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Non-kissing complexes for gentle algebras

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Abstract. We introduce the non-kissing complex of any gentle bound quiver. This complex provides a powerful combinatorial model for support τ-tilting theory over gentle algebras, and it generalizes and unifies the previously considered situations of quivers defined from subsets of the grid or from dissections of a polygon (both generalizing the classical associahedron). In this extended abstract, we report on lattice theoretic and geometric properties of finite non-kissing complexes: we show that their flip graphs are Hasse diagrams of congruence-uniform lattices, and that they can be realized by convex polytopes.

Résumé. Nous introduisons le complexe platonique d'un carquois aimable. Ce complexe offre un modèle combinatoire pour la théorie du τ-basculement à support des algèbres aimables et il généralise et unifie les cas particuliers définis à partir de sous-ensembles de la grille ou de dissections de polygones (contenant notamment le cas de l'associaèdre classique). Dans ce résumé étendu, nous présentons des propriétés combinatoires et géométriques des complexes platoniques finis: nous montrons que leurs graphes de flips sont les diagrammes de Hasse de treillis congruence-uniformes, et qu'ils peuvent être réalisés par des polytopes convexes.

1 Motivation: Non-kissing versus support τ-tilting

The non-kissing complex is a simplicial complex of pairwise non-kissing paths in a fixed shape of a grid. It was introduced by T. K. Petersen, P. Pylyavskyy and D. Speyer in [14] for a staircase shape, studied by F. Santos, C. Stump and V. Welker [17] for rectangular shapes, and extended by T. McConville in [11] for arbitrary shapes. This complex is known to be a simplicial sphere, and it was actually realized as a polytope using successive edge stellations and suspensions in [11, Section 4]. Moreover, the dual graph of the non-kissing complex has a natural orientation which equips its facets with a lattice structure [11, Theorem 1.1, Sections 5–8]. Further lattice theoretic and geometric aspects of this complex were recently developed by A. Garver and T. McConville in [7].

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The interest for non-kissing complexes is motivated by relevant instances arising from particular shapes. As already observed in [11, Section 10], when the shape is a ribbon, the non-kissing complex is an associahedron, and the non-kissing lattice is a type A Cambrian lattice of N. Reading [15]. In particular, the straight ribbon corresponds to the Tamari lattice, an object at the heart of a deep research area [12]. When the shape is a rectangle (or even a staircase), the non-kissing complex was studied in [14, 17] as the Grassmann associahedron, in connection to non-crossing subsets of $[n]$.

Other instances of such complexes arise naturally from the representation theory of associative algebras. The notion of support τ-tilting module over an algebra was introduced by T. Adachi, O. Iyama and I. Reiten in [1], and has proved to be a successful generalization of tilting and cluster-tilting theory. Over a given algebra, indecomposable τ-rigid modules form a complex. For an account of the various algebraic interpretations of this complex, we refer the reader to [3]. For example, in the case of the path algebra of a straight line quiver, the support τ-tilting complex is, again, an associahedron.

Our original motivation was to provide common interpretations to these different complexes. First, we realized any non-kissing complex as the support τ-tilting complex of a well-chosen associative algebra. The algebras that occur are certain gentle algebras, a special case of the well-studied string algebras of M. C. R. Butler and C. Ringel [4]. Conversely, starting from any gentle bound quiver $\bar{Q}$, we defined its blossoming quiver $\bar{Q}^{❀}$ and a non-kissing relation on the walks in $\bar{Q}^{❀}$ so that the following interpretation holds.

Theorem 1.1. *For any gentle bound quiver $\bar{Q} = (Q, I)$, the non-kissing complex of walks in the blossoming quiver $\bar{Q}^{❀}$ is isomorphic to the support τ-tilting complex of the gentle algebra kQ/I.*

In short, to any walk in $\bar{Q}^{❀}$ corresponds a representation of $\bar{Q}$, and this correspondence takes non-kissing walks to τ-compatible representations. This theorem provides a dictionary between the combinatorially-flavored non-kissing complex and the algebraically-flavoured support τ-tilting complex, thus opening a bridge to go back and forth between the two worlds. It allows us, for instance, to combinatorially define mutation of support τ-tilting modules. This seems worthwhile, as the mutation of support τ-tilting modules over an arbitrary algebra is generally difficult to carry out explicitly.

The precise statement and the proof of Theorem 1.1 can be found in the long version of this paper [13], as well as further representation-theoretic aspects of the project. In this extended abstract, we focus on combinatorial and geometric aspects. We first define in Section 2 the non-kissing complex of a gentle bound quiver and show that this complex is a pseudomanifold (meaning in particular that there is a well-defined notion of flips in non-kissing facets). In Section 3, we show that the graph of increasing flips is the Hasse diagram of a congruence-uniform lattice and describe its canonical join complex. Finally, Section 4 is devoted to the geometry of finite non-kissing complexes: we construct their **g**-vector fans and show that these fans are normal fans of convex polytopes. We refer to [13] for detailed proofs and further properties of non-kissing complexes.

2 Non-kissing complexes of gentle bound quivers

2.1 Blossoming quivers and non-kissing walks

We fix a *gentle bound quiver* $\bar{Q} := (Q, I)$, where $Q := (Q_0, Q_1, s, t)$ is a *quiver* with vertices Q_0, arrows Q_1, and source and target maps $s, t : Q_1 \to Q_0$, and I is a set of quadratic *relations* $\alpha\beta = 0$ with $\alpha, \beta \in Q_1$ and $t(\alpha) = s(\beta)$ such that for any $\beta \in Q_1$ there is at most one $\alpha \in Q_1$ such that $t(\alpha) = s(\beta)$ and $\alpha\beta \in I$ (resp. $\alpha\beta \notin I$) and at most one $\gamma \in Q_1$ such that $t(\beta) = s(\gamma)$ and $\beta\gamma \in I$ (resp. $\beta\gamma \notin I$). See Figure 1. In all pictures, a relation $\alpha\beta = 0$ is indicated with an arc connecting the target of α to the source of β.

A *string* in $\bar{Q}$ is a word of the form $\rho = \alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\cdots\alpha_\ell^{\varepsilon_\ell}$, where

- $\alpha_i \in Q_1$ and $\varepsilon_i \in \{-1, 1\}$ for all $i \in [\ell]$,
- $t(\alpha_i^{\varepsilon_i}) = s(\alpha_{i+1}^{\varepsilon_{i+1}})$ for all $i \in [\ell - 1]$, (by convention $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$),
- there is no $\alpha\beta \in I$ such that $\alpha\beta$ or $\beta^{-1}\alpha^{-1}$ appears as a factor of ρ, and
- ρ is reduced, in the sense that no factor $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ appears in ρ, for $\alpha \in Q_1$.

There is also an empty string ε_v for each vertex $v \in Q_0$. Strings are considered undirected, meaning that we implicitly identify ρ with ρ^{-1}. Let $\mathcal{S}(\bar{Q})$ be the set of strings of $\bar{Q}$.

The *blossoming quiver* of $\bar{Q} = (Q, I)$ is the gentle bound quiver $\bar{Q}^{❀} = (Q^{❀}, I^{❀})$ obtained by adding arrows and relations at each vertex $v \in Q_0$, so that v has precisely 2 incoming and 2 outgoing arrows and still fulfills the gentle conditions. See Figure 1.

A *walk* in $\bar{Q}$ is a maximal string in $\bar{Q}^{❀}$, *i.e.* connecting two vertices of $Q_0^{❀} \smallsetminus Q_0$. A walk ω is *bending* if it has two opposite arrows and *straight* otherwise. For $v \in Q_0$, the *peak walk* v_{peak} (resp. the *deep walk* v_{deep}) is the walk with two outgoing (resp. incoming) arrows at vertex v and one incoming and one outgoing arrow at all its other vertices. A *substring* of $\omega = \omega_1^{\varepsilon_1} \ldots \omega_\ell^{\varepsilon_\ell}$ is a factor $\sigma = \omega_{m+1}^{\varepsilon_{m+1}} \ldots \omega_{n-1}^{\varepsilon_{n-1}}$ with $1 \le m < n \le \ell$. We say that σ is a *top* (resp. *bottom*) substring of ω if $\varepsilon_m = -1 = -\varepsilon_n$ (resp. $\varepsilon_m = 1 = -\varepsilon_n$), meaning that ω has two outgoing (resp. incoming) arrows at the endpoints of σ. Let $\Sigma_{\text{bot}}(\omega)$ and $\Sigma_{\text{top}}(\omega)$ be the sets of bottom and top substrings of ω respectively.

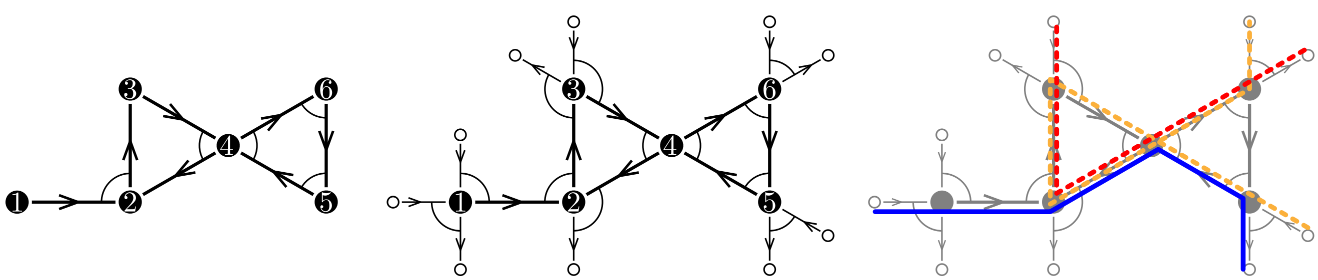


Figure 1: A gentle bound quiver $\bar{Q}$ (left), its blossoming quiver $\bar{Q}^{❀}$ (middle), and some walks in $\bar{Q}^{❀}$ (right). The dotted red and orange walks are non-kissing, but both are kissing the plain blue walk. See Figure 4 for examples of non-kissing facets.

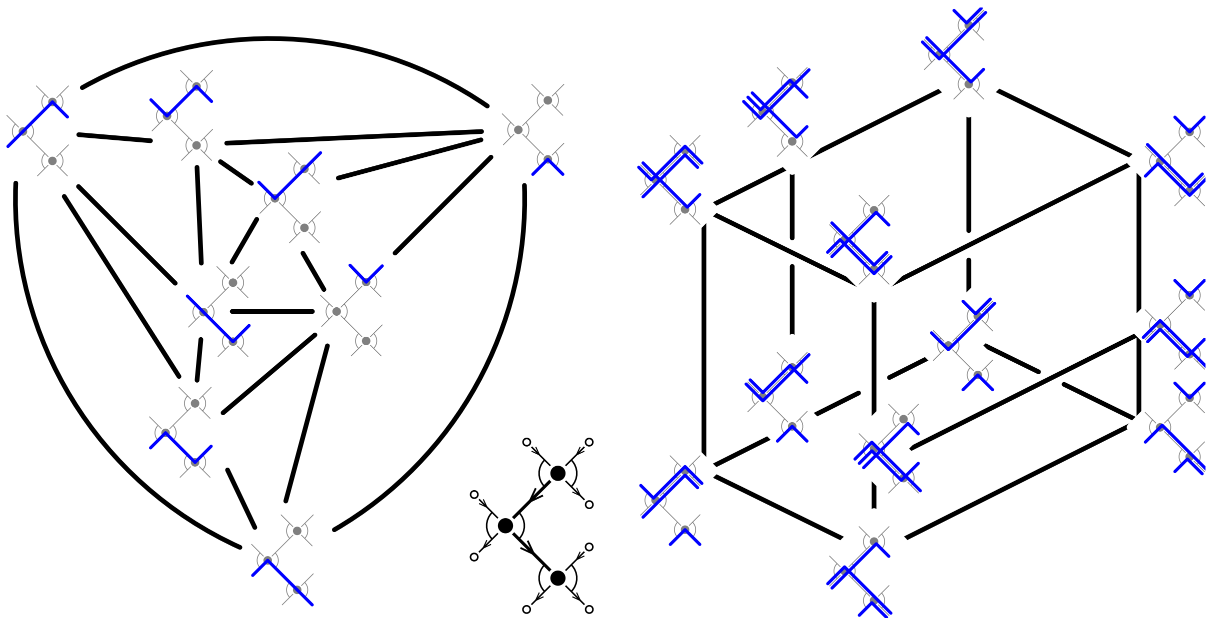


Figure 2: A reduced non-kissing complex (left) and its flip graph (right).

Consider two walks ω, ω' on $\bar{Q}$. We say that ω *kisses* ω' if $\Sigma_{\text{top}}(\omega) \cap \Sigma_{\text{bot}}(\omega') \neq \varnothing$, *i.e.* if there exists a common substring σ of ω and ω' such that ω has two outgoing arrows incident to σ while ω' has two incoming arrows incident to σ. See Figures 1 (right) & 3 (left). We authorize the case where σ is reduced to a vertex v, *i.e.* v is a peak of ω and a deep of ω'. Note that ω can kiss ω' several times, that ω and ω' can mutually kiss, and that ω can kiss itself. The *non-kissing complex* of $\bar{Q}$ is the simplicial complex $\mathcal{K}_{\text{nk}}(\bar{Q})$ whose faces are the collections of walks which are not self-kissing and pairwise non-kissing. Note that no straight walk can kiss another walk by definition, so that they appear in all facets of $\mathcal{K}_{\text{nk}}(\bar{Q})$. We thus consider the *reduced non-kissing complex* $\mathcal{C}_{\text{nk}}(\bar{Q})$ to be the deletion of all straight walks from $\mathcal{K}_{\text{nk}}(\bar{Q})$. See Figure 2 (left).

Our definition of non-kissing complex is largely inspired from and specializes to simplicial complexes defined from subsets of the grid [14, 17, 11, 7] or from dissections of polygons [2, 5, 8, 10]. To the best of our knowledge, we actually provide the first connection between these two families, besides the observation that associahedra are instances of both. In fact, the example of Figure 2 is also a special case of both families.

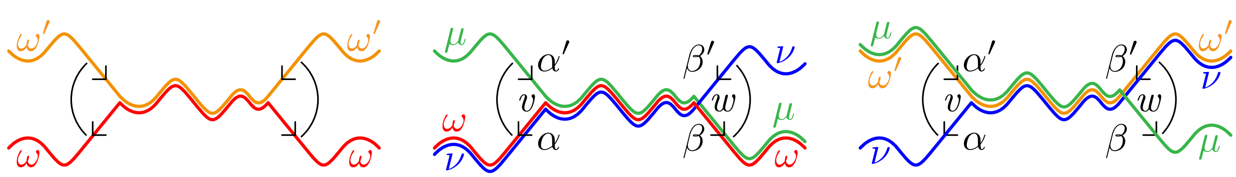


Figure 3: Two kissing walks (left) and a flip (right).

2.2 Distinguished arrows and flips

We now show that the non-kissing complex $\mathcal{C}_{\mathrm{nk}}(\bar{Q})$ is a *pseudomanifold*, *i.e.* that it is *pure* (all facets have the same dimension) and *thin* (there is a well-defined notion of flips).

A *marked walk* $\omega_\star$ is a walk $\omega = \alpha_1^{\varepsilon_1} \cdots \alpha_\ell^{\varepsilon_\ell}$ with a marked arrow $\alpha_i^{\varepsilon_i}$. Consider two distinct non-kissing walks $\mu_\star, \nu_\star$ marked at an arrow $\alpha^\varepsilon \in Q_1^{❀}$. Let σ denote their maximal common substring containing that occurrence of α. Since $\mu_\star \neq \nu_\star$, their common substring σ is strict, so that $\mu_\star$ and $\nu_\star$ split at least at one endpoint of σ. We define the *countercurrent order at* α by $\mu_\star \prec_\alpha \nu_\star$ when $\mu_\star$ enters and/or exits σ in the direction of α, while $\nu_\star$ enters and/or exits σ in the opposite direction. For a face F of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$, we call *distinguished walk* of F at an arrow α the $\prec_\alpha$-maximal walk $\mathsf{dw}(\alpha, F)$, and we call *distinguished arrows* of a walk $\omega \in F$ the arrows $\mathsf{da}(\omega, F) := \{\alpha \in \omega \mid \mathsf{dw}(\alpha, F) = \omega\}$. The following statement is inspired from [11, Theorem 3.2] and illustrated in Figure 4 (left).

Proposition 2.1. *Each bending (resp. straight) walk of a facet $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$ contains precisely* 2 *(resp.* 1*) distinguished arrows pointing in opposite directions (resp. in the direction of the walk).*

Corollary 2.2. *The reduced non-kissing complex $\mathcal{C}_{\mathrm{nk}}(\bar{Q})$ is pure of dimension $|Q_0|$.*

Define the *distinguished string* of a bending walk ω in a facet $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$ as the substring $\mathsf{ds}(\omega, F)$ of ω located between the two distinguished arrows of ω.

Proposition 2.3. *Consider a facet $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$ and a bending walk $\omega \in F$. Write $\omega = \rho\sigma\tau$ where $\sigma := \mathsf{ds}(\omega, F)$. Let $\{\alpha, \beta\} := \mathsf{da}(\omega, F)$, and α' and β' be the other two arrows of $Q_1^{❀}$ incident to the endpoints of σ and such that $\alpha'\alpha \in I$ or $\alpha\alpha' \in I$, and $\beta'\beta \in I$ or $\beta\beta' \in I$. Let $\mu := \mathsf{dw}(\alpha', F \smallsetminus \{\omega\})$ and $\nu := \mathsf{dw}(\beta', F \smallsetminus \{\omega\})$. See Figures 3 (right) & 4. Then*

(i) The string σ splits the walk μ into $\mu = \rho'\sigma\tau$ and the walk ν into $\nu = \rho\sigma\tau'$.

(ii) The walk $\omega' := \rho'\sigma\tau'$ is kissing ω but no other walk of F. Moreover, ω' is the only other walk besides ω which is not kissing any other walk of $F \smallsetminus \{\omega\}$.

We say that $F \triangle \{\omega, \omega'\}$ is obtained from F by flipping ω, and that the flip is supported by σ.

Corollary 2.4. *The reduced non-kissing complex $\mathcal{C}_{\mathrm{nk}}(\bar{Q})$ is a pseudomanifold without boundary.*

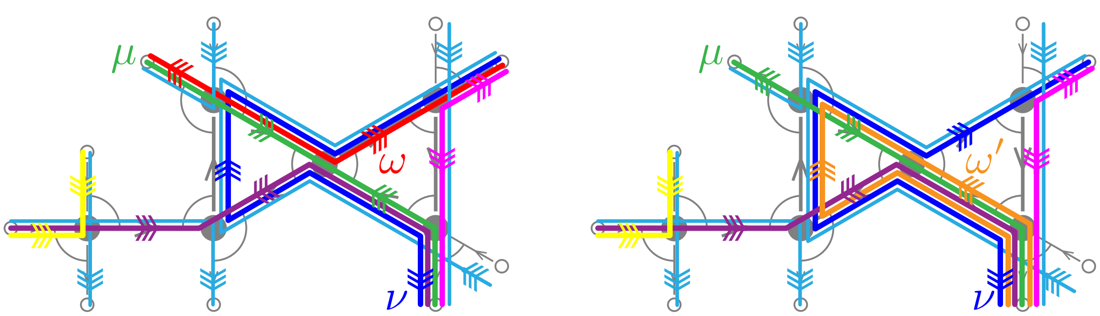


Figure 4: Flipping the red walk ω to the orange walk ω'. The walks μ, ν involved in the flip are the blue and green walks. Distinguished arrows are marked with triple arrows.

3 Non-kissing lattices

The flip of Proposition 2.3 and Figures 3 & 4 exchanges two kissing walks ω, ω'. The flip is *increasing* when their common substring is on top of ω and on the bottom of ω'. This yields the *increasing flip graph*, where vertices are non-kissing facets and arcs are increasing flips. See Figure 2 (right). The main result of this section is the following statement.

Theorem 3.1. *If $\mathcal{C}_{\mathrm{nk}}(\bar{Q})$ is finite, the increasing flip graph is the Hasse diagram of a congruence-uniform lattice, that we call non-kissing lattice and denote by $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$.*

Congruence-uniform lattices will be properly defined in Section 3.4. To achieve Theorem 3.1, we use a technique developed by T. McConville for grid quivers [11]: we identify the non-kissing lattice with a quotient of a lattice of biclosed sets of strings.

3.1 Biclosed sets of strings

A *closure operator* on a finite set $\mathcal{S}$ is a map $S \mapsto S^{\mathrm{cl}}$ on subsets of $\mathcal{S}$ such that $\varnothing^{\mathrm{cl}} = \varnothing$, $S \subseteq S^{\mathrm{cl}}$, $(S^{\mathrm{cl}})^{\mathrm{cl}} = S^{\mathrm{cl}}$ and $S \subseteq T \implies S^{\mathrm{cl}} \subseteq T^{\mathrm{cl}}$ for any $S, T \subseteq \mathcal{S}$. A subset $S \subseteq \mathcal{S}$ is *closed* if $S^{\mathrm{cl}} = S$, *coclosed* if $\mathcal{S} \smallsetminus S$ is closed, and *biclosed* if it is both closed and coclosed. Let $\mathsf{Bic}(\mathcal{S})$ be the inclusion poset of biclosed subsets of $\mathcal{S}$. In [11, Theorem 5.5], T. McConville gave simple sufficient conditions for $\mathsf{Bic}(\mathcal{S})$ to be a congruence uniform lattice. In [13, Theorem 3.21], we extended this criterion in the situation when the singletons of $\mathcal{S}$ are not biclosed so that we can apply it in our context of non-kissing complexes.

In a gentle bound quiver $\bar{Q}$, we define the *closure* S^{cl} of a set S of strings of $\bar{Q}$ as the set of all strings of the form $\sigma_1 \alpha_1^{\varepsilon_1} \sigma_2 \alpha_2^{\varepsilon_2} \dots \alpha_{\ell-1}^{\varepsilon_{\ell-1}} \sigma_\ell$ where $\sigma_i \in S, \alpha_i \in Q_1$ and $\varepsilon_i \in \{-1, 1\}$. Let $\mathsf{Bic}(\bar{Q})$ be the inclusion poset on biclosed sets of strings of $\bar{Q}$. For example, when $\bar{Q}$ is a path on n vertices with no relation, the strings are in bijection with pairs $(i,j) \in \binom{n+1}{2}$, the closure on strings translates to the concatenation $(i,j) \circ (k,\ell) = \delta_{j=k}(i,\ell)$, biclosed sets of strings are in bijection with inversion sets of permutations of $[n+1]$, so that $\mathsf{Bic}(\bar{Q})$ is isomorphic to the weak order on $\mathfrak{S}_{n+1}$. Figure 5 (left) illustrates the poset $\mathsf{Bic}(\bar{Q})$ for another gentle bound quiver, with the empty set in the bottom and the set of all strings of $\bar{Q}$ on top. The criterion of [13, Theorem 3.21] yields the following result.

Theorem 3.2. *When $\bar{Q}$ has finitely many strings, the inclusion poset of biclosed sets $\mathsf{Bic}(\bar{Q})$ is a congruence-uniform lattice.*

3.2 Lattice congruence

A *lattice congruence* of a lattice $(L, \leq, \wedge, \vee)$ is an equivalence relation $\equiv$ on L compatible with meets and joins: $x \equiv x'$ and $y \equiv y'$ implies $x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$. It defines a *lattice quotient* $L/\!\equiv$ on the congruence classes of $\equiv$ where the order relation is given by $X \leq Y$ if and only if there exists $x \in X$ and $y \in Y$ such that $x \leq y$ and the

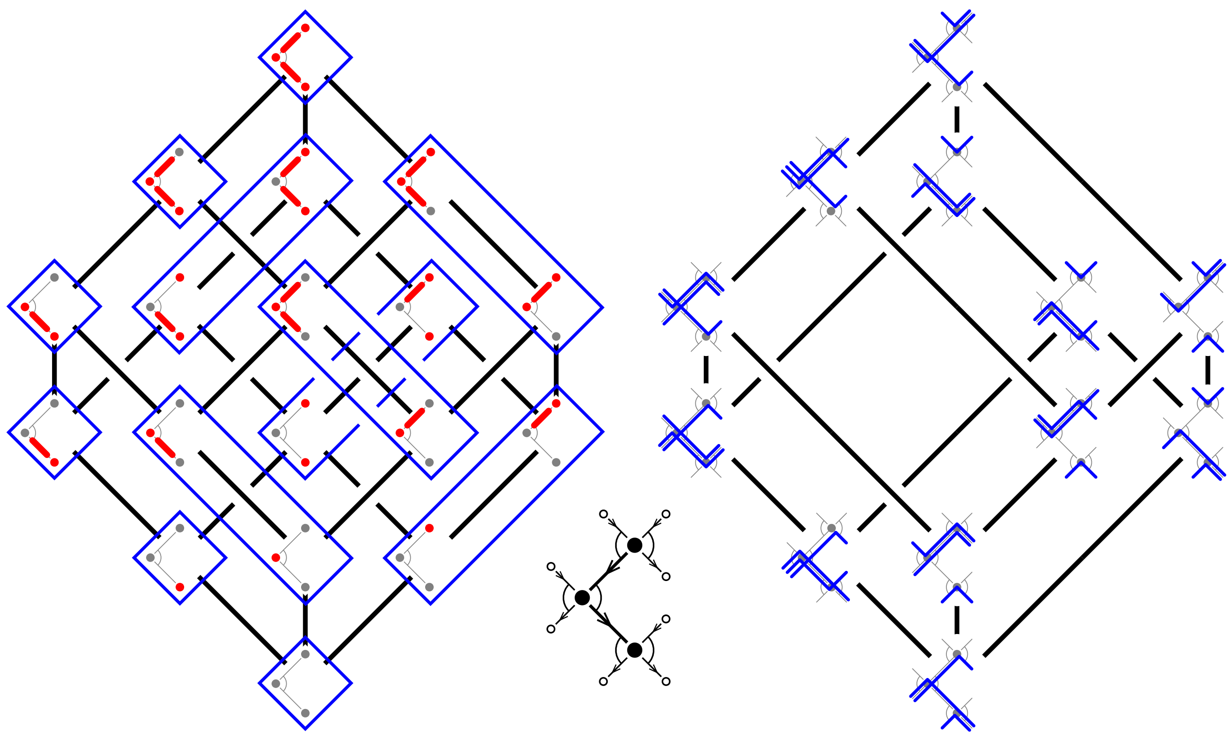


Figure 5: The inclusion lattice of biclosed sets $\mathsf{Bic}(\bar{Q})$ with congruence classes of $\equiv$ in blue (left), and the corresponding lattice of increasing flips on facets of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ (right).

meet $X \wedge Y$ (resp. the join $X \vee Y$) of two congruence classes X and Y is the congruence class of $x \wedge y$ (resp. of $x \vee y$) for arbitrary representatives $x \in X$ and $y \in Y$. For a finite lattice L, an equivalence relation $\equiv$ on L is a lattice congruence if and only if its congruence classes are intervals and the maps $\pi_\downarrow$ and $\pi^\uparrow$, sending an element $x \in L$ to the minimum and maximum of its congruence class respectively, are order preserving.

Following [11, Section 7], we associate to a biclosed set $S \in \mathsf{Bic}(\bar{Q})$ the sets

$$\pi_\downarrow(S) := \left\{ \sigma \in \mathcal{S}(\bar{Q}) \;\middle|\; \Sigma_{\mathrm{bot}}(\sigma) \subseteq S \right\} \quad \text{and} \quad \pi^\uparrow(S) := \left\{ \sigma \in \mathcal{S}(\bar{Q}) \;\middle|\; \Sigma_{\mathrm{top}}(\sigma) \cap S \neq \varnothing \right\}.$$

Here and throughout the paper, we denote by $\Sigma_{\mathrm{bot}}(\sigma)$ the set of bottom substrings for a string $\sigma = \alpha_1^{\varepsilon_1} \dots \alpha_\ell^{\varepsilon_\ell}$, *i.e.* the substrings $\alpha_m^{\varepsilon_m} \dots \alpha_n^{\varepsilon_n}$ for $1 \le m \le n \le \ell$ such that $m = 1$ or $\varepsilon_{m-1} = -1$, and $n = \ell$ or $\varepsilon_{n+1} = 1$ (and similarly for the set of top substrings $\Sigma_{\mathrm{top}}(\sigma)$).

Proposition 3.3. *For any $S \in \mathsf{Bic}(\bar{Q})$, the sets $\pi_\downarrow(S)$ and $\pi^\uparrow(S)$ are biclosed. Moreover,*
(i) $\pi_\downarrow(S) \subseteq S \subseteq \pi^\uparrow(S)$ for any element $S \in \mathsf{Bic}(\bar{Q})$,
(ii) $\pi_\downarrow \circ \pi_\downarrow = \pi_\downarrow \circ \pi^\uparrow = \pi_\downarrow$ and $\pi^\uparrow \circ \pi^\uparrow = \pi^\uparrow \circ \pi_\downarrow = \pi^\uparrow$,
(iii) $\pi_\downarrow$ and $\pi^\uparrow$ are order preserving.
Therefore, the fibers of $\pi^\uparrow$ and $\pi_\downarrow$ coincide and are the classes of a lattice congruence $\equiv$ on $\mathsf{Bic}(\bar{Q})$.

For example, if $\bar{Q}$ is an oriented path with no relation, $\equiv$ is a Cambrian congruence of the weak order [15]. The congruence classes of $\equiv$ appear as blue rectangles in Figure 5.

3.3 Non-kissing lattice

Coming back to our original problem, we now aim to show that the increasing flip graph on non-kissing facets is isomorphic to the Hasse diagram of the quotient of the lattice of biclosed set $\mathsf{Bic}(\bar{Q})$ of Section 3.1 by the lattice congruence of Section 3.2. The next two propositions provide explicit maps between biclosed sets of strings and non-kissing facets illustrated in Figure 6. It extends previous definitions of [11] for grid quivers.

Proposition 3.4. *For $S \in \mathsf{Bic}(\bar{Q})$ and $\alpha \in Q_1^{\circledast}$, let $\omega(\alpha, S) := \alpha_{-\ell}^{\varepsilon_{-\ell}} \cdots \alpha_{-1}^{\varepsilon_{-1}} \cdot \alpha \cdot \alpha_1^{\varepsilon_1} \cdots \alpha_r^{\varepsilon_r}$ be the directed walk containing α defined by*

- *$\varepsilon_i = -1$ if the string $\alpha_1^{\varepsilon_1} \cdots \alpha_{i-1}^{\varepsilon_{i-1}}$ belongs to S, and $\varepsilon_i = 1$ otherwise, for all $i \in [r]$,*
- *$\varepsilon_{-i} = 1$ if the string $\alpha_{-i+1}^{\varepsilon_{-i+1}} \cdots \alpha_{-1}^{\varepsilon_{-1}}$ belongs to S, and $\varepsilon_i = -1$ otherwise, for all $i \in [\ell]$.*

Then the set $\{\omega(\alpha, S) \mid \alpha \in Q_1^{\circledast}\}$ contains $2|Q_0| - |Q_1|$ straight walks and $|Q_0|$ pairs of inverse directed bending walks, which are all pairwise non-kissing. We thus obtain a facet $\eta(S)$ of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ by identifying these pairs of inverse directed bending walks.

Proposition 3.5. *For any facet $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$, the set $\zeta(F) := \Big(\bigcup_{\omega \in F} \Sigma_{\mathrm{bot}}(\omega)\Big)^{\mathrm{cl}}$ is biclosed.*

When the quiver is an oriented path with no relation, the map η should be thought of as the map from permutations to triangulations defined in [15]. Conversely, ζ maps a triangulation to the minimal permutation in its fiber under η. For the straight quiver, η plays the role of the binary search tree insertion while ζ selects the minimal linear extension of a binary tree. Using these maps, we show that the increasing flip graph on non-kissing facets is isomorphic to the Hasse diagram of the lattice quotient $\mathsf{Bic}(\bar{Q})/\equiv$.

Theorem 3.6. *The maps $\eta : \mathsf{Bic}(\bar{Q}) \to \mathcal{K}_{\mathrm{nk}}(\bar{Q})$ and $\zeta : \mathcal{K}_{\mathrm{nk}}(\bar{Q}) \to \mathsf{Bic}(\bar{Q})$ satisfy:*

- *$\eta\big(\zeta(F)\big) = F$ for any facet $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$,*
- *$\zeta\big(\eta(S)\big) = \pi_{\downarrow}(S)$ for any biclosed set $S \in \mathsf{Bic}(\bar{Q})$,*
- *for any facet $F' \in \mathcal{C}_{\mathrm{nk}}(\bar{Q})$ and $\sigma \in \zeta(F')$, there exists an increasing flip $F \to F'$ supported by σ if and only if $\zeta(F') \smallsetminus \{\sigma\}$ is biclosed.*

Therefore, the facets of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$ are in bijection with the congruence classes of $\equiv$ and the increasing flip graph is the Hasse diagram of the lattice quotient $\mathsf{Bic}(\bar{Q})/\equiv$.

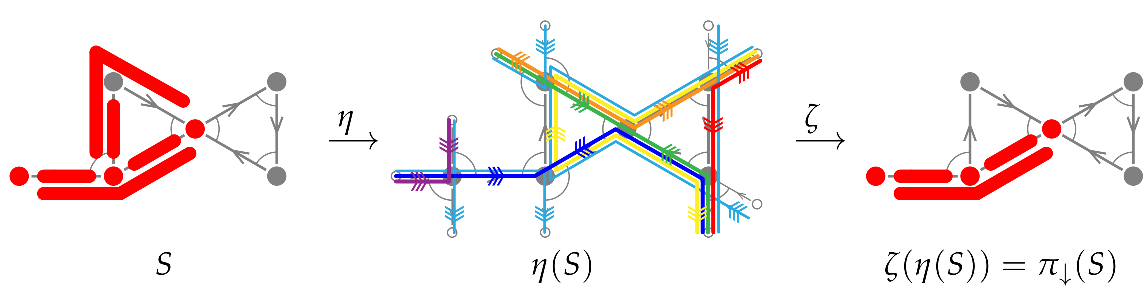


Figure 6: The maps η (left) and ζ (right) between non-kissing facets and biclosed sets.

3.4 Canonical join complex

In a lattice $(L, \leq, \wedge, \vee)$, a *join representation* of $x \in L$ is a subset $J \subseteq L$ such that $x = \bigvee J$. This representation is *irredundant* if $x \neq \bigvee J'$ for a strict subset $J' \subsetneq J$. The irredundant join representations of $x \in L$ are ordered by containment of the lower ideals of their elements, *i.e.* $J \leq J'$ if and only if for any $y \in J$ there exists $y' \in J'$ such that $y \leq y'$. When this order has a minimal element, it is called the *canonical join representation* of x. All elements of the canonical join representation $x = \bigvee J$ are then *join-irreducible*, *i.e.* cover a single element. Canonical meet representations and meet-irreducibles are defined dually.

A lattice L is *congruence-uniform* if its join-irreducible elements are in bijection with the join-irreducibles of its lattice of congruences, and similarly for meet-irreducibles. Congruence-uniform lattices behave nicely with join representations and congruence lattices. In particular, congruence-uniform lattices are semi-distributive, so that any element admits a canonical join representation. The collection of sets J which define canonical join representations in L is the *canonical join complex* of L.

To conclude our study of the non-kissing lattice $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$, we describe its canonical join complex. We say that a string $\sigma \in \mathcal{S}(\bar{Q})$ is *distinguishable* if there is a facet $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$ and a walk $\omega \in F$ such that $\sigma = \mathsf{ds}(\omega, F)$. These strings are characterized as follows.

Proposition 3.7. *A string $\sigma \in \mathcal{S}(\bar{Q})$ is distinguishable if and only if $\Sigma_{\mathrm{bot}}(\sigma) \cap \Sigma_{\mathrm{top}}(\sigma) = \{\sigma\}$.*

One checks that $\Sigma_{\mathrm{bot}}(\sigma)^{\mathrm{cl}}$ is biclosed so that we can define $\mathsf{ji}(\sigma) := \eta\big(\Sigma_{\mathrm{bot}}(\sigma)^{\mathrm{cl}}\big)$.

Proposition 3.8. *The map $\mathsf{ji} : \sigma \mapsto \mathsf{ji}(\sigma)$ defines a bijection between the distinguishable strings of $\bar{Q}$ and the join-irreducible elements of the non-kissing lattice $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$.*

Therefore, distinguishable strings are building blocks for canonical join representations in $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$. A *descent* of a facet $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$ is a string σ which is the distinguished string of a walk ω of F and is a bottom substring of ω (so that the flip of ω in F is a descent in the non-kissing lattice). We denote by $\mathrm{des}(F)$ the set of descents of F.

Proposition 3.9. *The canonical join representation of $F \in \mathcal{L}_{\mathrm{nk}}(\bar{Q})$ is given by $F = \bigvee_{\sigma \in \mathrm{des}(F)} \mathsf{ji}(\sigma)$.*

To conclude, we characterize which subsets of strings correspond to canonical join representations in the non-kissing lattice $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$. Following [7], we say that two strings are *non-friendly* if $\Sigma_{\mathrm{top}}(\sigma) \cap \Sigma_{\mathrm{bot}}(\tau) = \varnothing = \Sigma_{\mathrm{bot}}(\sigma) \cap \Sigma_{\mathrm{top}}(\tau)$. We call *non-friendly complex* the simplicial complex of sets of pairwise non-friendly distinguishable strings.

Theorem 3.10. *The following assertions are equivalent for a set Σ of distinguishable strings of $\bar{Q}$:*

- *Any two strings of Σ are non-friendly.*
- *$\{\mathsf{ji}(\sigma) \mid \sigma \in \Sigma\}$ is the canonical join-representation of a facet of $\mathcal{K}_{\mathrm{nk}}(\bar{Q})$.*
- *Σ is the descent set of a non-kissing facet $F \in \mathcal{K}_{\mathrm{nk}}(\bar{Q})$.*

In other words, the canonical join complex of $\mathcal{L}_{\mathrm{nk}}(\bar{Q})$ is isomorphic to the non-friendly complex.

For example, when the quiver is a straight path with no relation, the non-friendly complex is isomorphic to the non-crossing partition complex.

4 Gentle associahedra

In this section, we provide polyhedral realizations for finite non-kissing complexes, using tools inspired from the finite type cluster algebras of S. Fomin and A. Zelevinsky [6].

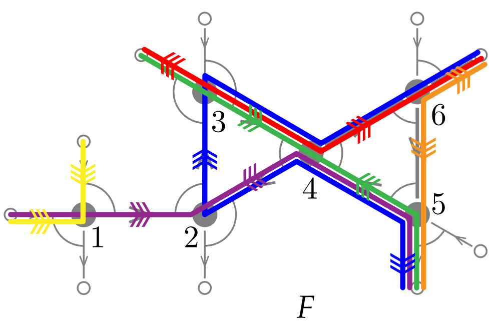
4.1 g-vectors and c-vectors

Let $\mathbf{m}_V := \sum_{i \in [m]} \mathbf{e}_{v_i} \in \mathbb{R}^{Q_0}$ be the *multiplicity vector* of a multiset $V = \{v_1, \dots, v_m\}$ of Q_0. For a string $\sigma \in \mathcal{S}(\bar{Q})$, let $\mathbf{m}_\sigma := \mathbf{m}_{V(\sigma)}$ where $V(\sigma)$ is the multiset of vertices of σ.

For a walk ω on $\bar{Q}$, we denote by $\mathsf{peaks}(\omega)$ (resp. by $\mathsf{deeps}(\omega)$) the (multi)set of peaks (resp. deeps) of ω. The **g**-*vector* of ω is the vector $\mathbf{g}(\omega) := \mathbf{m}_{\mathsf{peaks}(\omega)} - \mathbf{m}_{\mathsf{deeps}(\omega)} \in \mathbb{R}^{Q_0}$. For a set Ω of walks, $\mathbf{g}(\Omega) := \{\mathbf{g}(\omega) \,|\, \omega \in \Omega\}$. Note that $\mathbf{g}(\omega) = 0$ for a straight walk ω.

Consider a bending walk ω in a facet $F \in \mathcal{C}_{\mathrm{nk}}(\bar{Q})$. By Proposition 2.1, ω has two distinguished arrows $\mathsf{da}(\omega, F)$ around its distinguished string $\mathsf{ds}(\omega, F)$. The **c**-*vector* of $\omega \in F$ is the vector $\mathbf{c}(\omega \in F) := \varepsilon(\omega, F)\, \mathbf{m}_{\mathsf{ds}(\omega,F)} \in \mathbb{R}^{Q_0}$, where $\varepsilon(\omega, F) := 1$ if $\mathsf{ds}(\omega, F) \in \Sigma_{\mathrm{top}}(\omega)$ and $\varepsilon(\omega, F) := -1$ if $\mathsf{ds}(\omega, F) \in \Sigma_{\mathrm{bot}}(\omega)$. Denote by $\mathbf{c}(F) := \{\mathbf{c}(\omega \in F) \mid \omega \in F\}$.

Proposition 4.1. *For any non-kissing facet $F \in \mathcal{C}_{\mathrm{nk}}(\bar{Q})$, the set of* **g***-vectors $\mathbf{g}(F)$ and the set of* **c***-vectors $\mathbf{c}(F)$ form dual bases.*



$$\mathbf{g}(F) = \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array}\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{c}(F) = \begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array}\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 7: The **g**- and **c**-matrices of a facet F form dual bases.

4.2 g-vector fans and gentle associahedra

We now use **g**- and **c**-vectors to construct polyhedral realizations of finite non-kissing complexes. Since the **g**-vectors of the walks $\omega, \omega', \mu, \nu$ involved in the flip of Figure 3 satisfy the linear dependence $\mathbf{g}(\omega) + \mathbf{g}(\omega') = \mathbf{g}(\mu) + \mathbf{g}(\nu)$, we get the following statement.

Theorem 4.2. *For a gentle bound quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{C}_{\mathrm{nk}}(\bar{Q})$, the collection of cones $\mathcal{F}^{\mathbf{g}}(\bar{Q}) := \{\mathbb{R}_{\geq 0}\mathbf{g}(F) \;\big|\; F$ non-kissing face of $\mathcal{C}_{\mathrm{nk}}(\bar{Q})\}$ forms a complete simplicial fan, that we call the* **g***-vector fan of $\bar{Q}$.*

This fan is illustrated in Figure 8 (left). Note that it was constructed in [10] for dissection quivers and in [7] for grid quivers. Both constructions extend the type A Cambrian fans of N. Reading and D. Speyer [16] obtained for path quivers with no relations.

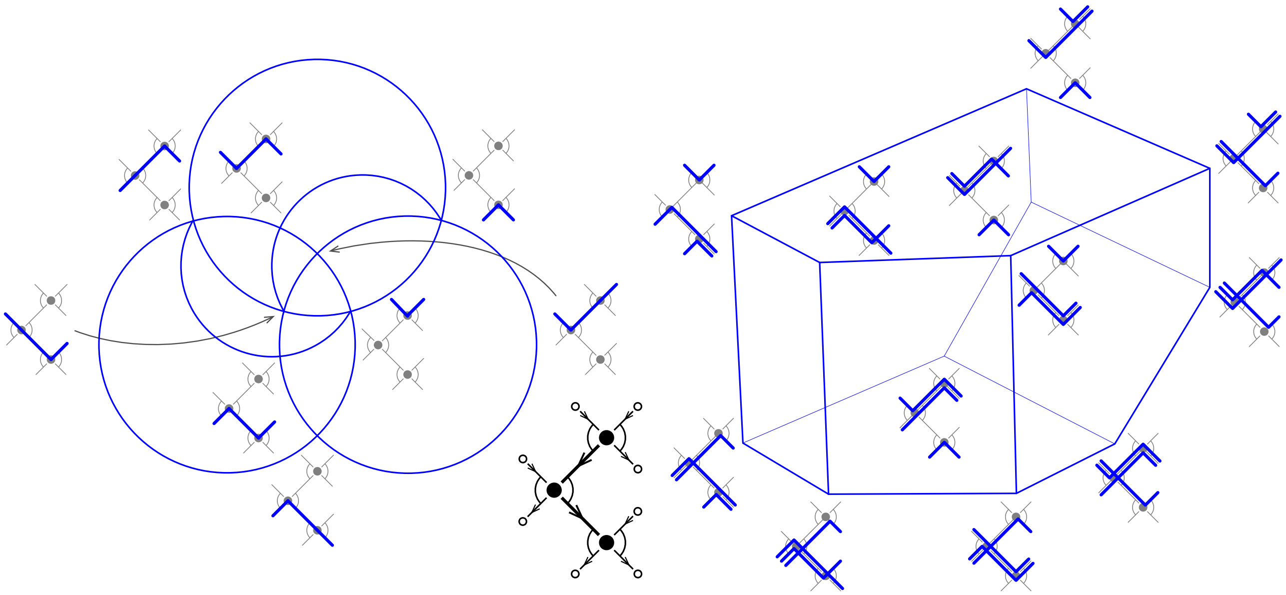


Figure 8: The $\mathbf{g}$-vector fan $\mathcal{F}^{\mathbf{g}}(\bar{Q})$ (left) and the gentle associahedra (right).

We now aim at constructing a polytope whose normal fan is the $\mathbf{g}$-vector fan of $\bar{Q}$. For two walks ω, ω' on $\bar{Q}$, denote by $\kappa(\omega, \omega')$ the number of distinct kisses of ω to ω'. The *kissing number* of ω and ω' is $\text{KN}(\omega, \omega') := \kappa(\omega, \omega') + \kappa(\omega', \omega)$. When $\mathcal{C}_{\text{nk}}(\bar{Q})$ is finite, we can define the *kissing number* of a walk ω on $\bar{Q}$ as $\text{KN}(\omega) := \sum_{\omega'} \text{KN}(\omega, \omega')$.

Theorem 4.3. *For a gentle bound quiver $\bar{Q}$ with finite non-kissing complex $\mathcal{C}_{\text{nk}}(\bar{Q})$, the $\mathbf{g}$-vector fan $\mathcal{F}^{\mathbf{g}}(\bar{Q})$ is the normal fan of the $\bar{Q}$-associahedron* $\mathsf{Asso}(\bar{Q})$ *defined equivalently as:*

(i) the convex hull of the points $\mathbf{p}(F) := \sum_{\omega \in F} \text{KN}(\omega)\, \mathbf{c}(\omega \in F)$ for all facets $F \in \mathcal{C}_{\text{nk}}(\bar{Q})$, or

(ii) the intersection of the halfspaces $\mathbf{H}^{\leq}(\omega) := \big\{ \mathbf{x} \in \mathbb{R}^{Q_0} \;\big|\; \langle \mathbf{g}(\omega) \mid \mathbf{x} \rangle \leq \text{KN}(\omega) \big\}$ for all walks ω on $\bar{Q}$.

For path quivers with no relation, we recover the associahedra of C. Hohlweg and C. Lange [9]. The latter are obtained by deleting inequalities in the facet description of the classical permutahedron. This property is lost for arbitrary gentle quivers: on the one hand, the Coxeter arrangement supporting the $\mathbf{g}$-vector fan $\mathcal{F}^{\mathbf{g}}(\bar{Q})$ is not necessarily of finite type; on the other hand, the $\bar{Q}$-associahedron is not always obtained by deleting inequalities in the facet description of the Minkowski sum of all $\mathbf{c}$-vectors. See [13] for a detailed discussion. The $\bar{Q}$-associahedron was constructed in [10] in the special case of dissection quivers. An example of $\mathsf{Asso}(\bar{Q})$ is shown in Figure 8. For grid quivers, our construction proves the polytopality conjecture for the $\mathbf{g}$-vector fan stated in [7].

This realization of the non-kissing complex has the following relevant property regarding the non-kissing lattice studied in Section 3.

Proposition 4.4. *When oriented in the linear direction $(-1, \dots, -1) \in \mathbb{R}^{Q_0}$, the graph of the $\bar{Q}$-associahedron is (isomorphic to) the increasing flip graph.*

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