilde{S}_k = \mathcal{Q}_k \rtimes S_k$, where $\mathcal{Q}_k \subset \mathcal{P}_k$ is the root lattice, in terms of bijections $\tilde{w} : \mathbb{Z} \to \mathbb{Z}$ that are subject to the following two conditions:

$$\tilde{w}(m+k) = \tilde{w}(m) + k,\ \forall m \in \mathbb{Z} \qquad \text{and} \qquad \sum_{i=1}^{k} \tilde{w}(m) = \binom{k}{2} . \tag{2.1}$$

As in the finite case the group multiplication is given by composition. This realisation of $\tilde{S}_k$ first appeared in [11] and has subsequently used by other authors [2]. The group is generated by the simple Weyl reflections $\{\sigma_0, \sigma_1, \ldots, \sigma_{k-1}\}$ which as maps $\mathbb{Z} \to \mathbb{Z}$ are defined as

$$\sigma_i(m) = \begin{cases} m+1, & m = i \bmod k \\ m-1, & m = i+1 \bmod k \\ m, & \text{otherwise} \end{cases} \tag{2.2}$$

and one verifies that they satisfy the familiar relations

$$\sigma_i \circ \sigma_i = \text{Id}, \qquad \sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1}, \qquad \sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i \quad \text{for } |i-j| > 1 ,$$

where all indices are understood modulo k.

2.2 The extended affine symmetric group

We now state a realisation of the *extended* affine symmetric group in terms of bijections $\mathbb{Z} \to \mathbb{Z}$. Define the shift operator $\tau : \mathbb{Z} \to \mathbb{Z}$ by $m \mapsto \tau(m) = m - 1$. Then one has the identities

$$\tau \circ \sigma_{i+1} = \sigma_i \circ \tau, \qquad i = 0, 1, \ldots, k-1 . \tag{2.3}$$

That is, the group generated by $\langle \tau, \sigma_0, \sigma_1, \ldots, \sigma_{k-1} \rangle$ is the extended affine symmetric group $\hat{S}_k = \mathcal{P}_k \rtimes S_k$. To see this more clearly, we introduce the additional generators $y_k = \tau \circ \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{k-1}$ and $y_i = \sigma_i \circ y_{i+1} \circ \sigma_i$ for $i = 1, 2, \ldots, k-1$. Then any element $\hat{w} \in \hat{S}_k$ can be written as $\hat{w} = w \circ y^\lambda$, where $\lambda \in \mathcal{P}_k$, $y^\lambda = y_1^{\lambda_1} \circ \cdots \circ y_k^{\lambda_k}$ and $w \in S_k$. Note that $\hat{w} : \mathbb{Z} \to \mathbb{Z}$ still satisfies the first property in (2.1) but not the second. Because any element $\hat{w} \in \hat{S}_k$ can be expressed as $\hat{w} = \tilde{w} \circ \tau^d$ for some $d \in \mathbb{Z}$ and $\tilde{w} \in \tilde{S}_k$, we have the following realisation of $\hat{S}_k$ generalising (2.1):

Lemma 2.1. *The extended affine symmetric group $\hat{S}_k$ can be realised as the bijections $\hat{w} : \mathbb{Z} \to \mathbb{Z}$ subject to the conditions*

$$\hat{w}(m+k) = \hat{w}(m) + k,\ \forall m \in \mathbb{Z} \qquad \textit{and} \qquad \sum_{m=1}^{k} \hat{w}(m) = \binom{k}{2} \mod k\ . \tag{2.4}$$

Our main interest in this realisation of $\hat{S}_k$ is that it naturally leads to the consideration of cylindric loops.

2.3 Cylindric loops and cylindric reverse plane partitions

Fix $n \in \mathbb{N}$. We are now generalising the notion of the weight lattice in order to define a level-n action of the extended affine symmetric group. Let $\mathcal{P}_{k,n}$ denote the set of functions $\lambda : \mathbb{Z} \to \mathbb{Z}$ subject to the constraint $\lambda_{i+k} = \lambda_i - n$ for all $i \in \mathbb{Z}$. Then the map $\mathcal{P}_{k,n} \times \hat{S}_k \to \mathcal{P}_{k,n}$ with $(\lambda, \hat{w}) \mapsto \lambda \circ \hat{w}$ defines a right action. One can convince oneself that this is the familiar level-n action of $\hat{S}_k$ on the weight lattice $\mathcal{P}_k$ by observing that each $\lambda \in \mathcal{P}_{k,n}$ is completely fixed by its values $(\lambda_1, \dots, \lambda_k)$ on the set $[k]$. In particular, the 'alcove'

$$\mathcal{A}_{k,n} = \{\lambda \in \mathcal{P}_{k,n} \mid n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0\} \tag{2.5}$$

is a fundamental domain with respect to this level-n action of $\hat{S}_k$ on $\mathcal{P}_{k,n}$. That is, for any $\lambda \in \mathcal{P}_{k,n}$ the orbit $\lambda\hat{S}_k$ intersects $\mathcal{A}_{k,n}$ in a unique point.

Given $\lambda \in \mathcal{A}_{k,n}$ and $d \in \mathbb{Z}$ denote by $\lambda[d]$ the (doubly) infinite sequence

$$\lambda[d] = (\dots, \lambda[d]_{-1}, \lambda[d]_0, \lambda[d]_1, \dots) = (\dots, \lambda_{-d-1}, \lambda_{-d}, \lambda_{1-d}, \dots)\,,$$

that is, the image $\lambda \circ \tau^d(\mathbb{Z})$ of the map $\lambda \circ \tau^d : \mathbb{Z} \to \mathbb{Z}$. This sequence defines a lattice path in $\mathbb{Z} \times \mathbb{Z}$ which repeats itself and is therefore called a *cylindric loop*. A *cylindric skew diagram* or *cylindric shape* can now be defined as the number of lattice points between two cylindric loops: let $\lambda, \mu \in \mathcal{A}_{k,n}$ be such that $\mu_i \leq \lambda_{i-d} = (\lambda \circ \tau^d)_i$ for all $i \in \mathbb{Z}$, then we write $\mu[0] \leq \lambda[d]$ and say that the set

$$\lambda/d/\mu = \{(i,j) \in \mathbb{Z}^2 \mid \mu[0]_i < j \leq \lambda[d]_i\}$$

is a cylindric skew diagram of degree d.

Definition 2.2. *A* cylindric reverse plane partition *of shape $\Theta = \lambda/d/\mu$ is a map $\hat{\pi} : \Theta \to \mathbb{N}$ such that for any $(i,j) \in \Theta$ one has $\hat{\pi}(i,j) = \hat{\pi}(i+k, j-n)$ together with*

$$\hat{\pi}(i,j) \leq \hat{\pi}(i+1,j) \quad \textit{and} \quad \hat{\pi}(i,j) \leq \hat{\pi}(i,j+1), \quad \textit{if } (i+1,j),(i,j+1) \in \Theta\ .$$

In other words, the entries in the squares between the cylindric loops $\mu[0]$ and $\lambda[d]$ are non-decreasing from left to right in rows and down columns.

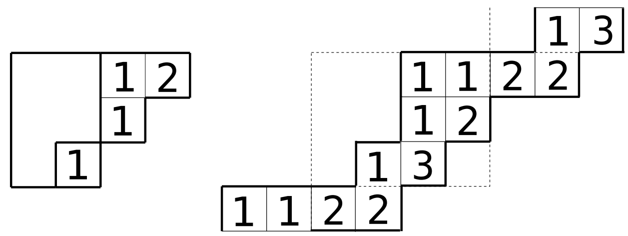

Figure 1: On the left a RPP of shape $(4,3,2)/(2,2,1)$ and weight $\theta = (3,1,0)$ and on the right a cylindric RPP of shape $(4,3,2)/1/(2,2,1)$ and weight $\theta = (4,3,1)$ with $n = 4$ and $k = 3$.

Alternatively, $\hat{\pi}$ can be defined as a sequence of cylindric loops

$$(\lambda^{(0)}[0] = \mu[0], \lambda^{(1)}[d_1], \ldots, \lambda^{(l)}[d_l] = \lambda[d]) \tag{2.6}$$

with $\lambda^{(i)} \in \mathcal{A}_{k,n}$ and $d_i - d_{i-1} \geq 0$ such that $\hat{\pi}^{-1}(i) = \lambda^{(i)}/(d_i - d_{i-1})/\lambda^{(i-1)}$ is a cylindric skew diagram; see Figure 1 for a simple example. The weight of $\hat{\pi}$ is the vector $\theta(\hat{\pi}) = (\theta_1, \ldots, \theta_l)$ where θ_i is the number of lattice points $(a,b) \in \lambda^{(i)}/(d_i - d_{i-1})/\lambda^{(i-1)}$ with $1 \leq a \leq k$.

2.4 Cylindric complete symmetric functions

Given $\mu \in \mathcal{A}_{k,n}$ note that $\mu_1 - \mu_k < n$ and, hence, its stabiliser group $S_\mu \subset S_k \subset \hat{S}_k$. Define $\tilde{S}^\mu$ as the minimal length representatives of the cosets $S_\mu \backslash \tilde{S}_k$. Consider now the conjugate partition μ', which is defined by $\mu'_i - \mu'_{i+1} = m_i(\mu)$ for $i = 1, \ldots, n$, where $m_i(\mu)$ is the multiplicity of the part i. Then define $\mu' : \mathbb{Z} \to \mathbb{Z}$ by setting $\mu'_{i+n} = \mu'_i - k$.

Lemma 2.3. *For $\lambda, \mu \in \mathcal{A}_{k,n}$ and $d \in \mathbb{Z}_{\geq 0}$, let $\chi_{\lambda/d/\mu}$ be the cardinality of the set*

$$\{\tilde{w} \in \tilde{S}^\mu \mid \mu \circ \tilde{w} \leq \lambda \circ \tau^d\} . \tag{2.7}$$

Then we have the following expression in terms of binomial coefficients and cylindric loops,

$$\chi_{\lambda/d/\mu} = \prod_{j=1}^{n} \binom{(\lambda \circ \tau^d)'_j - \mu'_{j+1}}{\mu'_j - \mu'_{j+1}} - \prod_{j=1}^{n} \binom{(\lambda \circ \tau^{d-1})'_j - \mu'_{j+1}}{\mu'_j - \mu'_{j+1}} . \tag{2.8}$$

Here the binomial coefficients are defined to be zero if any of their arguments is negative.

As in the non-cylindric case (1.4) we employ (2.8) to define weighted sums over cylindric RPP: given $\hat{\pi}$ set $\chi_{\hat{\pi}} = \prod_{i \geq 1} \chi_{\lambda^{(i)}/(d_i - d_{i-1})/\lambda^{(i-1)}}$, where the cylindric skew diagram $\lambda^{(i)}/(d_i - d_{i-1})/\lambda^{(i-1)}$ is the pre-image $\hat{\pi}^{-1}(i)$.

Definition 2.4. *For $\lambda, \mu \in \mathcal{A}_{k,n}$ and $d \in \mathbb{Z}_{\geq 0}$, introduce the* cylindric complete symmetric function $h_{\lambda/d/\mu}$ *as the weighted sum*

$$h_{\lambda/d/\mu}(x) = \sum_{\hat{\pi}} \chi_{\hat{\pi}} x^{\hat{\pi}} \tag{2.9}$$

over all cylindric reverse plane partitions $\hat{\pi}$ of shape $\lambda/d/\mu$.

Note that the cylindric complete symmetric functions introduced here are different from the cylindric skew Schur functions considered in [13] through the inclusion of the factor $\chi_{\hat{\pi}}$ and involve sums over binomial coefficients.

Theorem 2.5. *Let $\lambda, \mu \in \mathcal{A}_{k,n}$ and $d \in \mathbb{Z}_{\geq 0}$. The function $h_{\lambda/d/\mu}$ is symmetric and one has the expansion*

$$h_{\lambda/d/\mu} = \sum_{\nu \in \mathcal{P}_k^+} N_{\mu\nu}^{\lambda} \, h_\nu \tag{2.10}$$

into the basis $\{h_\nu\}_{\nu \in \mathcal{P}}$ of complete symmetric functions in Λ, where

(i) the sum is restricted to those $\nu \in \mathcal{P}_k^+$ for which $|\nu| = nd + |\lambda| - |\mu|$ and

(ii) $N_{\mu\nu}^{\lambda} \in \mathbb{Z}_{\geq 0}$ are given by the cardinality of the set

$$\{(w, w') \in S^\mu \times S^\nu \mid \mu \circ w + \nu \circ w' = \lambda \circ y^\alpha \text{ for some } \alpha \in \mathcal{P}_k \text{ with } |\alpha| = d\}, \tag{2.11}$$

with $y_i \in \hat{S}_k$ being the translations in the weight lattice introduced earlier.

Note that when setting $d = 0$ we recover the (non-cylindric) skew complete cylindric function, $h_{\lambda/0/\mu} = h_{\lambda/\mu}$. In particular, $N_{\lambda\mu}^{\nu} = f_{\lambda\mu}^{\nu}$ as defined in (1.6), since it follows from $\lambda, \mu \in \mathcal{A}_{k,n}$ that $\alpha_i = 0$ for all $i = 1, \ldots, k$.

Lemma 2.6. *Let $\lambda, \mu \in \mathcal{A}_{k,n}$ and $\nu \in \mathcal{P}_k^+$. Denote by $\check{\nu} \in \mathcal{A}_{k,n}$ the unique intersection point of the orbit $\nu\hat{S}_k$ with the alcove (2.5). Then the expansion coefficients for ν in (2.10) can be re-written in terms of those for $\check{\nu}$ and multinomial coefficients,*

$$N_{\mu\nu}^{\lambda} = N_{\mu\check{\nu}}^{\lambda} \binom{m_n(\check{\nu})}{m_0(\nu), m_n(\nu), m_{2n}(\nu), \ldots} \prod_{i=1}^{n-1} \binom{m_i(\check{\nu})}{m_i(\nu), m_{i+n}(\nu), m_{i+2n}(\nu), \ldots}, \tag{2.12}$$

where $m_j(\nu)$ and $m_j(\check{\nu})$ are the multiplicities of the part j in ν and $\check{\nu}$, respectively.

3 The generalised symmetric group and 2D TQFT

In view of the product and coproduct formulae (1.1) in Λ, one might ask whether similar formulae hold for the cylindric complete symmetric functions (2.9) in an appropriate ring or algebra. We now construct such an algebra as a particular quotient of Λ_k such that the expansion coefficients (2.10) are the structure constants with respect to the (projected) monomial symmetric functions. The quotient is finite-dimensional and can be endowed with the structure of a Frobenius algebra (more precisely a Frobenius extension). Based on work of Atiyah [1] Frobenius algebras are categorically equivalent to 2D topological quantum field theories which are functors from the category of 2-cobordisms to finite-dimensional vector spaces; see e.g. the textbook [7] for details.

3.1 Generalised Verlinde algebras

To motivate our construction let us consider the simplest case $k = 1$ first. Let z be an (invertible) indeterminate and consider the ring $\mathcal{V}_1(n) = \mathbb{C}[z^{\pm 1}][x]/\langle x^n - z\rangle$. The latter has basis $\{1, x, \dots, x^{n-1}\}$ together with the simple multiplication rule

$$x^a x^b = \sum_{c=0}^{n-1} z^{\frac{a+b-c}{n}} N_{ab}^c x^c, \qquad N_{ab}^c = \delta_{a+b \bmod n, c}$$

We have deliberately written the product rule as a sum to facilitate the comparison with the case $k > 1$ below. We shall refer to the structure constants N_{ab}^c as *fusion coefficients*. Consider the linear map $\varepsilon : \mathcal{V}_1(n) \to \mathbb{C}[z^{\pm 1}]$ fixed by $\varepsilon(x^a) = \delta_{a,0}/n$. Then for any $z_0 \in \mathbb{C}^\times$ the quotient $\mathcal{V}_1(n)/(z - z_0)\mathcal{V}_1(n)$ is a Frobenius algebra with the trace map induced by ε. In particular, setting $z_0 = 1$ the resulting Frobenius algebra is the $\widehat{\mathfrak{sl}}_n$-Verlinde algebra at level $k = 1$ [19]. It is closely related to the small quantum cohomology of projective space $QH^*(\mathbb{P}^{n-1})$ but both differ as Frobenius algebras as they are endowed with different trace maps.

Based on the work [10], which realises the Verlinde algebra for level $k \geq 1$ in terms of a quantum integrable model, q-deformed Verlinde algebras were introduced in [8]. The latter are Frobenius algebras that for $q = 0$ specialise to the known $\widehat{\mathfrak{sl}}_n$-Verlinde algebra at level k and for generic q are conjectured [8] to be related to Teleman's work [17] and [18]. Here we consider their $q = 1$ specialisation: let $p_r = m_{(r)}$ denote the rth power sum and consider the following quotient of Λ_k,

$$\mathcal{V}_k(n) = \mathbb{C}[z^{\pm 1}][x_1, \dots, x_k]^{S_k}/\langle p_n - zk, p_{n+1} - zp_1, \dots, p_{n+k-1} - zp_{k-1}\rangle\,. \tag{3.1}$$

For $k = 1$ this ring specialises to the above example $\mathcal{V}_1(n)$ and we therefore refer to (3.1) as generalised Verlinde algebra.

Theorem 3.1. *(i) The monomial symmetric functions $\{m_\lambda\}_{\lambda \in \mathcal{A}_{k,n}}$ are a basis of $\mathcal{V}_k(n)$ and together with the linear map $\varepsilon : \mathcal{V}_k(n) \to \mathbb{C}[z^{\pm 1}]$ given by $\varepsilon(m_\lambda) = z^k \delta_{\lambda, n^k}/n^k$ the pair $(\mathcal{V}_k(n), \varepsilon)$ forms a Frobenius extension. (ii) The following product rules hold in $\mathcal{V}_k(n)$,*

$$m_\lambda m_\mu = \sum_{\nu \in \mathcal{A}_{k,n}} z^{\frac{|\lambda|+|\mu|-|\nu|}{n}} N_{\lambda\mu}^\nu m_\nu \qquad \textit{and} \qquad h_\lambda m_\mu = \sum_{\nu \in \mathcal{A}_{k,n}} z^{\frac{|\lambda|+|\mu|-|\nu|}{n}} \chi_{\nu/d/\mu}(\lambda) m_\nu\,, \tag{3.2}$$

where the fusion coefficients $N_{\lambda\mu}^{\nu}$ *coincide with the non-negative integers in* (2.10) *and* $\chi_{\nu/d/\mu}(\lambda)$ *is the weighted sum over all cylindric RPP of shape* $\nu/d/\mu$ *and weight* λ*, i.e. the coefficient of* m_λ *when expanding* $h_{\nu/d/\mu}$ *defined in* (2.9) *into monomial symmetric functions.*

We wish to emphasise that the basis of monomial symmetric functions is distinguished in (3.1) by the fact that the fusion coefficients in (3.2) are non-negative. Other choices, for example Schur functions, lead to structure constants which can be negative.

3.2 Idempotents and the Verlinde formula

Provided that the roots $t^{\pm 1} = z^{\pm 1/n}$ exist, one can show that $\mathcal{V}_k(n)$ is semi-simple over $\mathbb{C}[t^{\pm 1}]$. Assume first that $z = 1$ and fix a primitive nth root of unity $\zeta \in \mathbb{S}^1 \subset \mathbb{C}$. Then each $\alpha \in \mathcal{A}_{k,n}$ determines a point $\zeta^\alpha = (\zeta^{\alpha_1}, \ldots, \zeta^{\alpha_k}) \in \mathbb{T}^k$ on the k-dimensional torus and the polynomials

$$\mathfrak{e}_\alpha(x_1, \ldots, x_k) = \frac{1}{n^k} \sum_{\lambda \in \mathcal{A}_{k,n}} \frac{|S_\lambda|}{|S_\alpha|} m_\lambda(\zeta^{-\alpha}) m_\lambda(x_1, \ldots, x_k) \tag{3.3}$$

obey $\mathfrak{e}_\alpha(\zeta^\beta) = \delta_{\alpha\beta}$ and form a complete set of idempotents of $\mathcal{V}_k(n)/(z-1)\mathcal{V}_k(n)$. The idempotents for general z are obtained by rescaling the roots, $\zeta^{\pm\alpha_i} \mapsto t^{\pm 1}\zeta^{\pm\alpha_i}$.

Theorem 3.2. *For* $z = 1$ *the product rules* (3.2) *describe the multiplication of symmetric functions evaluated at roots of unity and we have the following residue formula for the fusion coefficients in* (2.10) *and* (3.2)*:*

$$N_{\lambda\mu}^{\nu} = \sum_{\alpha \in \mathcal{A}_{k,n}} \frac{\mathcal{S}_{\lambda\alpha}\mathcal{S}_{\mu\alpha}\mathcal{S}_{\nu\alpha}^{-1}}{\mathcal{S}_{\emptyset\alpha}}, \qquad \mathcal{S}_{\lambda\alpha} = \frac{m_\lambda(\zeta^\alpha)}{n^{\frac{k}{2}}}. \tag{3.4}$$

The last expression (3.4) is a generalisation of the celebrated Verlinde formula [19] for $k = 1$. The $\mathcal{S}$-matrix in (3.4) links the 2D TQFT $\mathcal{V}_k(n)/(z-1)\mathcal{V}_k(n)$ with the following representation of the modular group:

Proposition 3.3. *For* $z = 1$ *the matrices* $\mathcal{S}_{\lambda\mu}$ *and* $\mathcal{T}_{\lambda\mu} = \delta_{\lambda\mu}\zeta^{-\frac{kn(n-1)}{24}}\theta_\lambda$*, with* $\theta_\lambda = \prod_{i=1}^k \zeta^{\frac{\lambda_i(n-\lambda_i)}{2}}$ *and* $\lambda, \mu \in \mathcal{A}_{k,n}$*, yield a representation of the (double cover of the) modular group* $PSL_2(\mathbb{Z})$*. That is, we have the identities*

$$(\mathcal{S}\mathcal{T})^3 = \mathcal{S}^2 = \mathcal{C}, \qquad \mathcal{C}_{\lambda\mu} = \delta_{\lambda\mu^*}, \tag{3.5}$$

where $\lambda^* \in \mathcal{A}_{k,n}$ *is the image of* $(n-\lambda_k, \ldots, n-\lambda_2, n-\lambda_1)$ *under* $\tau^{m_n(\lambda)}$ *and* $\mathcal{C}^2 = \mathrm{Id}$*. In addition, we have unitarity,* $\mathcal{S}^{-1} = \mathcal{S}^*$ *and the relations* $\mathcal{S}^* = \mathcal{C}\mathcal{S} = \mathcal{S}\mathcal{C}$ *and* $\mathcal{C}\mathcal{T}\mathcal{C} = \mathcal{T}$*.*

This representation of the modular group is different from the one for the $\widehat{\mathfrak{sl}}_n$-Verlinde algebra in terms of Schur functions if $k > 1$; see [6, Proposition 6.15] and the discussion in [10, Section 6.4]. To facilitate the comparison, observe that each $\lambda \in \mathcal{A}_{k,n}$ defines a $\widehat{\mathfrak{sl}}_n$-weight $m_n(\lambda)\omega_0 + \sum_{i=1}^{n-1} m_i(\lambda)\omega_i$ at level k, where the ω_i are the fundamental $\widehat{\mathfrak{sl}}_n$-weights.

3.3 The fusion product in terms of the generalised symmetric group

Recall the definition of the generalised symmetric group $S(n,k) = \mathcal{C}_n^{\times k} \rtimes S_k$ as the wreath product of the cyclic group of order n with S_k. That is, the group of all complex $k \times k$ matrices that have at most one nonzero entry $\in \mathcal{C}_n$ in each row and column. The simple modules $\mathcal{L}(\boldsymbol{\lambda})$ of $S(n,k)$ are labelled in terms of n-multipartitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ of k, i.e. $|\boldsymbol{\lambda}| = \sum_i |\lambda^{(i)}| = k$; see e.g. [14] for details. Consider the following exact sequence of groups, $1 \to \mathcal{C}_n^{\times k} \hookrightarrow S(n,k) \twoheadrightarrow S_k \to 1$, and denote by y_i the image of the generator of the ith copy of $\mathcal{C}_n$ of the normal subgroup $\mathcal{N} \cong \mathcal{C}_n^{\times k}$ in $S(n,k)$.

Lemma 3.4 ([14]). *The characters of $\mathcal{L}(\boldsymbol{\lambda})$ restricted to the normal subgroup $\mathcal{N}$ are given by*

$$\chi_{\boldsymbol{\lambda}}(y^\alpha) = \mathrm{Tr}_{\mathcal{L}(\boldsymbol{\lambda})}\, y^\alpha = f_{\boldsymbol{\lambda}}\, m_\lambda(\zeta^\alpha), \quad \forall \alpha \in \mathbb{Z}_n^{\times k}, \quad f_{\boldsymbol{\lambda}} = \prod_{i=1}^n f_{\lambda^{(i)}}\,, \tag{3.6}$$

where $f_{\lambda^{(i)}}$ is the number of standard tableaux of shape $\lambda^{(i)}$ and $\lambda \in \mathcal{A}_{k,n}$ is the unique partition with $m_i(\lambda) = |\lambda^{(i)}|$.

Let $\mathrm{Rep}\, S(n,k)$ denote the representation ring and $c^{\boldsymbol{\nu}}_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ its structure constants with respect to the simple modules $\mathcal{L}(\boldsymbol{\lambda})$.

Corollary 3.5. *We have the following alternative expression for the fusion coefficients in* (2.10), (2.12) *and* (3.2),

$$N^\nu_{\lambda\mu} = \sum_{\boldsymbol{\nu}} c^{\boldsymbol{\nu}}_{\boldsymbol{\lambda}\boldsymbol{\mu}} \frac{f_{\boldsymbol{\nu}}}{f_{\boldsymbol{\lambda}} f_{\boldsymbol{\mu}}}, \tag{3.7}$$

where the sum runs over all multipartitions $\boldsymbol{\nu}$ such that $|\nu^{(i)}| = m_i(\nu)$ with $\nu \in \mathcal{A}_{k,n}$. Similarly, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ is any pair of multipartitions which satisfy the analogous condition for $\lambda, \mu \in \mathcal{A}_{k,n}$.

We can interpret this result as stating that the generalised Verlinde algebra (3.1) describes for $z = 1$ the representation ring of the normal subgroup $\mathcal{N}$ inside $\mathrm{Rep}\, S(n,k)$.

3.4 Schur–Weyl duality and the principal subalgebra

Define the following elements of the loop algebra $\mathfrak{sl}_n[z,z^{-1}] = \mathfrak{sl}_n(\mathbb{C}) \otimes \mathbb{C}[z,z^{-1}]$ in terms of the unit matrices $\{e_{ij} \mid 1 \le i,j \le n\} \subset \mathfrak{gl}_n(\mathbb{C})$,

$$P_r = \sum_{i-j=r} e_{ij} + z \sum_{j-i=n-r} e_{ij}\,, \qquad r = 1, \dots, n-1, \tag{3.8}$$

and set $P_{r+n} = zP_r$ for all other $r \in \mathbb{Z}\backslash n\mathbb{Z}$. The latter generate a commutative subalgebra of $\mathfrak{sl}_n[z,z^{-1}]$ with respect to the usual Lie bracket, $[f(z)x, g(z)y] := f(z)g(z)[x,y]$, where $f, g \in \mathbb{C}[z,z^{-1}]$ and $x, y \in \mathfrak{sl}_n(\mathbb{C})$. This commutative subalgebra is the image of the *principal Heisenberg subalgebra* in the affine Lie algebra $\widehat{\mathfrak{gl}}_n = \mathfrak{gl}_n[z,z^{-1}] \oplus \mathbb{C}\underline{k}$ under the projection defined via $0 \to \mathbb{C}\underline{k} \to \widehat{\mathfrak{gl}}_n \xrightarrow{\pi} \mathfrak{gl}_n[z, z^{-1}] \to 0$, where $\underline{k}$ is the central element; see [5] for further details.

Recall the canonical left action of the universal enveloping algebra $U = U(\mathfrak{sl}_n[z, z^{-1}])$ on the tensor product $\mathbb{C}[z, z^{-1}] \otimes (\mathbb{C}^n)^{\otimes k}$: let $x \in \mathfrak{sl}_n$ and $v_1, \ldots, v_k \in \mathbb{C}^n$, then

$$f(z) \otimes x.g(z)v_1 \otimes \cdots \otimes v_k = f(z)g(z) \sum_{i=1}^{k} v_1 \otimes \cdots \otimes x.v_i \otimes \cdots \otimes v_k \,.$$

This U-action commutes with the canonical right action of the symmetric group S_k on $(\mathbb{C}^n)^{\otimes k}$. Thus, the U-action leaves the kth symmetric power $\mathbb{C}[z, z^{-1}] \otimes S^k(\mathbb{C}^n)$ invariant.

Theorem 3.6. *Denote by $\mathbb{V}_k(n) \subset \mathrm{End}(\mathbb{C}[z^{\pm 1}] \otimes S^k(\mathbb{C}^n))$ the commutative ring generated by the $\{P_r\}_{r \in \mathbb{Z}}$ with $P_{mk} = z^m k\,\mathrm{Id}$. Then the map $\mathbb{V}_k(n) \to \mathcal{V}_k(n)$ fixed by $P_r \mapsto p_r$ and $P_{-r} \mapsto z^{-1} p_{n-r}$ for $r = 1, \ldots, n-1$, where $p_r = m_{(r)}$ are the power sums, is a ring isomorphism.*

The analogous statement for the antisymmetric power $\bigwedge^k(\mathbb{C}^n)$ yields a ring isomorphism with the small quantum cohomology ring $QH^*(\mathrm{Gr}_k(\mathbb{C}^n))$ of the Grassmannian, where $z = (-1)^{k-1} q$ and q is the deformation parameter [9]. The last theorem shows that the generalised Verlinde algebra (3.1) naturally arises as a counterpart of $QH^*(\mathrm{Gr}_k(\mathbb{C}^n))$ within the context of Schur–Weyl duality and that in this setting the cylindric complete symmetric functions (2.9) play a role analogous to Postnikov's toric Schur functions [15].

References

[1] M.F. Atiyah. "Topological quantum field theory". *Publ. Math. IHES* **68** (1988), pp. 175–186. URL.

[2] A. Björner and F. Brenti. "Affine permutations of type A". *Electron. J. Combin.* **3**.2 (1996). The Foata Festschrift, Research Paper 18, approx. 35 pp. URL.

[3] L.M. Butler and A.W. Hales. "Nonnegative Hall polynomials". *J. Algebraic Combin.* **2**.2 (1993), pp. 125–135. DOI: 10.1023/A:1022407523839.

[4] I.M. Gessel and C. Krattenthaler. "Cylindric partitions". *Trans. Amer. Math. Soc.* **349**.2 (1997), pp. 429–479. DOI: 10.1090/S0002-9947-97-01791-1.

[5] V.G. Kač. *Infinite-dimensional Lie algebras*. Vol. 44. Cambridge University Press, 1994.

[6] V.G. Kač and D.H. Peterson. "Infinite-dimensional Lie algebras, theta functions and modular forms". *Adv. Math.* **53**.2 (1984), pp. 125–264. DOI: 10.1016/0001-8708(84)90032-X.

[7] J. Kock. *Frobenius algebras and 2D topological quantum field theories*. Vol. 59. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2004, xiv+240 pp.

[8] C. Korff. "Cylindric versions of specialised Macdonald functions and a deformed Verlinde algebra". *Comm. Math. Phys.* **318**.1 (2013), pp. 173–246. DOI: 10.1007/s00220-012-1630-9.

[9] C. Korff and D. Palazzo. “Cylindric symmetric functions and positivity”. 2018. arXiv: 1804.05647.

[10] C. Korff and C. Stroppel. “The $sl(n)_k$-WZNW fusion ring: a combinatorial construction and a realisation as quotient of quantum cohomology”. *Adv. Math.* **225**.1 (2010), pp. 200–268. DOI: 10.1016/j.aim.2010.02.021.

[11] G. Lusztig. “Some examples of square integrable representations of semisimple p-adic groups”. *Trans. Amer. Math. Soc.* **277**.2 (1983), pp. 623–653. DOI: 10.2307/1999228.

[12] I.G. Macdonald. *Symmetric functions and Hall polynomials*. Second. Oxford Mathematical Monographs. With contributions by A. Zelevinsky, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995, p. 475.

[13] P. McNamara. “Cylindric skew Schur functions”. *Adv. Math.* **205**.1 (2006), pp. 275–312. DOI: 10.1016/j.aim.2005.07.011.

[14] M. Osima. “On the representations of the generalized symmetric group”. *Math. J. Okayama Univ* **4**.1 (1954), pp. 39–56.

[15] A. Postnikov. “Affine approach to quantum Schubert calculus”. *Duke Math. J.* **128**.3 (2005), pp. 473–509. DOI: 10.1215/S0012-7094-04-12832-5.

[16] R.P. Stanley. *Enumerative Combinatorics, Vol. 2*. Cambridge Stud. Adv. Math. 62. Cambridge: Cambridge University Press, 1999.

[17] C. Teleman. “K-theory and the moduli space of bundles on a surface and deformations of the Verlinde algebra”. *Topology, geometry and quantum field theory*. Vol. 308. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2004, pp. 358–378. DOI: 10.1017/CBO9780511526398.015.

[18] C. Teleman and C.T. Woodward. “The index formula for the moduli of G-bundles on a curve”. *Ann. of Math.* 2 (2009), pp. 495–527. DOI: 10.4007/annals.2009.170.495.

[19] E. Verlinde. “Fusion rules and modular transformations in 2D conformal field theory”. *Nuclear Phys. B* **300**.3 (1988), pp. 360–376. DOI: 10.1016/0550-3213(88)90603-7.