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Variations of the Catalan numbers from some nonassociative binary operations

Nickolas Hein*[1] and Jia Huang†[2]

[1]*Department of Mathematics and Computer Science, Benedictine College, Atchison, KS 66002, USA*
[2]*Department of Mathematics and Statistics, University of Nebraska, Kearney, NE 68849, USA*

Abstract. We investigate certain nonassociative binary operations that satisfy a four-parameter generalization of the associative law. From this we obtain variations of the ubiquitous Catalan numbers and connections to many interesting combinatorial objects such as binary trees, plane trees, lattice paths, and permutations.

Keywords: Binary operation, binary tree, Catalan number, nonassociativity

1 Introduction

We consider an overlooked yet natural question: to what degree is a given binary operation nonassociative? We use $*$ to denote a binary operation on a set A and use $a_0, a_1, a_2, \ldots$ to denote A-valued indeterminates. Let $\mathcal{P}_{*,n}$ be the set of all parenthesizations of the otherwise ambiguous expression $a_0 * \cdots * a_n$. The set $\mathcal{P}_{*,n}$ is in bijection with the set $\mathcal{T}_n$ of (full) binary trees with $n+1$ leaves. Both sets $\mathcal{P}_{*,n}$ and $\mathcal{T}_n$ are among the hundreds of families of objects enumerated by the ubiquitous *Catalan number* $C_n := \frac{1}{n+1}\binom{2n}{n}$; see, e.g., Stanley [11]. Figure 1 below illustrates the bijection $\mathcal{P}_{*,3} \leftrightarrow \mathcal{T}_3$.

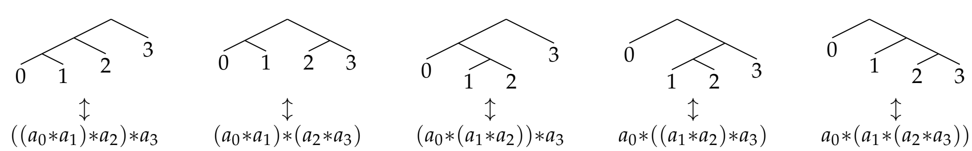

Figure 1: binary trees and parenthesizations

We investigate two nonassociativity measurements for the operation $*$. Given binary trees $t, t' \in \mathcal{T}_n$, write $t \sim_* t'$ if the corresponding parenthesizations are equal as functions from A^{n+1} to A. This is an equivalence relation on a set of Catalan objects, and for brevity we say its equivalence classes are *$(*, n)$-classes*. We define $C_{*,n}$ to be the number

*nhein@benedictine.edu
†huangj2@unk.edu

of $(*, n)$-classes, and observe that $1 \le C_{*,n} \le C_n$. We now have an alternate definition of associativity that specializes to the traditional meaning: the operation $*$ is associative if $C_{*,n} = 1$ for all $n \ge 0$. Thus $C_{*,n}$ measures the failure of $*$ to be associative. We say $*$ is *totally nonassociative* if $C_{*,n}$ attains its theoretical upper bound, $C_{*,n} = C_n$.

Alternatively, one may quantify nonassociativity by computing the cardinality $\widetilde{C}_{*,n}$ of the largest $(*, n)$-class for each n. Again, we have $1 \le \widetilde{C}_{*,n} \le C_n$. One may see that $\widetilde{C}_{*,n} = 1$ if and only if $*$ is totally nonassociative and $\widetilde{C}_{*,n} = C_n$ if and only if $*$ is associative. Moreover, $1 \le C_{*,n} + \widetilde{C}_{*,n} - 1 \le C_n$.

In earlier work [5], we investigated the nonassociativity of a 1-parameter family (depending on a positive integer k) of binary operations which generalize addition ($k = 1$) and subtraction ($k = 2$). Now we further generalize this to a four-parameter family (depending on positive integers d, e, k, ℓ) of binary operations which are neither associative nor totally nonassociative. Our prototypical example is in the quotient $\mathbb{C}[x, y]/I$ of the polynomial ring $\mathbb{C}[x, y]$ by its ideal $I := (x^{d+k} - x^d, y^{e+\ell} - y^e)$. We define the binary operation $*$ by the following rule:

$$f * g := xf + yg, \quad \forall f, g \in \mathbb{C}[x, y]/I. \tag{1.1}$$

The relations imposed on x and y are motivated by the relation satisfied by an element in a finite semigroup; see, e.g., Steinberg [12, Section 1.2]. A parenthesization corresponding to a binary tree $t \in \mathcal{T}_n$ has the form

$$x^{\delta_0(t)} y^{\rho_0(t)} f_0 + \cdots + x^{\delta_n(t)} y^{\rho_n(t)} f_n. \tag{1.2}$$

Here we list the leaves of t as $0, 1, \ldots, n$ according to *preorder* and define the *left depth* $\delta_i(t)$ (resp., *right depth* $\rho_i(t)$) of i to be the number of left (resp., right) steps along the unique path from the root of t down to i. The map sending each $t \in \mathcal{T}_n$ to its *left depth* $\delta(t) := (\delta_0(t), \ldots, \delta_n(t))$ is one-to-one [5, Section 2.1], and by symmetry, so is the map sending each $t \in \mathcal{T}_n$ to its *right depth* $\rho(t) := (\rho_0(t), \ldots, \rho_n(t))$.

For example, the third binary tree in Fig. 1 has left depth $\delta = (2, 2, 1, 0)$ and right depth $\rho = (0, 1, 2, 1)$, and the corresponding parenthesization $(f_0 * (f_1 * f_2)) * f_3$ can be written as $x^2 f_0 + x^2 y f_1 + x y^2 f_2 + y f_3$.

For $*$ defined by (1.1), comparing expressions for $t, t' \in \mathcal{T}_n$ of the form (1.2) gives

$$t \sim_* t' \quad \text{if and only if} \quad \delta(t) \sim_k^d \delta(t') \quad \text{and} \quad \rho(t) \sim_\ell^e \rho(t'). \tag{1.3}$$

Here for any two integer sequences $\mathbf{b} = (b_0, \ldots, b_n)$ and $\mathbf{c} = (c_0, \ldots, c_n)$ we write

(1) $\mathbf{b} \sim_k \mathbf{c}$ if $b_i \equiv c_i \pmod{k}$ for $i = 0, \ldots, n$,
(2) $\mathbf{b} \sim^d \mathbf{c}$ if $\min\{b_i, c_i\} < d$ implies $b_i = c_i$ for $i = 0, \ldots, n$, and
(3) $\mathbf{b} \sim_k^d \mathbf{c}$ if $\mathbf{b} \sim_k \mathbf{c}$ and $\mathbf{b} \sim^d \mathbf{c}$.

We write $C_{k,\ell,n}^{d,e} := C_{*,n}$ and $\widetilde{C}_{k,\ell,n}^{d,e} := \widetilde{C}_{*,n}$ for any binary operation $*$ satisfying (1.3). Observe that if $d \le d'$, $e \le e'$, $k \mid k'$, and $\ell \mid \ell'$, then $C_{k,\ell,n}^{d,e} \le C_{k',\ell',n}^{d',e'}$ and $\widetilde{C}_{k,\ell,n}^{d,e} \ge \widetilde{C}_{k',\ell',n}^{d',e'}$. We symmetrically have $C_{k,\ell,n}^{d,e} = C_{\ell,k,n}^{e,d}$ and $\widetilde{C}_{k,\ell,n}^{d,e} = \widetilde{C}_{\ell,k,n}^{e,d}$.

Note that the relations $\sim_1^d$ and $\sim_k^1$ coincide with $\sim^d$ and $\sim_k$, respectively, on left and right depths of binary trees in $\mathcal{T}_n$. In earlier work [5], we determined $C_{k,n} := C_{k,1,n}^{1,1}$ using plane trees, Dyck paths, and Lagrange inversion. We call $C_{k,n}$ a *(k-)modular Catalan number* as for any binary operation $*$ satisfying (1.3) with $d = e = \ell = 1$, the $(*, n)$-relation is the same as the congruence relation modulo k on left depths of binary trees in $\mathcal{T}_n$. We also determined $\widetilde{C}_{k,n} := \widetilde{C}_{k,1,n}^{1,1}$ and enumerated $(*, n)$-classes with this largest size. The "if" part of (1.3) with $d = e = \ell = 1$ is equivalent to *k-associativity*, given by the rule $(a_0 * \cdots * a_k) * a_{k+1} = a_0 * (a_1 * \cdots * a_{k+1})$, where the $*$'s in parentheses are evaluated from left to right [5, Proposition 2.11]. This gives a one-parameter generalization of the usual associativity ($k = 1$).

We will see in Section 2.1 that, for $e = k = \ell = 1$, the "if" part of the $(*, n)$-relation (1.3) can be viewed as *associativity at left depth d*, that is, $t \sim_* t'$ if t can be obtained from t' by a finite sequence of moves, each of which replaces the maximal subtree rooted at a node of left depth at least $d - 1$ by another binary tree with the same number of leaves. Here a *subtree rooted at a node v* is a subtree whose root is v, and the *maximal subtree rooted at v* is the subtree consisting of all descendants of v, including v itself.

Motivated by the two single-parameter generalizations of associativity above, we say a binary operation $*$ is *(k, ℓ)-associative at depth (d, e)* if it satisfies the "if" part of (1.3). We focus on two special cases, $k = \ell = 1$ and $e = \ell = 1$, each giving a two-parameter generalization of the usual associativity with connections to many interesting integer sequences and combinatorial objects.

In Section 2 we study the case $k = \ell = 1$. In this case the "if" part of (1.3) can be viewed as *associativity at left depth d and right depth e*, which recovers the associativity at left depth d when $e = 1$. We determine $\widetilde{C}_n^{d,e} := \widetilde{C}_{1,1,n}^{d,e}$ and enumerate $(*, n)$-classes with this largest size for any binary operation $*$ satisfying (1.3) with $k = \ell = 1$. We prove that the cardinality of each $(*, n)$-class is a product of Catalan numbers. We provide a recursive formula for the generating function $C^{d,e}(x) := \sum_{n \geq 0} C_n^{d,e} x^{n+1}$ of the number $C_n^{d,e} := C_{1,1,n}^{d,e}$ of $(*, n)$-classes and give closed formulas for $C^{d,e}(x)$ and $C_n^{d,e}$ when $e = 1, 2$. It turns out that $C^{d,1}(x)$ is a well-known continued fraction and $C_n^{d,1}$ enumerates many families of objects (see Andrews–Krattenthaler–Orsina–Luigi–Papi [1], de Bruijn–Knuth–Rice [3], Flajolet [4], Kitaev–Remmel–Tiefenbruck [7], Kreweras [8], and The OEIS [10, A080934]). Our formula for $C_n^{d,1}$ is different from the existing formulas [1, 3]. We find no existing result on $C_n^{d,e}$ for $d, e \geq 2$ except $C_n^{2,2}$ and $C_n^{3,2}$ [10, A045623,A142586].

In Section 3 we study the case $e = \ell = 1$. In this case the "if" part of (1.3) can be viewed as *k-associativity at left depth d*, which recovers the k-associativity when $d = 1$ and recovers the associativity at left depth d when $k = 1$. We show that the number $C_{k,n}^d := C_{k,1,n}^{d,1}$ enumerates binary trees and Dyck paths with certain constraints, and establish a recursive formula for its generating function $C_k^d(x) := \sum_{n \geq 0} C_{k,n}^d x^{n+1}$. We have $C_1^d(x) = C^{d,1}(x)$ and $C_2^d(x) = C^{d+1,1}(x)$ for $d, n \geq 0$. The number $C_{3,n}^d$ appears in The OEIS [10, A005773, A054391–A054394] for $d = 1, \ldots, 5$. Barcucci, Del Lungo, Pergola, and Pinzani [2] studied $C^d_{3,n}$ in terms of pattern avoidance in permutations and obtained a closed formula for $C^d_3(x)$, but no closed formula for $C^d_{3,n}$. We provide a different formula for $C^d_3(x)$ and derive a closed formula for $C^d_{3,n}$ from it. We also give closed formulas for $C^2_{k,n}$ using Lagrange inversion and our earlier work [5] on $C^1_{k,n}$.

Another 2-parameter specialization of (1.3) is obtained by taking $d = e = 1$ and computations suggest a conjecture: $C^{1,1}_{k,\ell,n} = C_{k+\ell-1,n}$ for all $k, \ell \geq 1$ and $n \geq 0$. We will also explore the case $k, \ell, d, e > 1$ in the future.

2 Associativity at left depth d and right depth e

In this section we study $C^{d,e}_n := C_{*,n}$ and $\widetilde{C}^{d,e}_n := \widetilde{C}_{*,n}$ for any operation $*$ satisfying (1.3) with $k = \ell = 1$, i.e., when $t \sim_* t' \Leftrightarrow \delta(t) \sim^d \delta(t')$ and $\rho(t) \sim^e \rho(t')$ for all $t, t' \in \mathcal{T}_n$.

We first introduce some notation. If a node in a binary tree has left depth $\delta \geq d-1$ and right depth $\rho \geq e-1$ then we say this node is *(d,e)-contractible*, or simply *contractible* if d and e are clear from the context. We call a contractible node *maximal* if its parent is not contractible. One sees that a node with left depth δ and right depth ρ is a maximal contractible node if and only if $\delta = d-1$ and $\rho \geq e-1$ when v is the left child of its parent, or $\delta \geq d-1$ and $\rho = e-1$ when v is the right child of its parent.

Let $t \in \mathcal{T}_n$ and assign each leaf weight one. For each maximal contractible node v, we contract its subtree to a single node and assign v a weight equal to the number of leaves in this subtree. Denote by $\phi(t)$ the resulting weighted binary tree. This gives a map $\phi : \mathcal{T}_n \to \mathcal{T}^{d,e}_n$ by $t \mapsto \phi(t)$, where $\mathcal{T}^{d,e}_n$ is the set of all leaf-weighted binary trees such that every contractible leaf is maximal and has a positive integer weight, every non-contractible leaf has weight one, and the sum of all leaf weights is $n+1$.

Conversely, if $\bar{t} \in \mathcal{T}^{d,e}_n$ has leaves $v_0, \ldots, v_r$ with weights $m_0, \ldots, m_r$, respectively, then replacing v_i by $t_i \in \mathcal{T}_{m_i-1}$ for $i = 0, \ldots, r$ gives a binary tree $\phi^{-1}(\bar{t}; t_0, \ldots, t_r)$.

Lemma 2.1. *(i) We have a surjection $\phi : \mathcal{T}_n \twoheadrightarrow \mathcal{T}^{d,e}_n$.*
(ii) For each $\bar{t} \in \mathcal{T}^{d,e}_n$ whose leaves $v_0, \ldots, v_r$ are weighted $m_0, \ldots, m_r$, its fiber is

$$\phi^{-1}(\bar{t}) = \{\phi^{-1}(\bar{t}; t_0, \ldots, t_r) : t_i \in \mathcal{T}_{m_i-1}\}.$$

(iii) We have $t \sim_* t'$ whenever $\phi(t) = \phi(t')$.

Theorem 2.2. *Let $*$ be a binary operation satisfying (1.3) with $k = \ell = 1$. Then the fibers of ϕ are precisely the $(*, n)$-classes. Thus the cardinality of a $(*, n)$-class is a product of Catalan numbers $C_{m_0-1} \cdots C_{m_r-1}$ for some positive integers $m_0, \ldots, m_r$ satisfying $m_0 + \cdots + m_r = n+1$.*

Theorem 2.3. *Let $d, e \geq 1$. If $0 \leq n < d+e$ then $\widetilde{C}^{d,e}_n = 1$. If $n \geq d+e$ then $\widetilde{C}^{d,e}_n = C_{n+2-d-e}$ and the number of $(*, n)$-classes with this size is $\binom{d+e-2}{d-1}$.*

For $d,e \geq 1$ let $C^{d,e}(x) := \sum_{n\geq 0} C_n^{d,e} x^{n+1}$, $C^{0,e}(x) := C^{1,e}(x)$, and $C^{d,0}(x) := C^{d,1}(x)$.

Proposition 2.4. *For $d,e \geq 1$ we have $C^{d,e}(x) = x + C^{d-1,e}(x)C^{d,e-1}(x)$.*

We apply Proposition 2.4 to study $C^{d,e}(x)$ for $e = 1,2$, respectively, in the next two subsections. We find no result on $C_n^{d,3}$ for $d \geq 3$ in The OEIS [10].

2.1 Associativity of left depth d: The case $e = k = \ell = 1$

Assume $e = k = \ell = 1$ in this subsection. Then the "if" part of the $(*, n)$-relation (1.3) may be regarded as *associativity at left depth d*, as Theorem 2.2 implies that two trees $t, t' \in \mathcal{T}_n$ satisfy $t \sim_* t'$ if and only if t can be obtained from t' by a finite sequence of moves, each of which replaces the maximal subtree rooted at a node of left depth at least $d-1$ by another binary tree with the same number of leaves.

We use Proposition 2.4 to determine the number $C_n^d := C_n^{d,1}$ of $(*, n)$-classes from its generating function $C^d(x) := C^{d,1}(x)$ for all $d \geq 1$. We need the *Fibonacci polynomials* defined by $F_n(x) := F_{n-1}(x) - xF_{n-2}(x)$ for $n \geq 2$ with $F_i(x) := i$ for $i = 0,1$. For $n \geq 1$ we have [3, (8), (9), (10)]

$$\begin{aligned} F_n(x) &= \frac{1}{\sqrt{1-4x}} \left(\left(\frac{1+\sqrt{1-4x}}{2} \right)^n - \left(\frac{1-\sqrt{1-4x}}{2} \right)^n \right) \\ &= \sum_{0 \leq i \leq (n-1)/2} \binom{n-1-i}{i} (-x)^i = \prod_{1 \leq j \leq (n-1)/2} (1 - 4x\cos^2(j\pi/n)). \end{aligned}$$

Corollary 2.5 (Kreweras [8])**.** *For $d \geq 1$ we have (with $C^0(x) := x$)*

$$C^d(x) = \frac{x}{1 - C^{d-1}(x)} = \frac{xF_{d+1}(x)}{F_{d+2}(x)}.$$

Corollary 2.5 follows from Proposition 2.4 and implies that $C^d(x)$ is a well-known continued fraction [3, 4]:

$$C^1(x) = \frac{x}{1-x},\ C^2(x) = \frac{x}{1 - \frac{x}{1-x}} = \frac{x(1-x)}{1-2x},\ C^3(x) = \frac{x}{1 - \frac{x}{1 - \frac{x}{1-x}}} = \frac{x(1-2x)}{1-3x+x^2}, \ldots.$$

Hence $(C_n^d)_{d\geq 1, n\geq 0}$ coincides with an array in The OEIS [10, A080934] and enumerates
(i) Dyck paths of length $2n$ with height at most d (Flajolet [4], Kreweras [8, page 38]),
(ii) permutations in $\mathfrak{S}_n$ avoiding 132 and $12\cdots(d+1)$ (Kitaev–Remmel–Tiefenbruck [7]),
(iii) plane trees with $n+1$ nodes of depth at most d (de Bruijn–Knuth–Rice [3]), and
(iv) ad-nilpotent ideals of the Borel subalgebra of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of order at most $d-1$ (Andrews–Krattenthaler–Orsina–Papi [1]).

One has $C_n^2 = 2^{n-1}$, $C_n^3 = F_{2n-1}$, and $C_n^4 = \frac{1}{2}(1+3^{n-1})$ for $n \geq 1$ [1, 7]. In general,

$$\begin{aligned} C_n^d &= \sum_{i\in\mathbb{Z}} \frac{2i(d+2)+1}{2n+1}\binom{2n+1}{n-i(d+2)} && \text{[1, Theorem 4.5]} \\ &= \det\left[\binom{i-\max\{-1,j-d\}}{j-i+1}\right]_{i,j=1}^{n-1} && \text{[1, Theorem 4.5]} \\ &= \sum_{0=i_0\leq i_1\leq\cdots\leq i_{d-1}\leq i_d=n}\ \prod_{0\leq j\leq d-2}\binom{i_{j+2}-i_j-1}{i_{j+1}-i_j} && \text{[1, Corollary 4.3]} \\ &= \frac{2^{2n+1}}{d+2}\sum_{1\leq j\leq d+1}\sin^2(j\pi/(d+2))\cos^{2n}(j\pi/(d+2)). && \text{[3, (14)]} \end{aligned}$$

Now we derive a closed formula for C_n^d from the generating function $C^d(x)$. We write $\alpha \models n$ if α is a *composition* of n, i.e., if $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ is a sequence of positive integers such that $\alpha_1 + \cdots + \alpha_\ell = n$. We also define $\ell(\alpha) := \ell$ and $\max(\alpha) := \max\{\alpha_1, \ldots, \alpha_\ell\}$.

Proposition 2.6. *For $n, d \geq 1$ we have*

$$C_n^d = \sum_{\substack{\alpha\models n \\ \max(\alpha)\leq(d+1)/2}} (-1)^{n-\ell(\alpha)}\binom{d-\alpha_1}{\alpha_1-1}\prod_{2\leq r\leq\ell(\alpha)}\binom{d+1-\alpha_i}{\alpha_i}.$$

This alternating formula differs from the other known formulas. For example, it gives $C_3^4 = 1\cdot4\cdot4-2\cdot4-1\cdot3 = 5$ and the determinant formula gives $C_3^4 = 2\cdot3-1 = 5$.

Next, we give an interpretation of the number C_n^d which is very similar to the one by Andrews–Krattenthaler–Orsina–Papi [1]. We do not know any direct way (e.g., using the exponential map) to convert one to the other.

Let $\mathcal{U}_n$ be the algebra of n-by-n upper triangular matrices over a field $\mathbb{F}$, with the usual matrix product. A nilpotent (two-sided) ideal of $\mathcal{U}_n$ can be represented by an n-by-n matrix whose entries are either zero or arbitrary such that these two kinds of entries are separated by a lattice path above the main diagonal from the upper left corner to the lower right corner. Thus the number of nilpotent ideals of $\mathcal{U}_n$ is the Catalan number C_n. L. Shapiro [9] showed that the number of commutative ideals of $\mathcal{U}_n$ is 2^{n-1}. Motivated by the observation that an ideal I of $\mathcal{U}_n$ is commutative if and only if $I^2 = 0$, we generalize Shapiro's result below, using the *(nilpotent) order* of a nilpotent ideal I, which is defined as $\inf\{d : I^d = 0\}$. We are grateful to Brendon Rhoades for giving a proof for this result.

Proposition 2.7. *For $d \geq 1$, nilpotent ideals of order at most d in $\mathcal{U}_n$ are enumerated by C_n^d.*

2.2 Associativity at left depth d and right depth $e = 2$

Now we give closed formulas for the generating function $C^{d,2}(x)$ and the number $C_n^{d,2}$.

Proposition 2.8. *For $d, n \geq 2$ we have*

$$C^{d,2}(x) = C^d(x) + \frac{x^{d+2}}{(1-2x)F_{d+2}(x)} \quad \textit{and}$$

$$C_n^{d,2} = C_n^d + \sum_{1 \leq i \leq n-d} 2^{i-1} \sum_{\substack{\alpha \models n-d-i \\ \max(\alpha) \leq (d+1)/2}} (-1)^{n-d-i-\ell(\alpha)} \prod_{1 \leq j \leq \ell(\alpha)} \binom{d+1-\alpha_j}{\alpha_j}.$$

Using Proposition 2.8 we find $C_n^{2,2}$ and $C_n^{3,2}$ in The OEIS. The sequence $\{C_n^{2,2} : n \geq 0\}$ is the binomial transformation of $1, 1, 2, 2, 2, 3, 3, \ldots$, it has a simple formula $C_n^{2,2} = (n+2)2^{n-3}$ for $n \geq 2$, and it enumerates a few families of objects. These objects include copies of r in all compositions of $n+r$ for any positive integer r, weak compositions of $n-1$ with exactly one zero, triangulations of a regular $(n+3)$-gon in which each triangle contains at least one side of the polygon, and more [10, A045623]. The sequence $(C_n^{3,2})_{n \geq 0}$ is the binomial transformation of $(\lfloor (\frac{1+\sqrt{5}}{2})^n \rfloor)_{n \geq 0}$ and satisfies the formula $C_n^{3,2} = \left(\frac{1+\sqrt{5}}{2}\right)^{2n-2} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n-2} - 2^{n-2}$ for $n \geq 2$ [10, A142586]. We do not see $C_n^{4,2}$ in The OEIS, but it also has a simple formula.

Proposition 2.9. *For $n \geq 3$ we have $C_n^{4,2} = 1 + 5 \cdot 3^{n-3} - 2^{n-3}$.*

3 k-associativity of left depth d

In this section we study $C_{k,n}^d := C_{*,n}$ and $\widetilde{C}_{k,n}^d := \widetilde{C}_{*,n}$ for any binary operation $*$ satisfying (1.3) with $e = \ell = 1$, i.e., when $t \sim_* t'$ if and only if $\delta(t) \sim_k^d \delta(t')$ for all $t, t' \in \mathcal{T}_n$.

We first study $(*, n)$-classes using rotations on binary trees. Given two binary trees s and t, we say *t contains s at left depth d* if t contains s as a (possibly non-maximal) subtree rooted at a node of left depth d and say *t avoids s at left depth d* otherwise.

Given binary trees s and t, write $s \wedge t$ for the binary tree whose root has left and right maximal subtrees s and t, respectively. Let v be a node of t. If the maximal subtree of t rooted at v can be written as $(t_0 \wedge \cdots \wedge t_k) \wedge t_{k+1}$, where $t_0, \ldots, t_{k+1}$ are binary trees, then replacing this subtree with $t_0 \wedge (t_1 \wedge \cdots \wedge t_{k+1})$ in t gives another binary tree t'. Here the operations $\wedge$ in parentheses are performed left-to-right. We call the operation $t \mapsto t'$ a *right k-rotation at v* (see Fig. 2), and call the inverse operation $t' \mapsto t$ a *left k-rotation at v*.

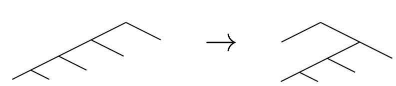

Figure 2: A right 3-rotation

Lemma 3.1. *A left or right k-rotation at a node v in a binary tree t produces another binary tree t' satisfying $t \sim_k^d t'$ if and only if the left depth of v in t is at least $d-1$.*

If $s \in \mathcal{T}_n$ can be obtained from $t \in \mathcal{T}_n$ by finitely many left k-rotations at nodes of left depths at least $d-1$ then we say $s \le_k^d t$. We call this partial order on $\mathcal{T}_n$ the *$\binom{d}{k}$-order*, which includes the k-associative order introduced in earlier work [5] as a special case ($d=1$). A binary tree minimal or maximal under the $\binom{d}{k}$-order is called *$\binom{d}{k}$-minimal* or *$\binom{d}{k}$-maximal*. For each $k \ge 1$ we define $\mathrm{comb}_k := t_0 \wedge \cdots \wedge t_k$ and $\mathrm{comb}_k^1 := t_0 \wedge \mathrm{comb}_k$ where $t_0 = \cdots = t_k$ is the unique tree in $\mathcal{T}_0$.

Proposition 3.2. *Let t be a binary tree. For $d, k \ge 1$ we have*
(i) t is $\binom{d}{k}$-minimal if and only if it avoids comb_k^1 at each left depth at least $d-1$, and
(ii) t is $\binom{d}{k}$-maximal if and only if it avoids comb_{k+1} at each left depth at least $d-1$.

Theorem 3.3. *The $(*, n)$-classes are precisely the connected components of the Hasse diagram of the $\binom{d}{k}$-order on $\mathcal{T}_n$. Every $(*, n)$-class has a unique $\binom{d}{k}$-minimal element.*

A subpath L' of a lattice path L is *at height* h if the initial point of L' has height h. We say L *avoids* L' *at height* h if L contains no subpath L' at height h.

Proposition 3.4. *For $k, d \ge 1$ and $n \ge 0$, the number $C_{k,n}^d$ enumerates*
(i) binary trees with $n+1$ leaves avoiding comb_k^1 at any left depth at least $d-1$, and
(ii) Dyck paths of length $2n$ avoiding DU^k at height at least d.

Remark 3.5. For any fixed n and k, the limit of $C_{k,n}^d$ as $d \to \infty$ is the Catalan number C_n since the constraints in Proposition 3.4 are redundant if d is large enough.

Let $M_{k,n}^d$ be the number of binary trees in $\mathcal{T}_n$ avoiding comb_{k+1} at any node of left depth at least $d-1$. This number counts $\binom{d}{k}$-maximal elements in $\mathcal{T}_n$ by Proposition 3.2, and is closely related to $C_{k,n}^d$. The number $M_{k,n} := M_{k,n}^1$ was called a *generalized Motzkin number* in our previous work [5] and also studied by Takács [13]. For $d, k \ge 1$ we define

$$C_k^d(x) := \sum_{n \ge 0} C_{k,n}^d x^{n+1} \quad \text{and} \quad M_{k-1}^d(x) := \sum_{n \ge 0} M_{k-1,n}^d x^{n+1}.$$

Proposition 3.6. *For $m, n, d \ge 0$ and $k \ge 1$ we have (with $C_k^0(x) := M_{k-1}^1(x)$)*

$$C_k^{d+1}(x) = x \Big/ \left(1 - C_k^d(x)\right) \quad \textit{and}$$

$$[x^{n+m}] C_k^{d+1}(x)^m = \sum_{0 \le i \le n} \binom{m+i-1}{i} [x^n] C_k^d(x)^i.$$

Proposition 3.7. *For $d, k \ge 1$ and $n \ge 0$ we have $M_{k-1}^d(x) = C_k^{d-1}(x)$, $M_{k-1,n}^d = C_{k,n}^{d-1}$, and $M_{k-1,n}^d \le C_{k,n}^d \le M_{k,n}^d$. Consequently, we have the following additional inequalities for $d, k \ge 1$:*

$$\cdots \le C_{k,n}^d \le C_{k,n}^{d+1} \le C_{k+1,n}^d \le C_{k+1,n}^{d+1} \le C_{k+2,n}^d \le C_{k+2,n}^{d+1} \le \cdots.$$

Proposition 3.8. *For $d \ge 1$ and $n \ge 0$ we have $M_1^d(x) = C_1^d(x)$ and $M_{1,n}^d = C_{1,n}^d$.*

3.1 The case $k \le 3$

We studied $C^d_{1,n} = C^d_n = C^{d,1}_n$ in Section 2.1. We now give their relationship to $C^d_{2,n}$.

Proposition 3.9. *For $d, n \ge 0$ we have $C^d_2(x) = C^{d+1}_1(x)$ and $C^d_{2,n} = C^{d+1}_{1,n}$.*

Next, we study $C^d_{3,n}$. It follows from our earlier work [5] that $C^0_{3,n}$ is the *Motzkin number* [10, A001006], which has many closed formulas, and its generating function is

$$C^0_3(x) = \frac{1 - x - \sqrt{1-2x-3x^2}}{2x} = \cfrac{x}{1 - x - \cfrac{x^2}{1 - x - \cfrac{x^2}{\cdots}}}.$$

Applying Proposition 3.6 to this gives a continued fraction

$$C^d_3(x) = \cfrac{x}{1 - \cfrac{x}{1 - \cdots \cfrac{x}{1 - C^0_3(x)}}} \tag{3.1}$$

where the number of ones is d. Equation (3.1) is a special case of the generating function studied by Flajolet [4] for *labeled positive paths*. Such a path L starts at $(0,0)$ and stays weakly above the line $y = 0$, with three kinds of steps $U = (1,1)$, $D = (1,-1)$, and $H = (1,0)$. Each step is labeled with some weight, and the *total weight* of L is the sum of all weights of the steps. The *height* of L is the largest y-coordinate of a point on L.

By Equation (3.1) or Remark 3.5, the number $C^d_{3,n}$ interpolates between the Motzkin number $C^0_{3,n}$ and the Catalan number $C_n = \lim_{d\to\infty} C^d_{3,n}$. For $d = 1, \ldots, 5$, the sequences $\{C^d_{3,n}\}$ are recorded in The OEIS [10, A005773, A054391–A054394]. For an arbitrary d, Barcucci, Del Lungo, Pergola, and Pinzani [2] studied $C^d_{3,n}$ in terms of permutations avoiding certain *barred patterns* and provided a closed formula [2, p. 47] for the generating function $C^d_3(x)$, but no formula for the number $C^d_{3,n}$.

Proposition 3.10. *For $d, n \ge 0$ the number $C^d_{3,n}$ enumerates*
(i) labeled positive paths with no H-step strictly below $y = d$ and with total weight n, where each U-step or D-step weakly below $y = d$ has a weight $1/2$ and each other step has a weight 1, and
(ii) permutations of $1, 2, \ldots, n$ avoiding 321 and $(d+3)\bar{1}(d+4)23\cdots(d+2)$ (barred pattern).

We give a new closed formula for $C^d_3(x)$ and derive a closed formula for $C^d_{3,n}$ from that. We have not found the sequences $\{C^d_{k,n}\}$ for $k \ge 4$ (and $d \ge 2$) in The OEIS.

Theorem 3.11. *For $d \ge 0$ we have*

$$C^d_3(x) = \frac{2xF_{d+1}(x)F_{d+2}(x) - x^d - x^{d+1} + x^d\sqrt{1-2x-3x^2}}{2(F_{d+2}(x)^2 - x^d - x^{d+1})}.$$

Theorem 3.12. *For $d \geq 1$ and $n \geq 0$ we have*

$$C_{3,n}^{d} = \sum_{\substack{\alpha \models n+1 \\ h>1 \Rightarrow \alpha_h \leq d+1}} -\left(C_{3,\alpha_1-d-2}^{0} + \frac{\delta_{\alpha_1,d}}{2} + (-1)^{\alpha_1} \sum_{i+j=\alpha_1-1} \binom{d-i}{i} \binom{d+1-j}{j} \right)$$
$$\cdot \prod_{h\geq 2} \left(\left(\delta_{\alpha_h,d} + (-1)^{\alpha_h - 1} \sum_{i+j=\alpha_h} \binom{d+1-i}{i} \binom{d+1-j}{j} \right) \right)$$

where

$$C_{3,m}^{0} := \begin{cases} 1/2, & m=-1, \\ -1/2, & m=-2, \\ 0, & m \leq -3, \end{cases} \quad \text{and} \quad \delta_{m,d} := \begin{cases} 1, & m \in \{d, d+1\}, \\ 0, & \text{otherwise.} \end{cases}$$

3.2 The case $d \leq 2$

As the second formula in Proposition 3.6 gives a way to obtain the number $C_{k,n}^{d+1}$ from $C_k^d(x)^m$, we study $[x^{n+m}]C_k^d(x)^m$ for fixed d. We first generalize the closed formulas [5, (9) and (11)] for $C_k^0(x) = M_{k-1}(x)$ and $C_k^1(x) = C_k(x)$. Recall that the *monomial symmetric function* $m_\lambda(x_1, \ldots, x_n)$ indexed by a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is the sum of $x_1^{a_1} \cdots x_n^{a_n}$ for all rearrangement $(a_1, \ldots, a_n)$ of the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$.

Proposition 3.13. *For $k, m \geq 1$ and $n \geq 0$, the number of plane forests with m components and $n+m$ nodes, each of degree less than k, is*

$$[x^{n+m}]M_{k-1}(x)^m = \frac{m}{n+m} \sum_{0 \leq j \leq n/k} (-1)^j \binom{n+m}{j} \binom{2n+m-jk-1}{n+m-1}$$
$$= \frac{m}{n+m} \sum_{\substack{\lambda \subseteq (k-1)^{n+m} \\ |\lambda| = n}} m_\lambda(1^{n+m}).$$

Remark 3.14. We have $([x^{n+m}]M_0(x)^m)_{n \geq 0} = (1, 0, 0, \ldots)$ for any fixed $m \geq 0$ and also $[x^{n+m}]M_1(x)^m = \binom{m+n-1}{n}$ for all $m, n \geq 0$. For $k = 3$ the array $([x^{n+m}]M_{k-1}(x)^m)_{m,n \geq 0}$ gives the diagonals of the Motzkin triangle [10, A026300]; see [10, A002026, A005322–A005325] for $m = 2, \ldots, 6$. We find no result in The OEIS when $k \geq 4$ and $m \geq 2$.

Proposition 3.15. *For $k, m, n \geq 1$, the number of plane forests with m components and n non-root nodes, each of degree less than k, is*

$$[x^{n+m}]C_k(x)^m = \frac{m}{n} \sum_{0 \leq j \leq (n-1)/k} (-1)^j \binom{n}{j} \binom{2n+m-jk-1}{n+m}$$
$$= \sum_{\lambda \subseteq (k-1)^n} \frac{n-|\lambda|}{n} \binom{m+n-|\lambda|-1}{n-|\lambda|} m_\lambda(1^n).$$

A *weak composition* of n with m parts is a sequence of m nonnegative integers whose sum is n. For $k = 1, 2$, Proposition 3.15 is related to weak compositions.

Corollary 3.16. *For $m, n \geq 0$, weak compositions of n with m parts are enumerated by*

$$[x^{n+m}]C_1(x)^m = \binom{m+n-1}{n}$$

and weak compositions of n with $m-1$ zero parts are enumerated by

$$[x^{n+m}]C_2(x)^m = \sum_{0\leq i\leq m} \binom{m}{i}\binom{n-1}{n-i}2^{n-i} = \sum_{0\leq i\leq n}\binom{m+i-1}{i}\binom{n-1}{n-i}.$$

The case $k = 1$ of Corollary 3.16 is well known and the case $k = 2$ has been studied by Janjić and Petković [6] with a different approach from ours. For $k = 2$ and $1 \leq m \leq 10$ see the sequences [10, A011782, A045623, A058396, A062109, A169792–A169797]. We find no result for $k \geq 3$ and $m \geq 2$ except the special case $(k, m) = (3, 2)$ [10, A036908].

Now we study the case $d = 2$.

Proposition 3.17. *Let $m, n \geq 0$ and $k \geq 1$. Then*

$$\begin{aligned}
[x^{n+m}]C_k^2(x)^m &= \binom{m+n-1}{n} + \sum_{i=1}^{n-1}\binom{m+i-1}{i}\frac{i}{n-i}\sum_{0\leq j\leq\frac{n-i-1}{k}}(-1)^j\binom{n-i}{j}\binom{2n-i-jk-1}{n}\\
&= \binom{m+n-1}{n} + \sum_{i=1}^{n-1}\binom{m+i-1}{i}\sum_{\lambda\subseteq(k-1)^{n-i}}\frac{n-i-|\lambda|}{n-i}\binom{n-|\lambda|-1}{n-|\lambda|-i}m_\lambda(1^{n-i}).
\end{aligned}$$

In particular,

$$\begin{aligned}
C_{k,n}^2(x) &= 1 + \sum_{i=1}^{n-1}\frac{i}{n-i}\sum_{0\leq j\leq(n-i-1)/k}(-1)^j\binom{n-i}{j}\binom{2n-i-jk-1}{n}\\
&= 1 + \sum_{i=1}^{n-1}\sum_{\lambda\subseteq(k-1)^{n-i}}\frac{n-i-|\lambda|}{n-i}\binom{n-|\lambda|-1}{n-|\lambda|-i}m_\lambda(1^{n-i}).
\end{aligned}$$

Proposition 3.18. *Let $m, n \geq 0$. Then*

$$[x^{n+m}]C_2^2(x)^m = \sum_{0\leq i\leq n}\binom{m+i-1}{i}\sum_{0\leq j\leq n-i}\binom{i+j-1}{j}\binom{n-i-1}{n-i-j}.$$

In particular,

$$C_{2,n}^2 = \sum_{0\leq j\leq n}\binom{n+j-1}{2j} = F_{2n-1}.$$

Remark 3.19. Let $C(x) := \sum_{n\geq 0} C_n x^{n+1}$ be the generating function of the Catalan numbers. The limit of $[x^{n+m}]C_k^d(x)^m$ as $k\to\infty$ or $d\to\infty$ is below, which gives the diagonals of Catalan's triangle [10, A009766]:

$$[x^{n+m}]C(x)^m = \frac{m}{n+m}\binom{2n+m-1}{n}.$$

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References

[1] G.E. Andrews, C. Krattenthaler, L. Orsina, and P. Papi. "ad-nilpotent $\mathfrak{b}$-ideals in $\mathrm{sl}(n)$ having a fixed class of nilpotence: combinatorics and enumeration". *Trans. Amer. Math. Soc.* **354**.10 (2002), pp. 3835–3853. DOI: 10.1090/S0002-9947-02-03064-7.

[2] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. "From Motzkin to Catalan permutations". *Discrete Math.* **217**.1-3 (2000). Formal power series and algebraic combinatorics (Vienna, 1997), pp. 33–49. DOI: 10.1016/S0012-365X(99)00254-X.

[3] N.G. de Bruijn, D.E. Knuth, and S.O. Rice. "The average height of planted plane trees" (1972). *Graph theory and computing*, pp. 15–22.

[4] P. Flajolet. "Combinatorial aspects of continued fractions". *Discrete Math.* **32**.2 (1980), pp. 125–161. DOI: 10.1016/0012-365X(80)90050-3.

[5] N. Hein and J. Huang. "Modular Catalan numbers". *European J. Combin.* **61** (2017), pp. 197–218. DOI: 10.1016/j.ejc.2016.11.004.

[6] M. Janjić and B. Petković. "A counting function generalizing binomial coefficients and some other classes of integers". *J. Integer Seq.* **17**.3 (2014), Article 14.3.5, 23 pp. URL.

[7] S. Kitaev, J. Remmel, and M. Tiefenbruck. "Quadrant marked mesh patterns in 132-avoiding permutations". *Pure Math. Appl. (PU.M.A.)* **23**.3 (2012), pp. 219–256.

[8] G. Kreweras. "Sur les éventails de segments". *Cahiers du Bureau universitaire de recherche opérationnelle Série Recherche* **15** (1970), pp. 3–41. URL.

[9] L.W. Shapiro. "Upper triangular rings, ideals, and Catalan numbers". *Amer. Math. Monthly* **82** (1975), pp. 634–637. DOI: 10.2307/2319698.

[10] N.J.A. Sloane. *The On-Line Encyclopedia of Integer Sequences*. https://oeis.org.

[11] R.P. Stanley. *Catalan numbers*. Cambridge University Press, 2015, pp. viii+215. URL.

[12] B. Steinberg. *Representation theory of finite monoids*. Universitext. Springer, Cham, 2016, pp. xxiv+317. DOI: 10.1007/978-3-319-43932-7.

[13] L. Takács. "Enumeration of rooted trees and forests". *Math. Sci.* **18**.1 (1993), pp. 1–10.