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Winnie-the-Pooh and the Strange Expectations

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Abstract. We prove that the expected norm of a weight in a highest weight representation $\mathfrak{g}(\lambda)$ of a complex simple Lie algebra $\mathfrak{g}$ is $\frac{1}{h+1}(\lambda, \lambda + 2\rho)$ by relating it to the "Winnie-the-Pooh problem." This proof method applies to all types except A and C; the same formula holds in these two remaining types, but we are forced to provide a direct computation.

Résumé. Nous prouvous que l'espérance de la norme d'un poids dans une représentation $\mathfrak{g}(\lambda)$ d'une algèbre de Lie $\mathfrak{g}$ complexe et simple est $\frac{1}{h+1}(\lambda, \lambda + 2\rho)$ en le rapportant au "problème de Winnie l'ourson." La méthode de preuve s'applique à tous les types sauf A et C; la même formule s'applique dans ces deux cas, mais nous sommes obligés de fournir un calcul direct.

Keywords: Lie algebra, highest weight representation, weight lattice, expected value

1 Introduction

The representation theory of complex simple Lie algebras is a cornerstone of algebraic and enumerative combinatorics, leading to tableaux and plane partitions, symmetric functions, quantum groups and crystal theory, the plactic monoid and RSK. Having chosen a Cartan subalgebra $\mathfrak{h}$, the finite-dimensional irreducible representations of a complex simple Lie algebra $\mathfrak{g}$ are completely classified by the dominant weights λ in its weight lattice $\Lambda \subset \mathfrak{h}^*$. Using the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$ induced by the Killing form and writing ρ for the half-sum of the positive roots Φ^+, the Weyl dimension formula asserts that

$$\dim(\mathfrak{g}(\lambda)) = \prod_{\alpha \in \Phi^+} \frac{\langle \alpha, \lambda + \rho \rangle}{\langle \alpha, \rho \rangle}. \tag{1.1}$$

Motivated by the recent interest in the number of boxes in simultaneous cores (see Section 2), the purpose of this abstract is to give a formula for the average norm of a weight in a highest weight representation.

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Theorem 1.1. *For $\mathfrak{g}$ a complex simple Lie algebra with $\mathfrak{g}(\lambda)$ its finite-dimensional irreducible representation of highest weight λ, the expected norm squared of a weight in $\mathfrak{g}(\lambda)$ is*

$$\frac{1}{\dim(\mathfrak{g}(\lambda))} \sum_{\mu \in \mathfrak{g}(\lambda)} n_\lambda^\mu \langle \mu, \mu \rangle = \frac{1}{h+1} \langle \lambda, \lambda + 2\rho \rangle,$$

where n_λ^μ is the multiplicity of the weight μ in $\mathfrak{g}(\lambda)$ and h is the Coxeter number of $\mathfrak{g}$.

Theorem 1.1 is illustrated in Figure 1 for a highest weight representation for $\mathfrak{sl}_3$ and for $\mathfrak{sp}_4$. Although types A and C are ordinarily the easiest types of Lie algebras for most computations and results, they turn out to be the hardest types for which to establish Theorem 1.1. Our proof is therefore broken into two parts: we first settle Theorem 1.1 for almost all cases, leaving only $\mathfrak{sl}_n$ with n composite and $\mathfrak{sp}_{2n}$ with $n \neq 2^m$; we then combinatorially prove the theorem for those two remaining cases.

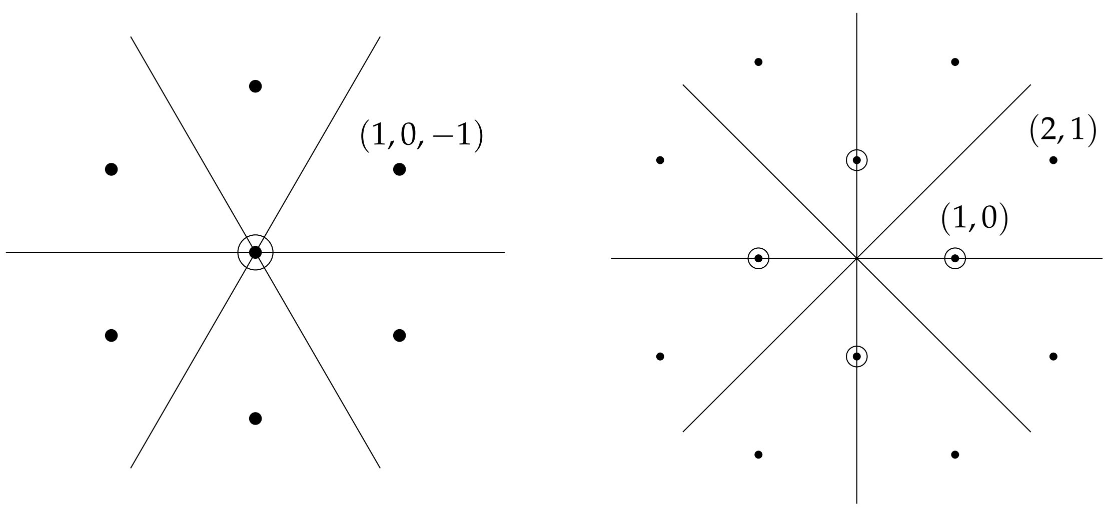

Figure 1: On the left is the highest weight representation $\mathfrak{sl}_3(\lambda)$ for $\lambda = \rho = (1, 0, -1)$. There are eight ($= 2^3$) weights, and the average norm—which may be computed in coordinates with the dot product—is $\frac{6 \cdot 2 + 2 \cdot 0}{8} = \frac{3}{2} = \frac{1}{3+1} \langle \lambda, \lambda + 2\rho \rangle$. On the right is the highest weight representation $\mathfrak{sp}_4(\lambda)$ for $\lambda = \rho = (2, 1)$. There are sixteen ($= 2^4$) weights, and the average norm is $\frac{8 \cdot 5 + 8 \cdot 1}{16} = 3 = \frac{1}{4+1} \langle \lambda, \lambda + 2\rho \rangle$.

2 Motivation

An *a-core* is an integer partition with no hook-length of size a. The study of simultaneous (a, b)-cores—that is, partitions that are both a-cores and b-cores—is a topic that has recently seen quite a lot of interest from the combinatorics community [13, 1]. When $\gcd(a,b) = 1$, Anderson proved that the number of (a,b)-cores has the simple expression

$$|\mathsf{core}(a,b)| = \frac{1}{a+b}\binom{a+b}{b}$$

by giving a bijection to Dyck paths in an $a \times b$ rectangle [2]. It is well-known that the dominant alcoves in the affine symmetric group $\widetilde{S}_a$ are naturally indexed by a-cores, and in this language Anderson's result had previously been proven in the generality of affine Weyl groups by both Haiman and Suter [6, 14].

While investigating the interpretation of q,t-statistics and the zeta map using the affine symmetric group [3, 4], Armstrong was led to conjecture that the expected number of boxes of a simultaneous core (its "size") had the beautiful formula

$$\mathop{\mathbb{E}}_{\lambda \in \mathsf{core}(a,b)}(\mathsf{size}(\lambda)) = \frac{(a-1)(b-1)(a+b+1)}{24}. \tag{2.1}$$

This was first proven by Johnson using Ehrhart theory [7]. In [15], building on Johnson's approach, we showed that the statistic size could be interpreted as a slight modification of the natural norm on the weight space (see Figure 2), and generalized the result to all simply-laced affine Weyl groups (we now have a generalization to all affine Weyl groups).

In short, by composing the bijection between a-cores and dominant alcoves in $\widetilde{S}_a$, and the natural bijection between dominant alcoves and coroot points, one obtains a bijection between simultaneous (a,b)-cores and $Q^\vee \cap bA_0$—coroot points inside a b-fold dilation of the fundamental alcove in $\widetilde{S}_a$. When a is coprime to b, the cyclic symmetry of the affine Dynkin diagram is an affine isometry that partitions the weights inside bA_0 to a orbits, each of which contain a single coroot. One may then apply Ehrhart theory to the norm of the weights in bA_0 to prove (generalizations of) Armstrong's conjecture.

For $\mathfrak{sl}_a$, there is a bijection between weights inside the b-fold dilation of the fundamental alcove bA_0, and weights in the highest weight representation $\mathfrak{sl}_a(b\omega_1)$, where ω_1 is the first fundamental weight—as illustrated in Figure 2, both are counted by the binomial coefficient $\binom{a+b}{b}$. Given the success of studying moments of norms of weights in bA_0, we found it a reasonable extension to ask for the expected norm of a weight in a highest weight representation.

3 Background

3.1 Lie Algebras and their Representation Theory

Fix a complex simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$; recall that all Cartan subalgebras of $\mathfrak{g}$ are conjugate. Given a complex representation $V : \mathfrak{g} \to \mathfrak{gl}(V)$, we say

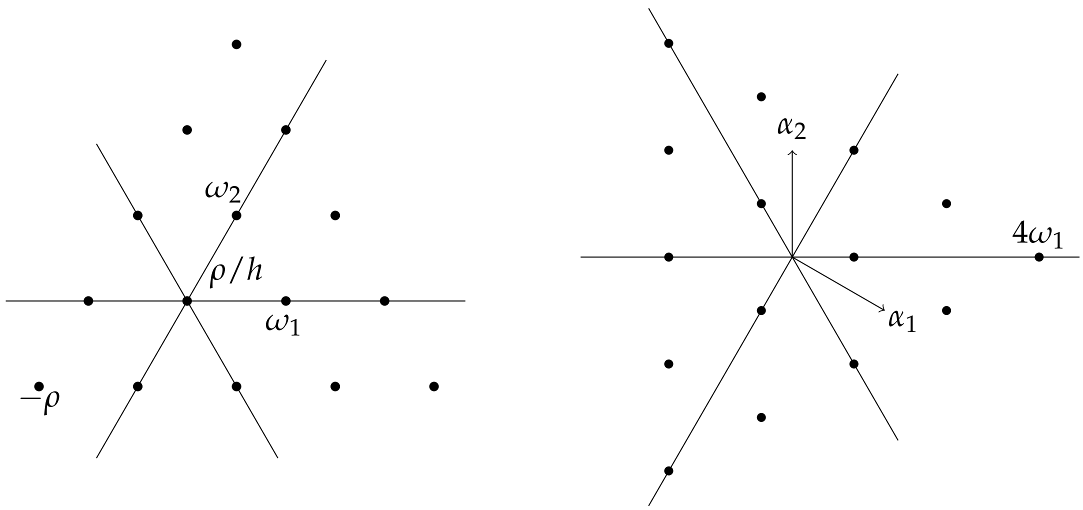

Figure 2: The weights inside a 4-fold dilation of the fundamental alcove in $\mathfrak{sl}_3$, and the weights in the highest weight representation $\mathfrak{sl}_3(4\omega_1)$. The statistic size on the weights in the dilation of the fundamental alcove is a quadratic form that is a slight modification of the norm, centered on the fundamental alcove (the point ρ/h).

that the *weight space* for $\lambda \in \mathfrak{h}^*$ is the subspace

$$V_\lambda = \{v \in V : H \cdot v = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}.$$

The adjoint representation of $\mathfrak{g}$ has non-zero weights called *roots*, and we obtain the decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha. \tag{3.1}$$

The *Killing form* is the nondegenerate symmetric bilinear form defined by $B(X,Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y))$; we choose to normalize the Killing form so that the norm of a long root is fixed to be 2, and we will write this normalized form as $\langle \cdot, \cdot \rangle$. Restricting the Killing form to $\mathfrak{h}$ and writing $\dim(\mathfrak{h}) = n$ allows us to view weights and roots as points in $\mathbb{R}^n$, and the *Weyl group* for $\mathfrak{g}$ is the reflection group generated by the reflections perpendicular to the roots $\alpha \in \Phi$.

3.2 Casimir Elements and the Universal Enveloping Algebra

The Harish-Chandra isomorphism is an isomorphism between the center of the universal enveloping algebra of $\mathfrak{g}$ $Z(U(\mathfrak{g}))$ and W-invariant polynomials $S(\mathfrak{h})^W$. By the Shephard–Todd–Chevalley theorem, $S(\mathfrak{h})^W$ is a polynomial algebra with n generators, and the degrees $d_1, d_2, \ldots, d_n$ of these generators play an important numerological role:

for example, the highest degree is the *Coxeter number* h (the order of a Coxeter element of W), the dimension of the Lie algebra is $\dim(\mathfrak{g}) = n(h+1)$, the number of reflections in W is $\sum_{i=1}^n (d_i - 1)$, and the number of elements in W is $|W| = \prod_{i=1}^n d_i$.

We call an element of $Z(U(\mathfrak{g}))$ a *Casimir element*—the Harish-Chandra isomorphism combined with the Shephard–Todd–Chevalley theorem implies that there are n algebraically independent Casimir elements. Special emphasis is given to the Casimir element Ω of degree two, which may be defined as follows: fixing any basis $\{X_i\}_{i=1}^{n(h+1)}$, we obtain a dual basis $\{X^i\}_{i=1}^{n(h+1)}$ using the Killing form. Then

$$\Omega = \sum_{i=1}^{n(h+1)} X_i X^i \in Z(U(\mathfrak{g})). \tag{3.2}$$

As representations of $\mathfrak{g}$ coincide with modules for its universal enveloping algebra, the fact that Ω is in the center of $U(\mathfrak{g})$ means that it acts as a scalar on any highest weight representation of $\mathfrak{g}$. This scalar is well-known [12, 11].

Theorem 3.1. *Let λ be a dominant weight. Then Ω acts as multiplication by $(\lambda, \lambda + 2\rho)$ on $\mathfrak{g}(\lambda)$.*

3.3 Orthogonal Decompositions of Lie algebras

The usual decomposition of $\mathfrak{g}$ using a fixed Cartan subalgebra $\mathfrak{h}$ and the adjoint representation is given in Equation (3.1). Numerologically, this reflects the identity $n(h+1) = n + nh = \dim(\mathfrak{h}) + |\Phi|$.

But since $\dim(\mathfrak{h}) = n$ divides $\dim(\mathfrak{g}) = n(h+1)$, we might ask for a *different* decomposition of $\mathfrak{g}$ using a direct sum of $h+1$ Cartan subalgebras:

$$\mathfrak{g} = \bigoplus_{i=0}^{h} \mathfrak{h}_i, \text{ with } \mathfrak{h}_i \text{ a Cartan subalgebra of } \mathfrak{g} \text{ and } \mathfrak{h}_0 = \mathfrak{h}. \tag{3.3}$$

In fact, such a decomposition is always possible. More difficult is to require that these $h+1$ Cartan subalgebras are pairwise orthogonal with respect to the Killing form; such a decomposition is called an *orthogonal decomposition*. We refer the reader to [9, 10] for background and references, pausing only to remark that such decompositions have a number of applications, including Thompson's construction of his sporadic simple group from the Lie algebra of type E_8 [16] and the construction of mutually unbiased bases for quantum cryptography [5].

Theorem 3.2 ([10])**.** *A complex simple Lie algebra $\mathfrak{g}$ has an orthogonal decomposition, except possibly if*

- $\mathfrak{g} = \mathfrak{sl}_n$ *for n composite; or if*

- $\mathfrak{g} = \mathfrak{sp}_{2n}$ *for* $n \neq 2^m$.

Although types A and C are usually the easiest Lie algebras to deal with, it is widely believed that the exceptions *do not* have orthogonal decompositions; this problem is wide open, even for $\mathfrak{sl}_6$. It was dubbed the *Winnie-the-Pooh problem* in the Russian paper [9], due to a play on words found in Zahoder's translation of Milne's famous children's book "Winnie-the-Pooh" into Russian. Zahoder's play on words can be interpreted as the sequence of Cartan types A_5—corresponding to the smallest open case $\mathfrak{sl}_6$—then A_6, A_7, and A_8. This play on words apparently has no counterpart in Milne's original text, so when translating [9] into English, Queen also translated Zahoder's verse—while managing to preserve the pun [10].

We note that Kostant used a dual approach to the related numerological problem of trying to uniformly explain the duality between degrees and the heights of roots [8]. Kostant decomposed $\mathfrak{g}$ into direct sum of n irreducible representations of the principal three dimensional simple subalgebra (a distinguished copy of $\mathfrak{sl}_2$ inside $\mathfrak{g}$, unique up to conjugacy), reflecting the identity $n(h+1) = \sum_{i=1}^{n} 2d_i - 1$.

4 Proof of the main theorem

In this section, we prove our main Theorem 1.1.

4.1 Types not A or C

The strategy is to compute the trace of the degree two Casimir element Ω in two different ways on the representation $\mathfrak{g}(\lambda)$.

Proof. Suppose $\mathfrak{g}$ has an orthogonal decomposition $\mathfrak{g} = \bigoplus_{i=0}^{h} \mathfrak{h}_i$. For each $0 \leq i \leq h$, pick an orthonormal basis $\{X_{i,1}, \ldots, X_{i,n}\}$ of $\mathfrak{h}_i$. Then

$$\left\{X_{i,j} : 0 \leq i \leq h \text{ and } 1 \leq j \leq n\right\}$$

is an orthonormal basis of $\mathfrak{g}$, so we may write the degree two Casimir element Ω as

$$\Omega = \sum_{\substack{0 \leq i \leq h \\ 1 \leq j \leq n}} X_{i,j}^2.$$

We compute the trace of Ω on $\mathfrak{g}(\lambda)$ in two different ways. On the one hand, Ω acts as the scalar $\langle \lambda, \lambda + 2\rho \rangle$ by Theorem 3.1, so that

$$\mathrm{tr}_{\mathfrak{g}(\lambda)}(\Omega) = \langle \lambda, \lambda + 2\rho \rangle \dim(\mathfrak{g}(\lambda)).$$

On the other hand, for $0 \leq i \leq h$ define

$$\Omega_i = \sum_{j=1}^{n} X_{i,j}^2.$$

By definition, $X_{0,j}$ acts as $\mu(X_{0,j})$ on the μ-weight space of $\mathfrak{g}(\lambda)$, so

$$\mathrm{tr}_{\mathfrak{g}(\lambda)}(\Omega_0) = \sum_{\mu \in \mathfrak{g}(\lambda)} n_\lambda^\mu \sum_{j=1}^{n} \mu(X_{0,j})^2 = \sum_{\mu \in \mathfrak{g}(\lambda)} n_\lambda^\mu \langle \mu, \mu \rangle,$$

since $\{X_{0,1}, \ldots, X_{0,n}\}$ is an orthonormal basis of $\mathfrak{h} = \mathfrak{h}_0$. Since every $\mathfrak{h}_i$ is conjugate to $\mathfrak{h}_0$ under an inner automorphism of $\mathfrak{g}$ that leaves the Killing form invariant, we have that Ω_i is conjugate to Ω_0 for all $0 \leq i \leq h$. Therefore,

$$\begin{aligned} \mathrm{tr}_{\mathfrak{g}(\lambda)}(\Omega) &= \mathrm{tr}_{\mathfrak{g}(\lambda)} \left(\sum_{i=0}^{h} \Omega_i \right) \\ &= \sum_{i=0}^{h} \mathrm{tr}_{\mathfrak{g}(\lambda)} \left(\Omega_i \right) \\ &= (h+1)\mathrm{tr}_{\mathfrak{g}(\lambda)} \left(\Omega_0 \right) = (h+1) \sum_{\mu \in \mathfrak{g}(\lambda)} n_\lambda^\mu \langle \mu, \mu \rangle. \end{aligned}$$

The result now follows from equating the two expressions for $\mathrm{tr}_{\mathfrak{g}(\lambda)}(\Omega)$. □

By Theorem 3.2, this proof of Theorem 1.1 applies to all types except possibly if $\mathfrak{g} = \mathfrak{sl}_n$ for n composite; or if $\mathfrak{g} = \mathfrak{sp}_{2n}$ for $n \neq 2^m$. We prove Theorem 1.1 for types A and C by direct calculation.

4.2 Types A and C

Fix $\mathfrak{g} = \mathfrak{sl}_n$. In $\mathfrak{sl}_n$, dominant weights of $\mathfrak{h}$ may be parametrized as integer partitions

$$\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n] \vdash m,$$

where parts λ_i may be equal to zero. Fix a highest weight λ, and write

$$\overline{\lambda_i} = \lambda_i - \frac{|\lambda|}{n},\ \overline{\lambda} = \left[\overline{\lambda_1} \geq \cdots \geq \overline{\lambda_n}\right], \text{ and } \rho = [n-1, n-2, \ldots, 1, 0].$$

With these conventions, weights μ in the highest weight representation $\mathfrak{sl}_n(\lambda)$ may be thought of as certain points in $\mathbb{R}^n$ with positive entries and sum equal to m. Combinatorially, the multiplicity of μ in $\mathfrak{sl}_n(\lambda)$ is well-known to be given by the number of semistandard tableaux of shape λ on the alphabet $[n]$ with content μ; m is just the number of boxes in the Ferrers shape λ:

$$\mathrm{ch}(\mathfrak{sl}_n(\lambda)) = s_\lambda(x_1, \ldots, x_n) = \sum_{\substack{T \text{ semistandard} \\ \text{of shape } \lambda}} \mathbf{x}^T,$$

where $\mathbf{x}^T = \prod_{i=1}^n x_i^{|\{i \in T\}|}$ and $s_\lambda(x_1, \ldots, x_n)$ is a Schur polynomial. As a simple consequence of this combinatorial description of the character, we have Weyl's "interlacing" multiplicity-free formula for the branching of the representation $\mathfrak{sl}_n(\lambda)$ to $\mathfrak{sl}_{n-1}$:

$$\mathfrak{sl}_n(\lambda) = \bigoplus_\mu \mathfrak{sl}_{n-1}(\mu), \text{ where } \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n.$$

By symmetry of the Schur function, and since this branching rule exactly peels off the boxes containing the entry n (corresponding to the value of the coordinate x_n), one could imagine using this formula to determine the norm by computing

$$\sum_{\mu \in \mathfrak{sl}_n(\lambda)} n_\lambda^\mu \langle \mu, \mu \rangle = n \sum_\mu \dim(\mathfrak{sl}_{n-1}(\mu)) \left(|\lambda| - |\mu|\right)^2.$$

We do not follow this approach here, but instead isolate the boxes containing the entry n using the Pieri rule and an inclusion-exclusion argument.

Theorem 4.1. *Let $\mathfrak{g} = \mathfrak{sl}_n$. Suppose that λ is a dominant weight of $\mathfrak{h}$ and let $\mathfrak{sl}_n(\lambda)$ be the finite dimensional irreducible representation of $\mathfrak{g}$ of highest weight λ. For a weight μ, let n_μ be the multiplicity of μ in $\mathfrak{sl}_n(\lambda)$. Then*

$$\frac{1}{\dim(\mathfrak{sl}_n(\lambda))} \sum_{\mu \in \mathfrak{sl}_n(\lambda)} n_\lambda^\mu \langle \overline{\mu}, \overline{\mu} \rangle = \frac{1}{h+1} \langle \overline{\lambda}, \overline{\lambda + 2\rho} \rangle.$$

Proof. Some care is needed when we compute the norm squared of $\mu \in \mathfrak{sl}_n(\lambda)$—we wish to compute the norm squared of the *normalized* weight $\overline{\mu}$. Of course, there is a simple relationship between the norm squared of μ and of $\overline{\mu}$: $\langle \overline{\mu}, \overline{\mu} \rangle = \langle \mu, \mu \rangle - \frac{m^2}{n}$, where $m = \langle \mu, [1]^n \rangle$ (constant for all $\mu \in \mathfrak{sl}_n(\lambda)$). We may therefore compute with unnormalized weights using the relationship

$$\frac{1}{\dim(\mathfrak{sl}_n(\lambda))} \sum_{\mu \in \mathfrak{sl}_n(\lambda)} n_\lambda^\mu \langle \overline{\mu}, \overline{\mu} \rangle = -\frac{m^2}{n} + \frac{1}{\dim(\mathfrak{sl}_n(\lambda))} \sum_{\mu \in \mathfrak{sl}_n(\lambda)} n_\lambda^\mu \langle \mu, \mu \rangle.$$

Define n new partitions

$$\lambda^{(i)} = [\lambda_1 + 1 \geq \lambda_2 + 1 \geq \cdots \geq \lambda_{i-1} + 1 \geq \lambda_{i+1} \geq \cdots \geq \lambda_n] \text{ for } 1 \leq i \leq n.$$

Using the fact that Schur polynomials in the variables x_i are symmetric, conditioning on which boxes of λ contain the entry n, and using the Pieri rule allows us to write

$$\frac{1}{\dim(\mathfrak{sl}_n(\lambda))}\sum_{\mu\in\mathfrak{sl}_n(\lambda)} n_\lambda^\mu \langle\overline{\mu},\overline{\mu}\rangle = -\frac{m^2}{n}+$$

$$+\overbrace{\frac{n}{\dim(\mathfrak{sl}_n(\lambda))}}^{\text{Schur polynomial symmetry}}\sum_{j=1}^{n}\overbrace{(-1)^{j+1}}^{\text{inclusion-exclusion}}\Bigg(\underbrace{s_{\lambda^{(j)}}([1]^{n-1})\sum_{i=0}^{\lambda_j-(j-1)}h_i([1]^{n-1})}_{\text{Pieri rule; leftover boxes will contain } n}\underbrace{\left(\lambda_j-(j-1)-i\right)^2}_{\text{contribution to norm of boxes containing } n}\Bigg), \quad (4.1)$$

where the alternating sum reflects an inclusion-exclusion argument that removes the over-count of those partitions that aren't contained in λ.

We have the evaluations of the Schur and homogeneous functions at $x_i = 1$

$$s_\lambda([1]^n) = \prod_{1\le i<j\le n}\frac{\lambda_i-\lambda_j+j-i}{j-i} \text{ and } h_i([1]^n) = \binom{n+i-1}{i}. \quad (4.2)$$

Dividing by $\dim(\mathfrak{sl}_n(\lambda)) = s_\lambda([1]^n)$, using the formulas above, performing the obvious cancellations, and explicitly evaluating the sums, we obtain

$$\frac{1}{\dim(\mathfrak{sl}_n(\lambda))}\sum_{\mu\in\mathfrak{sl}_n(\lambda)} n_\lambda^\mu \langle\overline{\mu},\overline{\mu}\rangle =$$

$$= -\frac{m^2}{n} + n!\sum_{j=1}^{n}\left(\prod_{\substack{1\le i\le n\\ i\neq j}}\frac{1}{\lambda_j-\lambda_i+i-j}\right)\sum_{i=0}^{\lambda_j-(j-1)}\binom{n+i-2}{i}\left(\lambda_j-(j-1)-i\right)^2$$

$$= -\frac{m^2}{n} + \frac{n!}{n+1}\sum_{j=1}^{n}\left(\prod_{\substack{1\le i\le n\\ i\neq j}}\frac{1}{\lambda_j-\lambda_i+i-j}\right)\binom{n+\lambda_j-j}{\lambda_j-j}\left(n+2(\lambda_j-j)+1\right). \quad (4.3)$$

On the other hand,

$$\frac{1}{h+1}\langle\overline{\lambda},\overline{\lambda+2\rho}\rangle = \frac{1}{n+1}\left\langle \lambda-\frac{m}{n}[1]^n, \lambda-\frac{m}{n}[1]^n+2\rho-(n-1)[1]^n\right\rangle \quad (4.4)$$

$$= -\frac{m^2-mn+n^2}{n(n+1)} + \frac{1}{n+1}\left(\sum_{i=1}^{n}\lambda_i^2+2(n-i)\lambda_i\right) \quad (4.5)$$

Setting Equations (4.3) and (4.5) equal, multiplying by $(n+1)$, pushing the constants to one side, and writing

$$x_j = \lambda_j - j \text{ and } P(x_j) = \big(n + 2x_j + 1\big) \prod_{i=1}^{n} (x_j + i),$$

we must check that

$$2\left(\sum_{1\leq i\leq j\leq n} x_i x_j\right) + (1+n)^2 \left(\sum_{i=1}^{n} x_i\right) + \frac{n(n+1)(3n^2+5n+4)}{12} \tag{4.6}$$

$$= \sum_{j=1}^{n} \left(\prod_{\substack{1\leq i\leq n\\ i\neq j}} \frac{1}{x_j - x_i}\right) P(x_j). \tag{4.7}$$

Treating the x_i as formal variables, Equation (4.6) is clearly a symmetric polynomial of degree 2 in the x_j. For any $1 \leq i < j \leq n$, the ith and jth terms of the sum in Equation (4.6) are the only two terms with $x_i - x_j$ in the denominator—and the sum of these terms is multiplied by $P(x_j) - P(x_i)$. This is true for any i and j, and so all residues cancel. We conclude that since $P(x_j)$ has degree $n+1$, Equation (4.7) is also a symmetric polynomial in the x_j of degree at most 2. It remains to confirm that these are the same degree 2 symmetric function.

Setting $x_i = -i$ for $1 \leq i \leq n-2$, every term except the last two vanish in Equation (4.7). These remaining terms simplify to

$$\begin{aligned}
&\left(\frac{n+2x_n+1}{x_n - x_{n-1}} \frac{\prod_{i=1}^{n} x_n + i}{\prod_{i=1}^{n-2} x_n + i}\right) + \left(\frac{n+2x_{n-1}+1}{x_{n-1} - x_n} \frac{\prod_{i=1}^{n} x_{n-1} + i}{\prod_{i=1}^{n-2} x_{n-1} + i}\right)\\
&= \frac{(n+2x_n+1)\,(x_n+n-1)(x_n+n) - (n+2x_{n-1}+1)\,(x_{n-1}+n-1)(x_{n-1}+n)}{x_n - x_{n-1}}\\
&= 2\left(x_{n-1}^2 + x_{n-2}^2\right) + 2x_n x_{n-1} + (5n-1)(x_n + x_{n-1}) + 4n^2 - n - 1,
\end{aligned}$$

which proves that the coefficients of x_i^2 and $x_i x_j$ in Equation (4.7) agree with those in Equation (4.6):

$$\sum_{j=1}^{n} \left(\prod_{\substack{1\leq i\leq n\\ i\neq j}} \frac{1}{x_j - x_i}\right) P(x_j) = 2\left(\sum_{1\leq i\leq j\leq n} x_i x_j\right) + C_1 \left(\sum_{i=1}^{n} x_i\right) + C_0. \tag{4.8}$$

We now determine C_1. Setting $x_i = -i$ for $1 \leq i \leq n$ makes every term in Equation (4.7) vanish; setting $x_i = -i + 1$ leaves only the first term, which simplifies to $n(n+1)$. Specializing x_i to these values in Equation (4.8) and subtracting, we obtain

$$\left(\frac{n(n+1)(n+2)(3n+1)}{12} - \frac{n(n+1)}{2}C_1 + C_0\right) - \left(\frac{(n-1)n(n+1)(3n-2)}{12} - \frac{(n-1)n}{2}C_1 + C_0\right) = n^3 + n^2 - nC_1.$$

Equating $n^3 + n^2 - nC_1 = 0 - n(n+1)$, we obtain $C_1 = (n+1)^2$.

Again setting $x_i = -i$ for $1 \leq i \leq n$ so that Equation (4.7) is 0, we finally determine that $C_0 = \frac{n(n+1)(3n^2+5n+4)}{12}$ by computing

$$\left(2\left(\sum_{1\leq i\leq j\leq n} ij\right) - (n+1)^2\left(\sum_{i=1}^{n} i\right) + C_0\right) = -\frac{n(n+1)(3n^2+5n+4)}{12} + C_0 = 0.$$

□

We omit the proof for $\mathfrak{sp}_{2n}$ in this abstract.

5 Open problems

- The problem of determining higher moments of the norm squared is open.
- By the Harish-Chandra isomorphism, $\mathfrak{g}$ has n Casimir elements. Since the Casimir elements live in the center of $U(\mathfrak{g})$, they generalize the degree two Casimir by acting as a scalar on any highest weight representation $\mathfrak{g}(\lambda)$. These should be related to the expectations arising from evaluating other W-symmetric functions besides the norm on the weights of $\mathfrak{g}(\lambda)$.
- More speculatively, since Schubert polynomials generalize Schur functions, one could ask for the expected sum of squares of exponents in a Schubert polynomial.
- Generally, it seems reasonable to investigate the expected moments of the exponents of generating functions with product formulas.

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