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The cohomology of abelian Hessenberg varieties and the Stanley–Stembridge conjecture

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Abstract. We define a subclass of Hessenberg varieties called abelian Hessenberg varieties, inspired by the theory of abelian ideals in a Lie algebra developed by Kostant and Peterson. We prove that the cohomology of an abelian regular semisimple Hessenberg variety, with respect to the symmetric group action defined by Tymoczko, is a non-negative combination of tabloid representations. Our result implies that a graded version of the Stanley–Stembridge conjecture holds in the abelian case. As part of our arguments, we obtain inductive formulas for the Betti numbers of regular Hessenberg varieties.

Keywords: Stanley–Stembridge conjecture, symmetric functions, Hessenberg varieties

1 Introduction

Hessenberg varieties are subvarieties of the full flag variety $\mathcal{F}\ell ags(\mathbb{C}^n)$ of nested sequences of linear subspaces in $\mathbb{C}^n$. Their geometry and (equivariant) topology have been studied extensively since the late 1980s [2]. This subject lies at the intersection of, and makes connections between, many research areas such as geometric representation theory, combinatorics, and algebraic geometry and topology.

In this extended abstract, we are concerned with the connection between Hessenberg varieties and the famous Stanley–Stembridge conjecture in combinatorics, which states that the chromatic symmetric function of the incomparability graph of a so-called $(3+1)$-free poset is a non-negative linear combination of elementary symmetric polynomials [12, Conjecture 5.5]. Guay-Paquet has proved that this conjecture can be reduced to the statement that the chromatic symmetric function of the incomparability graph of a unit interval order is e-positive [5]. Shareshian and Wachs linked the Stanley–Stembridge conjecture to Hessenberg varieties via the "dot action" $\mathfrak{S}_n$-representation on the cohomology ring of regular semisimple Hessenberg varieties defined by Tymoczko [13].

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Specifically, Shareshian and Wachs established a bijection between Hessenberg functions and unit interval orders [9, Proposition 4.1]. They then formulated a conjecture relating the chromatic quasisymmetric function of the incomparability graph of a unit interval order to the dot action representation on the cohomology of an associated regular semisimple Hessenberg variety. This conjecture, known as the Shareshian–Wachs conjecture, provides the link between Hessenberg varieties and chromatic symmetric (and quasisymmetric) functions.

The Shareshian–Wachs conjecture was proved in 2015 by Brosnan and Chow [1] (also independently by Guay-Paquet [6]) by showing a remarkable relationship between the Betti numbers of different Hessenberg varieties. Since cohomology rings are graded by degree, the Shareshian–Wachs conjecture naturally suggests a stronger, "graded" conjecture based on the original Stanley–Stembridge conjecture (stated as Conjecture 2.3 below). In order to prove this graded Stanley–Stembridge conjecture, it suffices to prove that the cohomology $H^{2i}(\mathcal{H}ess(\mathsf{S},h))$ is a non-negative combination of the tabloid representations M^λ [3, Section 7.2] of $\mathfrak{S}_n$ for λ a partition of n. In other words, given the decomposition

$$H^{2i}(\mathcal{H}ess(\mathsf{S},h)) = \sum_{\lambda \vdash n} c_{\lambda,i} M^\lambda$$

in the representation ring $\mathcal{R}ep(\mathfrak{S}_n)$ of $\mathfrak{S}_n$, it suffices to prove that the coefficients $c_{\lambda,i}$ are non-negative [9, Conjecture 10.4].

The above discussion explains the motivation for this manuscript. We now describe our main results. Let $h : \{1,2,...,n\} \to \{1,2,...,n\}$ be a Hessenberg function. Our approach to the graded Stanley–Stembridge conjecture is by induction. From any Hessenberg function h one can construct a corresponding incomparability graph Γ_h. Previous results of Stanley show that the acyclic orientations of Γ_h, and their corresponding sets of sinks, encode information about the coefficients $c_{\lambda,i}$. The Hessenberg function also corresponds uniquely to a certain subset I_h of the negative roots of $\mathfrak{gl}(n,\mathbb{C})$. In this manuscript, in the special case when I_h is abelian, we give an inductive formula for the coefficients of the tabloid representations. Roughly, the idea is that the coefficients $c_{\lambda,i}$ associated to a Hessenberg variety in $\mathcal{F}\ell ags(\mathbb{C}^n)$ for $n \geq 3$, can be computed via coefficients associated to certain Hessenberg varieties in a smaller flag variety $\mathcal{F}\ell ags(\mathbb{C}^{n-2})$.

The technical core of this result consists of two inductive formulas for the Betti numbers of regular nilpotent Hessenberg varieties (Proposition 4.4) and of regular Hessenberg varieties (Proposition 4.5) which are also of independent interest. It is straightforward to prove the graded Stanley–Stembridge conjecture for the abelian case based on our inductive formula in Theorem 4.1, and we record this in Corollary 4.3. Our approach using abelian ideals is closely connected with the lower central series of an ideal. An ideal is abelian precisely when its lower central series has length one, so we can interpret Theorem 4.1 as the "base case" of an argument for the full graded Stanley–Stembridge conjecture, using induction on the length of the lower central series of I_h. We intend to explore this further in future work.

The "abelian case" considered in Corollary 4.3 corresponds, in combinatorial language, to the case when the vertices of the graph Γ_h can be partitioned into two disjoint cliques. The fact that the coefficients $c_\lambda = \sum_{i\geq 0} c_{\lambda,i}$ are non-negative in this case was stated in [11, Corollary 3.6] as a corollary to [11, Theorem 3.4]; moreover, this fact is also equivalent to [12, Remark 4.4]. However, [11, Theorem 3.4] is incorrect as stated [10], and the equivalence of [12, Remark 4.4] and [11, Corollary 3.6] is not explicit in [12, 11]. Thus, our Corollary 4.3 records a new and explicit proof of the non-negativity of the coefficients c_λ. Shareshian and Wachs also proved a graded version of Corollary 4.3 for the case when h satisfies $h(3) = \cdots = h(n) = n$, using very different techniques. This instance is a special case of our result.

2 Preliminaries

Let n be a positive integer, and $[n] = \{1, 2, \ldots, n\}$. We work in Lie type A throughout, so $G = GL(n, \mathbb{C})$ is the group of invertible $n \times n$ complex matrices and $\mathfrak{gl}(n, \mathbb{C})$ is the Lie algebra of $GL(n, \mathbb{C})$ consisting of all $n \times n$ complex matrices.

2.1 Hessenberg varieties

The full flag variety is the collection of sequences of nested linear subspaces:

$$\mathcal{F}\ell ags(\mathbb{C}^n) := \{V_\bullet = (\{0\} \subset V_1 \subset V_2 \subset \cdots V_{n-1} \subset V_{n-1} \subset \mathbb{C}^n) \mid \dim_{\mathbb{C}}(V_i) = i\}.$$

A Hessenberg variety in $\mathcal{F}\ell ags(\mathbb{C}^n)$ is specified by two pieces of data: a Hessenberg function and a choice of an element in $\mathfrak{gl}(n, \mathbb{C})$. A **Hessenberg function** is an increasing function $h : [n] \to [n]$ such that $h(i) \geq i$ for all $i \in [n]$. We frequently write a Hessenberg function by listing its values in sequence, i.e., $h = (h(1), h(2), \ldots, h(n))$.

Let $h : [n] \to [n]$ be a Hessenberg function and X be an $n \times n$ matrix in $\mathfrak{gl}(n, \mathbb{C})$, which we also consider as a linear operator $\mathbb{C}^n \to \mathbb{C}^n$. Then the **Hessenberg variety** $\mathcal{H}ess(\mathsf{X}, h)$ associated to h and X is defined to be

$$\mathcal{H}ess(\mathsf{X}, h) := \{V_\bullet \in \mathcal{F}\ell ags(\mathbb{C}^n) \mid \mathsf{X}V_i \subset V_{h(i)} \text{ for all } i \in [n]\}.$$

We focus on certain special cases of Hessenberg varieties. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ be a composition of n, i.e. $\lambda_1 + \cdots + \lambda_n = n$. A linear operator is regular of Jordan type λ if its standard Jordan canonical form has block sizes given by λ_1, λ_2, etc., and the eigenvalues from different blocks are distinct. Note that if $g \in \mathrm{GL}(n, \mathbb{C})$, then $\mathcal{H}ess(\mathsf{X}, h)$ and $\mathcal{H}ess(g\mathsf{X}g^{-1}, h)$ can be identified via the action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathcal{F}\ell ags(\mathbb{C}^n)$. For concreteness in what follows, for a given λ as above we let X_λ denote a (fixed) matrix in standard Jordan canonical form, which is regular of Jordan type λ.

Two special cases are of particular interest. Namely, if $\lambda = (n)$, then we may take the corresponding regular operator to be the regular nilpotent operator which we denote by N. The regular Hessenberg variety $\mathcal{H}ess(\mathsf{N}, h)$ is called a **regular nilpotent Hessenberg variety**. Similarly let S denote a regular semisimple matrix in $\mathfrak{gl}(n, \mathbb{C})$. This corresponds to the other extreme case, namely, $\lambda = (1, 1, 1, \ldots, 1)$. We call $\mathcal{H}ess(\mathsf{S}, h)$ a **regular semisimple Hessenberg variety**.

Denote the root system of $\mathfrak{gl}(n, \mathbb{C})$ by Φ. Then the negative roots Φ^- of $\mathfrak{gl}(n, \mathbb{C})$ are $\Phi^- = \{t_i - t_j \mid 1 \leq j < i \leq n\}$ where $t_i - t_j \in \Phi^-$ corresponds to the weight space of the adjoint action spanned by the elementary matrix E_{ij}. Let

$$\Phi_h^- := \{\gamma = t_i - t_j \in \Phi^- \mid i \leq h(j)\}$$

and $\Phi_h := \Phi_h^- \sqcup \Phi^+$. It is clear that h is uniquely determined by either Φ_h^- or Φ_h.

An **ideal** (also called an upper-order ideal) I of Φ^- is defined to be a collection of (negative) roots such that if $\alpha \in I$, $\beta \in \Phi^-$, and $\alpha + \beta \in \Phi^-$, then $\alpha + \beta \in I$. The definition of a Hessenberg function implies that $I_h := \Phi^- \setminus \Phi_h^-$ is an ideal in Φ^-. We call it the **ideal corresponding to** h.

Definition 2.1. We say that an ideal $I \subseteq \Phi^-$ is **abelian** if $\alpha + \beta \notin \Phi^-$ for all $\alpha, \beta \in I$.

The notion of abelian ideals is not new in the context of Lie theory. However, as far as we are aware, its use in the study of Hessenberg varieties is new.

Example 2.2. *Consider $h = (3, 4, 5, 6, 6, 6)$. This Hessenberg function corresponds to the abelian ideal*

$$I_h = \{t_4 - t_1, t_5 - t_1, t_5 - t_2, t_6 - t_1, t_6 - t_2, t_6 - t_3\} \subseteq \Phi^-.$$

We say that the Hessenberg variety $\mathcal{H}ess(\mathsf{X}, h)$ and the corresponding Hessenberg function h are abelian, if I_h is abelian. Abelian ideals of Φ^- (or equivalently, of Φ^+) are the source of many combinatorial and Lie-theoretic formulas. The number of abelian ideals in the negative roots of any Lie algebra $\mathfrak{g}$ is exactly $2^{\mathrm{rank}(\mathfrak{g})}$ [7, Theorem 2.1].

2.2 The Stanley–Stembridge conjecture

As discussed in the Introduction, the main motivation for this manuscript was to study the graded version of the Stanley–Stembridge conjecture (Conjecture 2.3 below), stated in terms of the $\mathfrak{S}_n$-representation on the cohomology rings of regular semisimple Hessenberg varieties through the dot action defined by Tymoczko [13].

A partition of n is a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$ of n such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If λ is a partition of n we write $\lambda \vdash n$. We say a partition $\lambda \vdash n$ has k parts if

$\lambda_k \neq 0$ and $\lambda_{k+1} = \cdots = \lambda_n = 0$. In this case, we write $\lambda = (\lambda_1, \ldots, \lambda_k)$. Moreover, for each $\nu \vdash n$, we let $\mathfrak{S}_\nu \subseteq \mathfrak{S}_n$ denote the Young subgroup of $\mathfrak{S}_n$ corresponding to ν.

It is well known that the set of tabloid representations $\{M^\lambda\}_{\lambda \vdash n}$ form a $\mathbb{Z}$-basis for the representation ring $\mathcal{R}ep(\mathfrak{S}_n)$ of $\mathfrak{S}_n$. Therefore there exist unique integers c_λ and $c_{\lambda,i}$ such that

$$H^*(\mathcal{H}ess(\mathsf{S}, h)) = \sum_{\lambda \vdash n} c_\lambda M^\lambda \quad \text{and} \quad H^{2i}(\mathcal{H}ess(\mathsf{S}, h)) = \sum_{\lambda \vdash n} c_{\lambda,i} M^\lambda \tag{2.1}$$

as elements in $\mathcal{R}ep(\mathfrak{S}_n)$. Note that, a priori, the coefficients c_λ and $c_{\lambda,i}$ may be negative. We also have $c_\lambda = \sum_{i \geq 0} c_{\lambda,i}$ for all $\lambda \vdash n$. We can now formulate the **graded Stanley–Stembridge conjecture** which motivates this manuscript.

Conjecture 2.3. *Let n be a positive integer and $h : [n] \to [n]$ be a Hessenberg function. Then the integers $c_{\lambda,i}$ appearing in (2.1) are non-negative.*

Finally, we recall a fundamental result of Brosnan and Chow which identifies the dimension of the subspaces $H^*(\mathcal{H}ess(\mathsf{S}, h))^{\mathfrak{S}_\nu}$ with the dimension of the cohomology of the regular Hessenberg variety of Jordan type ν [1, Theorem 76]. This result is an essential tool in the proof of Theorem 4.1.

Theorem 2.4. *Let n be a positive integer and $h : [n] \to [n]$ be a Hessenberg function. Let $\nu \vdash n$ be a partition of n, X_ν a regular operator of Jordan type ν, and S a regular semisimple operator. Then for each non-negative integer i,*

$$\dim(H^{2i}(\mathcal{H}ess(\mathsf{S}, h)))^{\mathfrak{S}_\nu} = \dim H^{2i}(\mathcal{H}ess(\mathsf{X}_\nu, h)).$$

When I_h is abelian, the corresponding restriction on the partitions that can appear in the right hand side of (2.1) is quite striking. One can prove the following lemma using results of Gasharov [4] which describe the decomposition of the representation $H^*(\mathcal{H}ess(\mathsf{S}, h))$ into irreducible representations.

Lemma 2.5. *If $h : [n] \to [n]$ is an abelian Hessenberg function, then $c_\lambda = c_{\lambda,i} = 0$ for all $\lambda \vdash n$ with more than 2 parts and for all $i \geq 0$.*

3 Acyclic orientations

In this section we recall some graph-theoretic data which can be constructed from a Hessenberg function. We define the **incomparability graph** $\Gamma_h = (V(\Gamma_h), E(\Gamma_h))$ associated to the Hessenberg function h as follows. The vertex set $V(\Gamma_h)$ is $[n] = \{1, 2, \ldots, n\}$. The edge set $E(\Gamma_h)$ is defined as follows: $\{i, j\} \in E(\Gamma_h)$ if $1 \leq j < i \leq n$ and $i \leq h(j)$.

Example 3.1. *The incomparability graphs for $h = (2, 4, 4, 4)$ and $h = (3, 4, 5, 5, 5)$ are given below.*

Recall that an orientation ω of a graph is an assignment of a direction to each edge $e \in E(\Gamma_h)$. Equivalently, ω assigns to each edge e a source and a target; we notate the source (respectively target) of e according to the orientation ω by $\mathrm{src}_\omega(e)$ (respectively $\mathrm{tgt}_\omega(e)$). A (directed) cycle is a sequence of vertices starting and ending at the same vertex whose edges are oriented consistently with the order of the vertices in the sequence. We say that an orientation ω is acyclic if there are no (directed) cycles in the corresponding oriented graph. Let

$$\mathcal{A}(\Gamma_h) := \{\, \omega \mid \omega \text{ is an acyclic orientation of } \Gamma_h \}$$

denote the set of all acyclic orientations of Γ_h. Moreover, given an orientation ω, a sink associated to ω is a vertex v of the graph such that $\mathrm{tgt}_\omega(e) = v$ for all edges e adjacent to v. It will be important to pay close attention to the *number* of sinks associated to a given orientation. Thus we define

$$\mathcal{A}_k(\Gamma_h) := \{\, \omega \in \mathcal{A}(\Gamma_h) \mid \omega \text{ has exactly } k \text{ sinks}\}.$$

Since every acyclic orientation has at least one sink, we have $\mathcal{A}(\Gamma_h) = \bigsqcup_{k \geq 1} \mathcal{A}_k(\Gamma_h)$.

The following is a result of Shareshian and Wachs [9, Theorem 5.3] which generalizes a theorem of Stanley [11, Theorem 3.3]. Following their terminology, for an orientation ω of Γ_h, we let

$$\mathrm{asc}(\omega) := |\{e = \{a, b\} \in E(\Gamma_h) \mid \mathrm{src}_\omega(e) = a, \mathrm{tgt}_\omega(e) = b, \text{ and } a < b\}|.$$

In other words, if Γ_h is drawn as in Example 3.1 with the labels of the vertices increasing from left to right, then $\mathrm{asc}(\omega)$ is the total number of edges which "point to the right".

Theorem 3.2. *Let n be a positive integer and $h : [n] \to [n]$ a Hessenberg function. Let $c_{\lambda,i}$ denote the coefficients appearing in (2.1). Then*

$$\sum_{\lambda \vdash n \text{ and } \lambda \text{ has } k \text{ parts}} c_{\lambda,i} = |\{\omega \mid \omega \in \mathcal{A}_k(\Gamma_h) \text{ and } \mathrm{asc}(\omega) = i\}|.$$

Since there is only one partition of n with exactly 1 part, namely $\lambda = (n)$, we may immediately conclude that $c_{(n),i} \geq 0$ for all i.

The result above makes it evident that the set of acyclic orientations, and the cardinalities of the sink sets associated to them, play a crucial role in determining the coefficients $c_{\lambda,i}$. We further develop this circle of ideas by analyzing the sink sets themselves. For a fixed Γ_h and acyclic orientation ω of Γ_h, let $\mathrm{sk}(\omega) := \{v \in V(\Gamma_h) \mid v \text{ is a sink of } \omega\}$ denote the **sink set** of ω. We let

$$\mathrm{SK}(\Gamma_h) := \{\, \mathrm{sk}(\omega) \mid \omega \in \mathcal{A}(\Gamma_h)\}$$

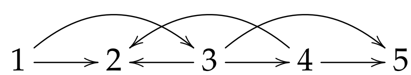
denote the set of all subsets of $V(\Gamma_h)$ which can arise as the sink set of some acyclic orientation. Similarly, we let $\mathrm{SK}_k(\Gamma_h)$ denote the subset of $SK(\Gamma_h)$ consisting of sink sets of cardinality k. By definition, we have

$$\mathcal{A}_k(\Gamma_h) = \bigsqcup_{T \in SK_k(\Gamma_h)} \{\omega \in \mathcal{A}_k(\Gamma_h) \mid \mathrm{sk}(\omega) = T\}.$$

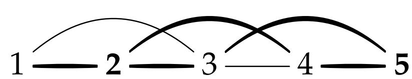
We call this the **sink set decomposition**. It isn't difficult to show that a set of vertices is a sink set of Γ_h so long as it is an independent set, i.e., no two of the vertices are connected by an edge. In fact, I_h is abelian if and only if any independent set of vertices in Γ_h contains at most two elements. Although this is case addressed in Theorem 4.1 below, the results of this section are more general.

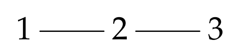
Let $k \geq 1$ and suppose $T \in SK_k(\Gamma_h)$. We define $\Gamma_h - T$ to be the induced subgraph corresponding to the vertices $V(\Gamma_h) \setminus T \subseteq V(\Gamma_h)$.

Example 3.3. *Consider the graph Γ_h for $h = (3,4,5,5,5)$ and let $T = \{2,5\}$. Then T is indeed a sink set, for the following acyclic orientation of Γ_h.*



We also draw the (unoriented) graphs Γ_h and $\Gamma_h - T$ in the figure below. In the figure for Γ_h, the vertices of T and all incident edges to T are in bold. The bold edges and vertices are then deleted to obtain $\Gamma_h - T$ (with re-labelled vertices).

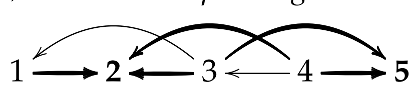


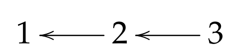


Lemma 3.4. *Let $T \in \mathrm{SK}_k(\Gamma_h)$ and let $\Gamma_h - T$ be defined as above. Then there exists a Hessenberg function $h_T : [n-k] \to [n-k]$ such that $\Gamma_h - T = \Gamma_{h_T}$.*

Finally, we observe that the construction of the smaller graph $\Gamma_h - T \cong \Gamma_{h_T}$ from the data of Γ_h also extends to orientations. Specifically, let $\omega \in \mathcal{A}_k(\Gamma_h)$ be any acyclic orientation such that $\mathrm{sk}(\omega) = T$. Then the orientation ω naturally induces, by restriction, an orientation on $\Gamma_h - T = \Gamma_{h_T}$ (since the edges of $\Gamma_h - T$ are a subset of those of Γ_h). We denote this acyclic orientation on Γ_{h_T} by ω_T.

Example 3.5. *We continue with Example 3.3. In the pictures below, we draw an orientation ω of Γ_h on the left, and its corresponding induced orientation ω_T of Γ_{h_T} on the right.*



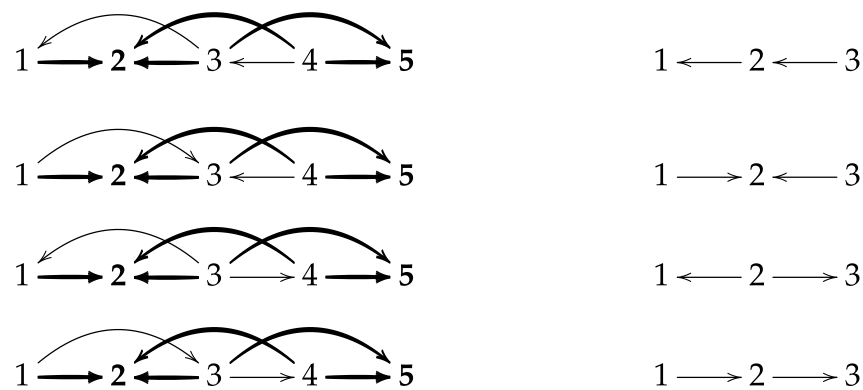


Suppose $T \in \mathrm{SK}(\Gamma_h)$. Any acyclic orientation ω with sink set T must have some number of edges oriented to the right, as determined by the vertices in T. Suppose $T \in \mathrm{SK}(\Gamma_h)$. We define the degree of T to be

$$\deg(T) := \min\{\, \mathrm{asc}(\omega) \mid \omega \in \mathcal{A}(\Gamma_h),\ \mathrm{sk}(\omega) = T\}.$$

In practice, it is easy to compute $\deg(T)$ for any $T \in \mathrm{SK}(\Gamma_h)$–it is the number of edges incident to the the vertices of T that are oriented to the right. We define the **maximum sink-set size** $m(\Gamma_h)$ to be the maximum of the cardinalities of the sink sets $\mathrm{sk}(\omega)$ associated to all possible acyclic orientations of Γ_h.

Example 3.6. *Consider the graph Γ_h for $h = (3,4,5,5,5)$ as in Example 3.3. In this example we have $m(\Gamma_h) = 2$ and $T = \{2,5\}$ is a sink set of maximal cardinality. In the figure below we draw all acyclic orientations $\omega \in \mathcal{A}(\Gamma_h)$ such that $\mathrm{sk}(\omega) = \{2,5\}$. The corresponding acyclic orientation of Γ_{h_T} is displayed to the right. In this example, we have $\deg(T) = 3$.*



In the example above, there is a bijection between acyclic orientations of Γ_{h_T} and acyclic orientations of Γ_h with sink set T. The next proposition shows that this is always the case when T is a sink set of maximal cardinality. Moreover, this natural bijection gives a tight relationship between the number of ascending edges $\mathrm{asc}(\omega)$ of the orientation ω of the original graph Γ_h with the number $\mathrm{asc}(\omega_T)$ of the induced orientation on the smaller graph Γ_{h_T}.

Proposition 3.7. *Let $h : [n] \to [n]$ be a Hessenberg function and let $m = m(\Gamma_h)$ be the maximum sink-set size for Γ_h. Let $T \in \mathrm{SK}_m(\Gamma_h)$. Then the restriction map*

$$\{\, \omega \in \mathcal{A}_m(\Gamma_h) \mid \mathrm{sk}(\omega) = T\} \to \mathcal{A}(\Gamma_{h_T}), \quad \omega \mapsto \omega_T$$

is a bijection. For any $\omega \in \mathcal{A}_m(\Gamma_h)$ with $\mathrm{sk}(\omega) = T$ we have $\mathrm{asc}(\omega) = \deg(T) + \mathrm{asc}(\omega_T)$.

4 The main theorem

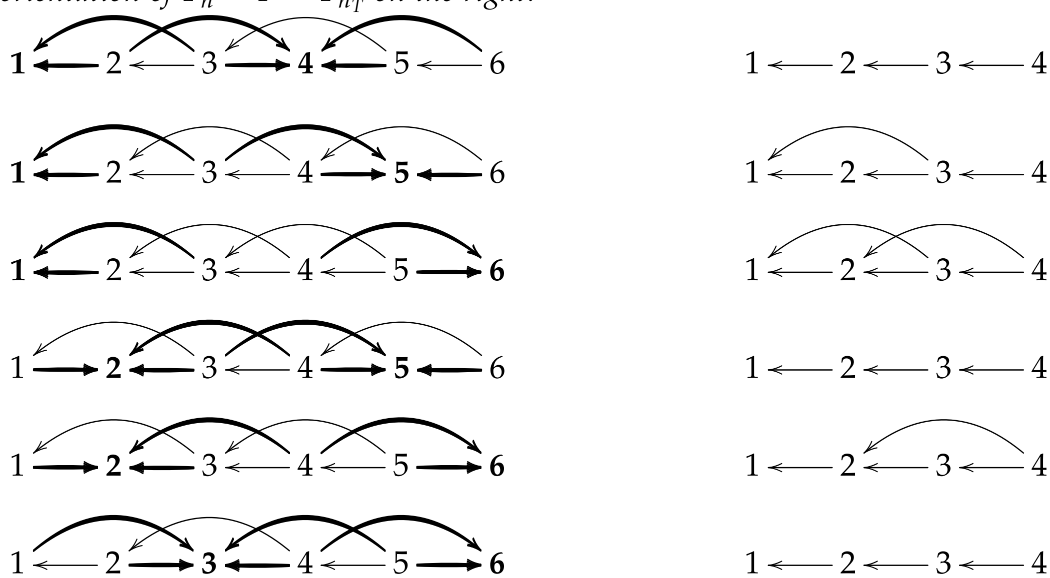
We now state our main theorem, which gives an inductive formula which, in the case when I_h is abelian, expresses Tymoczko's dot action representation on $H^{2i}(\mathcal{H}ess(\mathsf{S}, h))$ as a combination of trivial representations and a sum of tabloid representations with coefficients associated to Hessenberg varieties in $\mathcal{F}\ell ags(\mathbb{C}^{n-2})$.

Theorem 4.1. *Let n be a positive integer and $n \geq 3$. Let $h : [n] \to [n]$ be a Hessenberg function such that I_h is abelian and $i \geq 0$ be a non-negative integer. In the representation ring $\mathcal{R}ep(\mathfrak{S}_n)$ we have the equality*

$$H^{2i}(\mathcal{H}ess(\mathsf{S},h)) = c_{(n),i}M^{(n)} + \sum_{T \in SK_2(\Gamma_h)} \sum_{\substack{\mu \vdash (n-2) \\ \mu=(\mu_1,\mu_2)}} c^T_{\mu,i-\deg(T)} M^{(\mu_1+1,\mu_2+1)}.$$

To illustrate this result, we give an extended example when $n = 6$.

Example 4.2. *Let $n = 6$ and $h = (3,4,5,6,6,6)$. Then I_h is abelian, and there are six maximum dimensional sink sets in $\mathrm{SK}_2(\Gamma_h)$. The graphs below show an acyclic orientation $\omega \in \mathcal{A}_2(\Gamma_h)$ such that $\mathrm{asc}(\omega) = \deg(T)$ for each $T \in \mathrm{SK}_2(\Gamma_h)$. In each case, we display the corresponding acyclic orientation of $\Gamma_h - T \cong \Gamma_{h_T}$ on the right.*



Since the graphs are symmetric, $\Gamma - \{1,5\} \cong \Gamma - \{2,6\}$. Let S_T denote a regular semisimple element in $\mathfrak{gl}(n-2,\mathbb{C})$. The representation $H^*(\mathcal{H}ess(\mathsf{S}_T,h_T))$ for each Hessenberg function h_T with $T \in \mathrm{SK}_2(\Gamma_h)$ is shown in the table below.

Hessenberg function h_T:	$(2,3,4,4)$	$(3,3,4,4)$	$(3,4,4,4)$
$H^0(\mathcal{H}ess(\mathsf{S}_T,h_T))$	$M^{(4)}$	$M^{(4)}$	$M^{(4)}$
$H^2(\mathcal{H}ess(\mathsf{S}_T,h_T))$	$M^{(4)} + M^{(3,1)} + M^{(2,2)}$	$2M^{(4)} + M^{(3,1)}$	$3M^{(4)}$
$H^4(\mathcal{H}ess(\mathsf{S}_T,h_T))$	$M^{(4)} + M^{(3,1)} + M^{(2,2)}$	$2M^{(4)} + 2M^{(3,1)}$	$4M^{(4)} + M^{(3,1)}$
$H^6(\mathcal{H}ess(\mathsf{S}_T,h_T))$	$M^{(4)}$	$2M^{(4)} + M^{(3,1)}$	$4M^{(4)} + M^{(3,1)}$
$H^8(\mathcal{H}ess(\mathsf{S}_T,h_T))$		$M^{(4)}$	$3M^{(4)}$
$H^{10}(\mathcal{H}ess(\mathsf{S}_T,h_T))$			$M^{(4)}$

Next we see $\deg(\{1,4\}) = \deg(\{1,5\}) = \deg(\{1,6\}) = 2$, $\deg(\{2,5\}) = \deg(\{2,6\}) = 3$, *and* $\deg(\{3,6\}) = 4$ *from the graphs. We now have all the information we need to compute*

$H^*(\mathcal{H}ess(\mathsf{S},h))$ in all degrees as the shifted sum of $M^{(\mu_1+1,\mu_2+1)}$'s where $M^{(\mu_1,\mu_2)}$ appears in the representations above. The next two tables show how to shift these representations using $\deg(T)$ in order to obtain $H^*(\mathcal{H}ess(\mathsf{S},h))$.

$T \in \mathrm{SK}_2(\Gamma_h)$:	$\{1,4\}$	$\{1,5\}$	$\{1,6\}$
$H^4(\mathcal{H}ess(\mathsf{S},h))$	$M^{(5,1)}$	$M^{(5,1)}$	$M^{(5,1)}$
$H^6(\mathcal{H}ess(\mathsf{S},h))$	$M^{(5,1)}+M^{(4,2)}+M^{(3,3)}$	$2M^{(5,1)}+M^{(4,2)}$	$3M^{(5,1)}$
$H^8(\mathcal{H}ess(\mathsf{S},h))$	$M^{(5,1)}+M^{(4,2)}+M^{(3,3)}$	$2M^{(5,1)}+2M^{(4,2)}$	$4M^{(5,1)}+M^{(4,2)}$
$H^{10}(\mathcal{H}ess(\mathsf{S},h))$	$M^{(5,1)}$	$2M^{(5,1)}+M^{(4,2)}$	$4M^{(5,1)}+M^{(4,2)}$
$H^{12}(\mathcal{H}ess(\mathsf{S},h))$		$M^{(5,1)}$	$3M^{(5,1)}$
$H^{14}(\mathcal{H}ess(\mathsf{S},h))$			$M^{(5,1)}$

$T \in \mathrm{SK}_2(\Gamma_h)$:	$\{2,5\}$	$\{3,6\}$	$\{2,6\}$
$H^6(\mathcal{H}ess(\mathsf{S},h))$	$M^{(5,1)}$		$M^{(5,1)}$
$H^8(\mathcal{H}ess(\mathsf{S},h))$	$M^{(5,1)}+M^{(4,2)}+M^{(3,3)}$	$M^{(5,1)}$	$2M^{(5,1)}+M^{(4,2)}$
$H^{10}(\mathcal{H}ess(\mathsf{S},h))$	$M^{(5,1)}+M^{(4,2)}+M^{(3,3)}$	$M^{(5,1)}+M^{(4,2)}+M^{(3,3)}$	$2M^{(5,1)}+2M^{(4,2)}$
$H^{12}(\mathcal{H}ess(\mathsf{S},h))$	$M^{(5,1)}$	$M^{(5,1)}+M^{(4,2)}+M^{(3,3)}$	$2M^{(5,1)}+M^{(4,2)}$
$H^{14}(\mathcal{H}ess(\mathsf{S},h))$		$M^{(5,1)}$	$M^{(5,1)}$

For example, we get,

$$H^8(\mathcal{H}ess(\mathsf{S},h)) = c_{(6),4}M^{(6)} + 11M^{(5,1)} + 6M^{(4,2)} + 2M^{(3,3)}.$$

We can now prove the graded Stanley–Stembridge conjecture in the abelian case.

Corollary 4.3. *Let n be a positive integer and $h:[n]\to[n]$ a Hessenberg function such that I_h is abelian. Then the integers $c_{\lambda,i}$ appearing in (2.1) are non-negative.*

Proof. We argue by induction. Our base cases are $n=1$ and $n=2$. The case $n=1$ is trivial in the sense that the regular semisimple Hessenberg variety under consideration is just a single point, and the symmetric group is the trivial group. Hence the claim holds in this case. If $n=2$ then the corresponding flag variety $\mathcal{F}\ell ags(\mathbb{C}^2)\cong\mathbb{P}^1$ is non-trivial. In this case there are only two Hessenberg functions to consider: $h=(1,2)$ and $h=(2,2)$. Both cases correspond to abelian ideals and the reader may confirm that $H^0(\mathcal{H}ess(\mathsf{S},(1,2)))\cong M^{(1,1)}$, $H^0(\mathcal{H}ess(\mathsf{S},(2,2)))\cong M^{(2)}$, and $H^2(\mathcal{H}ess(\mathsf{S},(2,2)))\cong M^{(2)}$ are the corresponding representations.

The proof for $n\geq 3$ now follows from the inductive description of $H^{2i}(\mathcal{H}ess(\mathsf{S},h))$ given in Theorem 4.1 together with the fact that $c_{(n),i}\geq 0$ for all i. □

The proof of Theorem 4.1 relies on the following inductive formulas for the Betti numbers of a regular Hessenberg variety together with Brosnan and Chow's result, Theorem 2.4. Proposition 4.4 gives a formula for the Betti numbers of $\mathcal{H}ess(\mathsf{N},h)\subseteq\mathcal{F}\ell ags(\mathbb{C}^n)$ in terms of the Betti numbers of regular nilpotent Hessenberg varieties in $\mathcal{F}\ell ags(\mathbb{C}^{n-2})$, and Proposition 4.5 is of a similar flavor.

Proposition 4.4. *Let n be a positive integer and $n \geq 3$. Let $h : [n] \to [n]$ be a Hessenberg function such that I_h is abelian. Let N be a regular nilpotent element of $\mathfrak{gl}(n, \mathbb{C})$ and N' be a regular nilpotent element of $\mathfrak{gl}(n-2, \mathbb{C})$. Then the $2i$-th Betti number of $\mathcal{H}ess(\mathsf{N}, h)$ satisfies*

$$\dim H^{2i}(\mathcal{H}ess(\mathsf{N}, h)) = c_{(n),i} + \sum_{T \in \mathrm{SK}_2(\Gamma_h)} \dim H^{2i-2\deg(T)}(\mathcal{H}ess(\mathsf{N}', h_T)).$$

Proposition 4.5. *Let n be a positive integer and $n \geq 3$. Let $h : [n] \to [n]$ be a Hessenberg function such that I_h is abelian. Let X_ν be the regular element of $\mathfrak{gl}(n, \mathbb{C})$ associated to $\nu = (\mu_1 + 1, \mu_2 + 1) \vdash n$ and X_μ be a regular element of $\mathfrak{gl}(n-2, \mathbb{C})$ associated to $\mu = (\mu_1, \mu_2) \vdash (n-2)$. Then the $2i$-th Betti number of $\mathcal{H}ess(\mathsf{X}_\nu, h)$ satisfies*

$$\dim H^{2i}(\mathcal{H}ess(\mathsf{X}_\nu, h)) = \dim H^{2i}(\mathcal{H}ess(\mathsf{N}, h)) + \sum_{T \in \mathrm{SK}_2(\Gamma_h)} \dim H^{2i-2\deg(T)}(\mathcal{H}ess(\mathsf{X}_\mu, h_T)).$$

The proofs of Proposition 4.4 and 4.5 are combinatorial in nature, and rely on a formula for the Betti numbers of regular Hessenberg varieties obtained by the second author in [8] using geometric techniques. The aforementioned formula, together with the restriction to partitions with at most two parts, yields a simple presentation of these Betti numbers in the abelian case.

Although Theorem 4.1 holds for abelian Hessenberg varieties only, much of the framework and analysis in Section 3 is general. In particular the analysis of maximal sink sets of the graph Γ_h in Section 3 shows that every acyclic orientation of Γ_h corresponding to such a sink set T is obtained inductively from an acyclic orientation of the smaller graph Γ_{h_T} on $n - |T|$ vertices. Using Theorem 3.2, this indicates a correspondence between the representations $H^*(\mathcal{H}ess(\mathsf{S}, h))$ and $H^*(\mathcal{H}ess(\mathsf{S}_T, h_T))$, where S_T denotes a regular semisimple element in $\mathfrak{gl}(n - |T|, h_T)$.

Conjecture 4.6. *Let $h : [n] \to [n]$ be a Hessenberg function and $\lambda \vdash n$ be a partition with exactly $m = m(\Gamma_h)$ parts. Let $\mu = (\mu_1, \mu_2, ..., \mu_m)$ be a partition of $n - |T|$ such that $\lambda = (\mu_1 + 1, \mu_2 + 1, \cdots, \mu_m + 1)$. Then for all $i \geq 0$,*

$$c_{\lambda,i} = \sum_{T \in \mathrm{SK}_m(\Gamma_h)} c^T_{\mu, i-\deg(T)}.$$

This conjecture extends the results of Theorem 4.1 to arbitrary regular semisimple Hessenberg varieties. The formula given in Conjecture 4.6 does not determine the coefficients for M^λ unless λ has a maximal number of parts. To obtain a formula which fully generalizes Theorem 4.1 using similar methods, we need an inductive formula for the Betti numbers of an arbitrary regular Hessenberg variety.

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References

[1] P. Brosnan and T.Y. Chow. "Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties". *Adv. Math.* **329** (2018), pp. 955–1001. DOI: 10.1016/j.aim.2018.02.020.

[2] F. De Mari, C. Procesi, and M.A. Shayman. "Hessenberg varieties". *Trans. Amer. Math. Soc.* **332**.2 (1992), pp. 529–534. DOI: 10.2307/2154181.

[3] W. Fulton. *Young tableaux*. Vol. 35. London Mathematical Society Student Texts. Cambridge: Cambridge University Press, 1997.

[4] V. Gasharov. "Incomparability graphs of $(3+1)$-free posets are s-positive". *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994)*. Vol. 157. 1-3. 1996, pp. 193–197. DOI: 10.1016/S0012-365X(96)83014-7.

[5] M. Guay-Paquet. "A modular relation for the chromatic symmetric functions of (3+1)-free posets". 2013. arXiv: 1306.2400.

[6] M. Guay-Paquet. "A second proof of the Shareshian-Wachs conjecture, by way of a new Hopf algebra". 2016. arXiv: 1601.05498.

[7] B. Kostant. "The set of abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations". *Int. Math. Res. Not.* 5 (1998), pp. 225–252. URL.

[8] M. Precup. "Affine pavings of Hessenberg varieties for semisimple groups". *Selecta Math. (N.S.)* **19**.4 (2013), pp. 903–922. DOI: 10.1007/s00029-012-0109-z.

[9] J. Shareshian and M.L. Wachs. "Chromatic quasisymmetric functions". *Adv. Math.* **295** (2016), pp. 497–551. DOI: 10.1016/j.aim.2015.12.018.

[10] R.P. Stanley. Personal communication.

[11] R.P. Stanley. "A symmetric function generalization of the chromatic polynomial of a graph". *Adv. Math.* **111**.1 (1995), pp. 166–194. DOI: 10.1006/aima.1995.1020.

[12] R.P. Stanley and J.R. Stembridge. "On immanants of Jacobi-Trudi matrices and permutations with restricted position". *J. Combin. Theory Ser. A* **62**.2 (1993), pp. 261–279. DOI: 10.1016/0097-3165(93)90048-D.

[13] J.S. Tymoczko. "Permutation actions on equivariant cohomology of flag varieties". *Toric topology*. Vol. 460. Contemp. Math. Providence, RI: Amer. Math. Soc., 2008, pp. 365–384. DOI: 10.1090/conm/460.