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Prism Tableaux for Alternating Sign Matrix Varieties

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Abstract. A prism tableau is a set of reverse semistandard tableaux, each positioned within an ambient grid. Prism tableaux were introduced in joint work with A. Yong to provide a formula for the Schubert polynomials of A. Lascoux and M.-P. Schützenberger. This formula directly generalizes the well known expression for Schur polynomials as a sum over semistandard tableaux. Alternating sign matrix varieties generalize the matrix Schubert varieties of W. Fulton. We use prism tableaux to give a formula for the multidegree of an alternating sign matrix variety.

Keywords: Schubert polynomials, Alternating sign matrices, Prism tableaux

1 Introduction

An **alternating sign matrix** (ASM) is a square matrix with entries in $\{-1, 0, 1\}$ so that the nonzero entries in each row and column sum to 1 and alternate in sign. Let $\mathsf{ASM}(n)$ be the set of $n \times n$ ASMs. The enumeration of ASMs has drawn much interest, the sequence for $n \geq 1$ being

$$1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, \ldots.$$

There is a closed form expression for this sequence; the celebrated *alternating sign matrix conjecture* of W. H. Mills–D. P. Robbins–H. Rumsey [17] asserts that

$$|\mathsf{ASM}(n)| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

The original proof was given by D. Zeilberger [19]. G. Kuperberg gave a second proof using the six-vertex model of statistical mechanics [12]. See *Proofs and Confirmations: The Story of the Alternating-Sign Matrix Conjecture*, by D. Bressoud, for links between ASMs and hypergeometric series, plane partitions, and lattice paths [3].

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Fix $A = (a_{ij})_{i,j=1}^n \in \mathsf{ASM}(n)$. The **corner sum function** is $r_A(i,j) = \sum_{k=1}^{i}\sum_{\ell=1}^{j} a_{k\ell}$. Corner sums define a lattice structure on $\mathsf{ASM}(n)$; say

$$A \leq B \text{ if and only if } r_A(i,j) \geq r_B(i,j) \text{ for all } 1 \leq i,j \leq n. \tag{1.1}$$

Restricted to permutation matrices, (1.1) is the **Bruhat order** on the symmetric group $\mathcal{S}_n$. A. Lascoux and M.-P. Schützenberger showed that $\mathsf{ASM}(n)$ is the smallest lattice which contains $\mathcal{S}_n$ as an order embedding [15, Lemma 5.4].

A permutation w is **Grassmannian** if it has a unique descent, i.e. a position i so that $w(i) > w(i+1)$. A permutation u is **biGrassmannian** if both u and u^{-1} are Grassmannian. A. Lascoux and M.-P. Schützenberger showed that biGrassmannian permutations are the *basic elements* of $\mathcal{S}_n$, and hence $\mathsf{ASM}(n)$.

A **partition** $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a weakly decreasing sequence of nonnegative integers. Each λ_i is a **part** of lambda. The **length** $\ell(\lambda)$ is the number of positive parts of λ. Each partition λ has an associated (French) **Young diagram** which consists of left justified boxes with λ_1 boxes in the bottom row, λ_2 in the next, and so on.

A **reverse semistandard tableau** is a filling of the Young diagram of λ with positive integers so that labels weakly decrease within rows (from left to right) and strictly decrease (from bottom to top) within columns. Write $\mathtt{RSSYT}(\lambda, d)$ for the set of reverse semistandard fillings of λ which use labels from the set $[d] := \{1, 2, \ldots, d\}$.

Fix tuples of partitions and positive integers

$$\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(k)}) \text{ and } \mathbf{d} = (d_1, \ldots, d_k) \text{ so that } d_i \geq \ell(\lambda^{(i)}) \text{ for all } i. \tag{1.2}$$

Let

$$\mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d}) = \mathtt{RSSYT}(\lambda^{(1)}, d_1) \times \cdots \times \mathtt{RSSYT}(\lambda^{(k)}, d_k).$$

An element of $\mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$ is called a **prism tableau**. We associate to each $(\boldsymbol{\lambda}, \mathbf{d})$ an ASM, denoted $A_{\boldsymbol{\lambda},\mathbf{d}}$, which is the least upper bound of a list of Grassmannian permutations (see (2.2)). Conversely, for any $A \in \mathsf{ASM}(n)$, there exists some $(\boldsymbol{\lambda}, \mathbf{d})$ so that $A = A_{\boldsymbol{\lambda},\mathbf{d}}$. See Section 2 for details.

For the discussion which follows, it is not enough to think of a prism tableau as a tuple of reverse semistandard tableaux. Rather, we position each of the component tableaux in the $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ grid. We use matrix coordinates to refer boxes in the grid; (i,j) indicates the box in the ith row (from the top) and jth column (from the left) of the grid. The ith **antidiagonal** of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ consists of the boxes

$$\{(i,1), (i-1,2), \ldots, (1,i)\}.$$

We identify the shape of each $\lambda^{(i)}$ with

$$\lambda^{(i)} = \{(a,b) : b \leq \lambda^{(i)}_{d_i - a + 1}\} \subseteq \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}. \tag{1.3}$$

The **prism shape** for $(\boldsymbol{\lambda}, \mathbf{d})$ is obtained by overlaying the $\lambda^{(i)}$'s:

$$\mathbb{S}(\boldsymbol{\lambda}, \mathbf{d}) := \bigcup_{i=1}^{k} \{(a,b) : b \leq \lambda^{(i)}_{d_i - a + 1}\}. \tag{1.4}$$

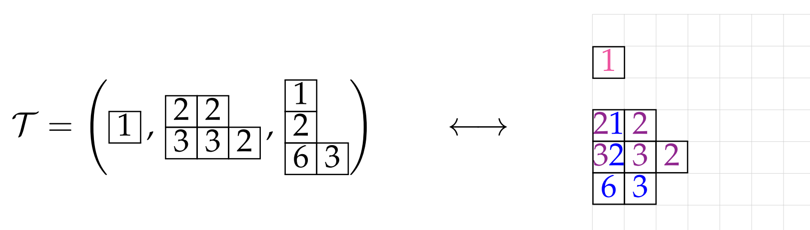
From this perspective, a prism tableau for $(\boldsymbol{\lambda}, \mathbf{d})$ is a filling of $\mathbb{S}(\boldsymbol{\lambda}, \mathbf{d})$ which assigns a label of color i from $[d_i]$ to each $(a,b) \in \lambda^{(i)}$ so that labels of color i weakly decrease along rows from left to right and strictly decrease along columns from bottom to top. Such fillings are in immediate bijection with $\texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$.

Weight $\mathcal{T}$ as follows:

$$\texttt{wt}(\mathcal{T}) = \prod_{i=1}^{\infty} x_i^{n_i}$$

where n_i is the number of antidiagonals which contain the label i (in any color).

Example 1.1. Let $\boldsymbol{\lambda} = ((1),(3,2),(2,1,1))$ and $\mathbf{d} = (2,5,6)$. Below, we give an example of $\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$.



The corresponding weight monomial is $\texttt{wt}(\mathcal{T}) = x_1^2 x_2^3 x_3^3 x_6$. □

Let $\texttt{deg}(\boldsymbol{\lambda}, \mathbf{d}) = \min\{\texttt{deg}(\texttt{wt}(\mathcal{T})) : \mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})\}$. Say $\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$ is **minimal** if $\texttt{deg}(\texttt{wt}(\mathcal{T})) = \texttt{deg}(\boldsymbol{\lambda}, \mathbf{d})$. Let ℓ_c be a label ℓ of color c. Labels $\{\ell_c, \ell_d, \ell'_e\}$ in the same antidiagonal form an **unstable triple** if $\ell < \ell'$ and replacing the ℓ_c with ℓ'_c gives a prism tableau. For instance, in Example 1.1, there is an unstable triple in the fifth antidiagonal; the blue 2 may be replaced with a 3. Write

$$\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d}) = \{\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d}) : \mathcal{T} \text{ is minimal and has no unstable triples}\}. \tag{1.5}$$

Let $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}} = \sum_{\mathcal{T} \in \texttt{Prism}(\boldsymbol{\lambda},\mathbf{d})} \texttt{wt}(\mathcal{T})$. Call $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}}$ an **ASM polynomial**.

If $\boldsymbol{\lambda} = (\lambda)$ and $\mathbf{d} = (d)$, the polynomial $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}}$ is the **Schur polynomial** $s_\lambda(x_1, \ldots, x_d)$. This follows immediately from the usual definition of s_λ as a weighted sum over *semistandard tableaux*. The *Schubert polynomials* $\{\mathfrak{S}_w : w \in \mathcal{S}_\infty\}$ of A. Lascoux and M.-P. Schützenberger [14] generalize Schur polynomials. The purpose of [18] was to provide a prism formula for Schubert polynomials. We prove the following generalization.

Theorem 1.2. $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}} = \sum_{w \in \texttt{MinPerm}(A_{\boldsymbol{\lambda},\mathbf{d}})} \mathfrak{S}_w.$

Here, $\texttt{MinPerm}(A)$ denotes the set permutations above A which have the minimum possible Coxeter length. Our proof of Theorem 1.2 is purely combinatorial; we give a bijection between $\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$ and the set of maximum dimensional facets in a union of the *subword complexes* of [11]. The Schubert polynomial is a weighted sum over the facets of its corresponding subword complex [1, 5, 10].

$\mathfrak{A}_{\lambda,\mathbf{d}}$ also has a geometric interpretation; it is the *multidegree* of an *alternating sign matrix variety*. Write $\mathsf{Mat}(n)$ for the space of $n \times n$ matrices over an algebraically closed field $\Bbbk$. Given $M \in \mathsf{Mat}(n)$, let $M_{[i],[j]}$ be the submatrix of M which consists of the first i rows and j columns of M. We define the **alternating sign matrix variety**

$$X_A := \{M \in \mathsf{Mat}(n) : \operatorname{rank}(M_{[i],[j]}) \leq r_A(i,j) \text{ for all } 1 \leq i,j \leq n\}. \tag{1.6}$$

If $w \in \mathcal{S}_n$, then X_w is a **matrix Schubert variety** as defined in [7].

ASM varieties are stable under multiplication by the group of invertible, diagonal matrices $\mathsf{T} \subset \mathsf{GL}(n)$. There is a corresponding $\mathbb{Z}^n$ grading and multidegree

$$\mathcal{C}(X_A; \mathbf{x}) \in \mathbb{Z}[x_1, \ldots, x_n].$$

Whenever $w \in \mathcal{S}_n$, we have $\mathfrak{S}_w = \mathcal{C}(X_w; \mathbf{x})$. This was shown in [10] and is equivalent to earlier statements in the language of equivariant cohomology [4] and degeneracy loci [7]. We show $\mathfrak{A}_{\lambda,\mathbf{d}}$ is the multidegree of the ASM variety $X_{A_{\lambda,\mathbf{d}}}$.

Theorem 1.3. *Fix λ and $\mathbf{d}$ as in (1.2). Then $\mathcal{C}(X_{A_{\lambda,\mathbf{d}}}; \mathbf{x}) = \mathfrak{A}_{\lambda,\mathbf{d}}$.*

The irreducible components of X_A are always matrix Schubert varieties. Theorem 1.3 follows from Theorem 1.2 and the additivity of multidegrees.

We also discuss the explicit connection of prism tableaux to the Gröbner geometry of X_A. Let $Z = (z_{ij})_{i,j=1}^n$ be the generic $n \times n$ matrix. Define the **ASM ideal** by

$$I_A := \langle \text{minors of size } r_A(i,j) + 1 \text{ in } Z_{[i],[j]} \rangle. \tag{1.7}$$

It is immediate that I_A provides set-theoretic equations for X_A. In fact, for any $A \in \mathsf{ASM}(n)$, we have that I_A is radical. This follows from the Frobenius splitting argument given in [9, Section 7.2]. We make the connection to ASM varieties explicit.

Proposition 1.4 ([9]). *Fix any antidiagonal term order $\prec$ on $\Bbbk[Z]$.*

1. *The essential (and hence defining) generators of I_A form a Gröbner basis under $\prec$.*
2. *I_A is radical and its initial ideal is a square-free monomial ideal.*
3. *The Stanley–Reisner complex of $\texttt{init}(X_A)$ is $\Delta(Q_{n\times n}, A)$.*

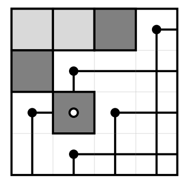
Since $\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$ is in weight preserving bijection with the facets of maximum dimension in $\Delta(Q_{n\times n}, A)$, this yields a second proof of Theorem 1.3.

2 Combinatorial Prism Models

We start by presenting a generalization of Rothe diagrams to ASMs. Plot $A \in \mathsf{ASM}(n)$ in the $n \times n$ grid by placing a black dot for each 1 in A and a white dot for each -1. Strike out hooks to the right and below each black dot which stop if they encounter the boundary of a box which contains a white dot. The boxes which remain form the **Rothe diagram** $D(A)$. Equivalently, $(i, j) \in D(A)$ if and only if (i, j) is an *inversion* of A (see [17] for this definition). The **essential set** $\mathcal{E}ss(A)$ consists of the southeast most corners of each connected component of $D(A)$.

Example 2.1.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad D(A) =$$



The boxes of the diagram of A are shaded gray. The essential boxes are dark gray. □

Notice that $D(A)$ is similar to the ASM diagram defined by A. Lascoux [13]. However, our conventions on inversions differ; we include the set of *negative inversions*. If w is a permutation matrix, $D(w)$ and $\mathcal{E}ss(w)$ coincide with the usual Rothe diagram and essential set, as defined in [7]. Any permutation is uniquely determined by the restriction of the corner sum function to its essential set [7, Lemma 3.10]. The same statement holds more generally for ASMs, see Proposition 2.2.

BiGrassmannian permutations in $\mathcal{S}_n$ are naturally labeled by triples of integers (i, j, r) which satisfy the following conditions:

(B1) $1 \leq i, j$

(B2) $0 \leq r < \min(i, j)$

(B3) $i + j - r \leq n$.

Let I_k denote the $k \times k$ identity matrix. Then we write

$$[i, j, r]_b := \left(\begin{array}{c|c|c|c} I_r & & & \\ \hline & & I_{i-r} & \\ \hline & I_{j-r} & & \\ \hline & & & I_{n-i-j+r} \end{array}\right) \tag{2.1}$$

for the (unique) biGrassmannian encoded by this triple. In the case $r = \min(i, j)$, let $[i, j, r]_b$ be the identity permutation.

Let $\mathtt{biGr}(A)$ be the set of maximal biGrassmannians below A in the lattice $\mathsf{ASM}(n)$. Then $A = \vee \mathtt{biGr}(A)$. The next proposition shows how to recover $\mathtt{biGr}(A)$ from $\mathcal{E}ss(A)$.

Proposition 2.2. $\texttt{biGr}(A) = \{[i,j,r_A(i,j)]_b : (i,j) \in \mathcal{E}ss(A)\}$.

Proposition 2.2 is discussed in [15, Section 5], using essential points of monotone triangles. It appears in a more general context in [6, Theorem 5.1]. As a consequence, A is determined by the restriction of r_A to $\mathcal{E}ss(A)$. This generalizes [7, Lemma 3.10].

Left justifying the Rothe diagram of a Grassmannian permutation u produces the (French) Young diagram of partition $\lambda^{(u)}$. If $d \geq \ell(\lambda)$, write $[\lambda, d]_g$ for the (unique) Grassmannian with descent at position d and associated partition λ. If $\lambda = ()$, then for any d we say $[\lambda, d]_g = \mathrm{id}$. Then given $(\boldsymbol{\lambda}, \mathbf{d})$ as in (1.2), we define

$$A_{\boldsymbol{\lambda},\mathbf{d}} = \vee\{[\lambda^{(1)}, d_1]_g, \ldots, [\lambda^{(k)}, d_k]_g\}. \tag{2.2}$$

We now describe two ways of taking $A \in \mathsf{ASM}(n)$ as a input and producing a pair $(\boldsymbol{\lambda}, \mathbf{d})$ so that $A = A_{\boldsymbol{\lambda},\mathbf{d}}$. Both procedures are entirely combinatorial. We start with BiGrassmannian prism tableaux, which were defined in [18].

Definition 2.3 (BiGrassmannian Prism Tableaux). Suppose

$$\mathcal{E}ss(A) = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}.$$

Let

$$\beta^{(\ell)} = (i_\ell - r_A(i_\ell, j_\ell)) \times (j_\ell - r_A(i_\ell, j_\ell)). \tag{2.3}$$

Define $\boldsymbol{\beta}_A = (\beta^{(1)}, \ldots, \beta^{(k)})$ and $\mathbf{b}_A = \{i_1, \ldots, i_k\}$. Elements of $\texttt{AllPrism}(\boldsymbol{\beta}_A, \mathbf{b}_A)$ are **biGrassmannian** prism tableaux.

We now introduce the parabolic prism model. Our definition uses the *monotone triangles* of W. H. Mills, D. P. Robbins, and H. Rumsey [17]. Given $A = (a_{ij})_{i,j=1}^n \in \mathsf{ASM}(n)$ let C_A be the matrix of partial column sums, i.e. $C_A(i,j) = \sum_{\ell=1}^i a_{\ell j}$. The ith row of m_A records (in increasing order) the positions of the 1s in the ith row of C_A. The array m_A is called a **monotone triangle**.

Example 2.4.

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad C_A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad m_A = \begin{matrix} & & & 3 & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{matrix}$$

□

Given A and $1 \leq \ell \leq n$, we define

$$\lambda^{(A,\ell)} = (m_A(\ell,\ell) - \ell, m_A(\ell,\ell-1) - (\ell-1), \ldots, m_A(\ell,1) - 1). \tag{2.4}$$

Since m_A strictly increases along rows, $\lambda^{(A,\ell)}$ is a partition.

Definition 2.5 (Parabolic Prism Tableaux). Write $\{i : (i,j) \in \mathcal{E}ss(A)\} = \{i_1, \ldots, i_k\}$ for the indices of essential rows of A. Let

$$\boldsymbol{\rho}_A = (\lambda^{(A,i_1)}, \lambda^{(A,i_2)}, \ldots, \lambda^{(A,i_k)}) \text{ and } \mathbf{p}_A = (i_1, \ldots, i_k).$$

Elements of $\texttt{AllPrism}(\boldsymbol{\rho}_A, \mathbf{p}_A)$ are **parabolic** prism tableaux.

Proposition 2.6. *1.* $A = A_{\boldsymbol{\beta}_A, \mathbf{b}_A}$.

2. $A = A_{\boldsymbol{\rho}_A, \mathbf{p}_A}$.

3 Subword complexes and prism tableaux

Let $\mathbb{P}(S)$ denote the power set of S. A **simplicial complex** Δ is a subset of $\mathbb{P}([N])$ so that whenever $f \in \Delta$ and $f' \subseteq f$, we have $f' \in \Delta$. An element $f \in \Delta$ is called a **face**. The **dimension** of f is $\dim(f) = |f| - 1$. Write $\dim(\Delta) = \max\{\dim(f) : f \in \Delta\}$. If $f \in \Delta$, the **codimension** of f is $\operatorname{codim}(f) = \dim(\Delta) - \dim(f)$. The set of faces of Δ ordered by inclusion form a poset. Let $F(\Delta)$ denote the set of **facets** of Δ, i.e. the maximal faces. Then define $F_{\texttt{max}}(\Delta) = \{f \in \Delta : \operatorname{codim}(f) = 0\}$. Necessarily, $F_{\texttt{max}}(\Delta) \subseteq F(\Delta)$. When this containment is an equality, Δ is called **pure**. Given two simplicial complexes $\Delta_1, \Delta_2 \subseteq \mathbb{P}([N])$, we may refer without ambiguity to the intersection (or union) of Δ_1 and Δ_2; it is precisely their intersection (or union) as sets. A straightforward verification shows that $\Delta_1 \cap \Delta_2$ and $\Delta_1 \cup \Delta_2$ are themselves simplicial complexes.

We now recall the definition of a subword complex, following [11]. Let Π be a Coxeter group generated by simple reflections Σ. A **word** is an ordered list $Q = (s_1, \ldots, s_m)$ of simple reflections in Σ. A **subword** of Q is an ordered subsequence $P = (s_{i_i}, \ldots, s_{i_k})$. Subwords of Q are naturally identified with faces of the simplicial complex $\mathbb{P}([m])$. A word P **represents** $w \in \Pi$ if $w = s_1 \cdots s_m$ and $\ell(w) = m$, i.e. the ordered product is a reduced expression for w. We say Q **contains** w if Q has a subword which represents w. Then define the **subword complex** $\Delta(Q, w) = \{Q - P : P \text{ contains } w\}$. We will abbreviate $\mathcal{F}_P := Q - P$. Immediately by definition, $\mathcal{F}_P \subseteq \mathcal{F}_{P'}$ if and only if $P \supseteq P'$. A well known characterization of the Bruhat order on $\mathcal{S}_n$ is via subwords:

$$w \geq v \text{ if and only if some (and hence every) reduced word for } w \text{ contains } v. \quad (3.1)$$

See [8, Section 5.10]. This is equivalent to the order on $\mathcal{S}_n$ as defined in (1.1). See [2, Theorem 2.1.5] for a proof.

The **Demazure algebra** of (Π, Σ) over a ring R is freely generated by $\{e_w : w \in \Pi\}$ with multiplication given by

$$e_w e_s = \begin{cases} e_{ws} & \text{if } \ell(ws) > \ell(w) \\ e_w & \text{if } \ell(ws) < \ell(w). \end{cases}$$

If $Q = (s_1, \dots, s_k)$, the **Demazure product** $\delta(Q)$ is defined by $e_{s_1} \cdots e_{s_m} = e_{\delta(Q)}$. The faces of $\Delta(Q, w)$ have a natural description in terms of the Demazure product.

Lemma 3.1 ([11, Lemma 3.4]). *$\delta(P) \geq w$ if an only if P contains w.*

Notice $\Delta(Q, w) = \{\mathcal{F}_P : \delta(P) \geq w\}$. This motivates the following definition. Let $\Pi = \mathcal{S}_n$ and $\Sigma = \{(i, i+1) : i = 1, \dots, n-1\}$ be the set of simple transpositions. Given $A \in \mathsf{ASM}(n)$, define $\Delta(Q, A) = \{\mathcal{F}_P : \delta(P) \geq A\}$. This is itself a simplicial complex, but need not be a subword complex. Immediately from the definition,

$$\text{if} \quad A \geq B \quad \text{then} \quad \Delta(Q, A) \subseteq \Delta(Q, B). \tag{3.2}$$

We will show that $\Delta(Q, A)$ is a union of subword complexes. In particular, if $A \in \mathsf{ASM}(m)$ with $m \leq n$ each of these subword complexes correspond to permutations in $\mathcal{S}_m$. Write $\mathtt{Min}(S)$ for the minimal elements in S. Let $\mathtt{Perm}(A) := \mathtt{MIN}(\{w \in \mathcal{S}_n : w \geq A\})$ and

$$\mathtt{MinPerm}(A) := \{w \in \mathtt{Perm}(A) : \ell(w) = \mathtt{deg}(A)\}. \tag{3.3}$$

Proposition 3.2. *Fix a word Q and $A \in \mathsf{ASM}(n)$.*

1. $\Delta(Q, A) = \bigcup_{w \in \mathtt{Perm}(A)} \Delta(Q, w).$
2. *If* $A = \vee\{A_1, \dots, A_k\}$ *then* $\Delta(Q, A) = \bigcap_{i=1}^{k} \Delta(Q, A_i).$
3. $F(\Delta(Q, A)) = \{\mathcal{F}_P : P$ *is a reduced expression for some* $w \in \mathtt{Perm}(A)\}.$

For the rest of this section, we focus on a fixed ambient word Q. Write s_i for the simple transposition $(i, i+1) \in \mathcal{S}_{2n}$. Define the **square word**

$$Q_{n\times n} = s_n \, s_{n-1} \cdots s_1 \quad s_{n+1} \, s_n \cdots s_2 \quad \cdots \quad s_{2n-1} \, s_{2n-2} \cdots s_n.$$

Order the boxes of the $n \times n$ grid by reading along rows from right to left, starting with the top row and working down to the bottom. This ordering identifies each letter of $Q_{n\times n}$ with a cell in the $n \times n$ grid.

A **plus diagram** is a subset of the $n \times n$ grid. We indicate (i, j) is in the plus diagram by marking its position in the grid with a $+$. The identification of the letters in $Q_{n\times n}$ with the grid defines a natural bijection between subwords of $Q_{n\times n}$ and plus diagrams. As such, we freely identify each word with its plus diagram.

Example 3.3. When $n = 3$, we have $Q_{n\times n} = s_3 s_2 s_1 s_4 s_3 s_2 s_5 s_4 s_3$. Below, we label the entries of the 3×3 grid with their corresponding simple transpositions. We also give a subword of $Q_{3\times 3}$ and its corresponding plus diagram.

$$\begin{matrix} s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \\ s_3 & s_4 & s_5 \end{matrix} \qquad s_3 - - - s_3 \, s_2 - - - \qquad \begin{matrix} \cdot & \cdot & + \\ + & + & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

Notice that P is not a reduced expression, $s_3s_3s_2 = s_2$. Therefore, it is not a facet of $\Delta(Q_{n\times n}, A)$ for any $A \in \mathsf{ASM}(n)$. □

For brevity, write $\Delta_A := \Delta(Q_{n\times n}, A)$. Assign $\mathcal{F}_P$ the weight $\mathtt{wt}(\mathcal{F}_P) = \prod_{i=1}^{n} x_i^{n_i}$ where $n_i = |\{j : (i,j) \in P\}|$. When $w \in \mathcal{S}_n$, the complex Δ_w is a pure simplicial complex. Its facets are in immediate bijection with pipe dreams (also known as RC-graphs).

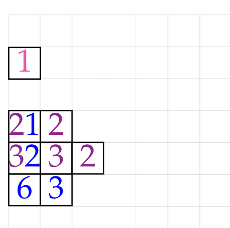
Theorem 3.4 ([1, 5, 10]). $\mathfrak{S}_w = \sum_{\mathcal{F}_P \in F(\Delta_w)} \mathtt{wt}(\mathcal{F}_P)$.

For permutations, Δ_w is the Stanley–Reisner complex of a degeneration of the Schubert determinantal ideal I_w [10, Theorem B]. The same holds for I_A and Δ_A. As a consequence of Theorem 3.4, we have the following corollary.

Corollary 3.5. *1.* $\sum_{w \in \mathtt{Perm}(A)} \mathfrak{S}_w = \sum_{\mathcal{F}_P \in F(\Delta_A)} \mathtt{wt}(\mathcal{F}_P)$.

2. $\sum_{w \in \mathtt{MinPerm}(A)} \mathfrak{S}_w = \sum_{\mathcal{F}_P \in F_{\mathtt{max}}(\Delta_A)} \mathtt{wt}(\mathcal{F}_P)$.

There is a natural map from $\mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$ to the set of plus diagrams: if $\mathcal{T}$ has the label i in an antidiagonal, place a plus in row i of this same antidiagonal. For instance, continuing Example 1.1, we have

$\mathcal{T} =$  $\mapsto$

$$P_{\mathcal{T}} = \begin{array}{ccccccc} \cdot & + & \cdot & + & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & + & \cdot & + & \cdot \\ \cdot & \cdot & + & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Our proof of Theorem 1.2 involves showing that the restriction of this map to $\mathtt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$ is a bijection onto the maximum dimensional facets of $\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}}$.

4 ASM varieties and determinantal ideals

Recall $\mathsf{Mat}(n)$ is the space of $n \times n$ matrices over an algebraically closed field $\Bbbk$. Write $\mathsf{GL}(n)$ for the invertible matrices in $\mathsf{Mat}(n)$ and T for the torus of diagonal matrices in $\mathsf{GL}(n)$. There is a natural action of $\mathsf{GL}(n)$, and hence T, on $\mathsf{Mat}(n)$ by left multiplication. A variety $X \subseteq \mathsf{Mat}(n)$ is T **stable** if $\mathsf{T}X \subseteq X$.

Recall $Z = (z_{ij})_{i,j=1}^n$. Write $\Bbbk[Z] = \Bbbk[z_{11}, z_{12}, \ldots, z_{nn}]$ for the coordinate ring of $\mathsf{Mat}(n)$. There is a *multigrading* defined by the action T on $\mathsf{Mat}(n)$. In particular, T stable subvarieties of $\mathsf{Mat}(n)$ have coordinate rings that are $\Bbbk[Z]$-graded modules and define a *multidegree*. Multidegrees are characterized by three properties, *normalization*, *additivity*, and *degeneration* [16, Theorem 8.44].

Fix $X \subseteq \mathsf{Mat}(n)$ with X being T stable. If $\Bbbk[Z]/I$ is the coordinate ring of X, write $\mathcal{C}(X; \mathbf{x}) := \mathcal{C}(\Bbbk[Z]/I; \mathbf{x})$ for its multidegree. By additivity, $\mathcal{C}(X; \mathbf{x}) = \sum_{i=1}^k \mathcal{C}(X_i; \mathbf{x})$ where $\{X_1, \ldots, X_k\}$ are the maximal dimensional irreducible components of X. Since I is radical, this is a multiplicity free sum.

Given an $n \times n$ matrix M, write $M_{[i],[j]}$ for the submatrix of M which consists of the first i rows and j columns of M. Fix $w \in \mathcal{S}_n$. The **matrix Schubert variety** is

$$X_w := \{M \in \mathsf{Mat}(n) : \operatorname{rank}(M_{[i],[j]}) \leq r_w(i,j) \ \text{ for all } \ 1 \leq i,j \leq n\}. \tag{4.1}$$

Matrix Schubert varieties generalize classical determinantal varieties. W. Fulton showed that they are irreducible [7]. By [10, Theorem A], when $w \in \mathcal{S}_n$, we have $\mathcal{C}(X_w; \mathbf{x}) = \mathfrak{S}_w$.

We define the **alternating sign matrix variety**

$$X_A := \{M \in \mathsf{Mat}(n) : \operatorname{rank}(M_{[i],[j]}) \leq r_A(i,j) \ \text{ for all } \ 1 \leq i,j \leq n\}.$$

Immediately by definition, if $A \leq B$ then $X_A \supseteq X_B$. Furthermore, X_A has the following set theoretic descriptions as unions and intersections of other ASM varieties.

Proposition 4.1. *1.* $X_A = \bigcup_{w \in \mathtt{Perm}(A)} X_w$.

2. If $A = \vee\{A_1, \ldots, A_k\}$*, then* $X_A = \bigcap_{i=1}^k A_i$.

W. Fulton showed that each X_w is defined by a smaller set of **essential** conditions,

$$X_w = \{M \in \mathsf{Mat}(n) : \operatorname{rank}(M_{[i],[j]}) \leq r_w(i,j) \ \text{ for all } \ (i,j) \in \mathcal{E}ss(w)\}. \tag{4.2}$$

By Proposition 4.1, $X_A = \bigcap_{u \in \mathtt{biGr}(A)} X_u$. Therefore, ASM varieties are defined by essential conditions. The rank of *any* submatrix is preserved under the T action, so X_w is T stable.

Proposition 4.2. $\mathcal{C}(X_A; \mathbf{x}) = \sum_{w \in \mathtt{MinPerm}(A)} \mathfrak{S}_w$.

Proof. As a consequence of Proposition 4.1, the top dimensional irreducible components of X_A are $\{X_w : w \in \mathtt{MinPerm}(A)\}$. Then using the additivity property of multidegrees we have $\mathcal{C}(X_A; \mathbf{x}) = \sum_{w \in \mathtt{MinPerm}(A)} \mathcal{C}(X_w; \mathbf{x}) = \sum_{w \in \mathtt{MinPerm}(A)} \mathfrak{S}_w$. $\square$

Theorem 1.3 follows as an immediate consequence of Proposition 4.2 and Theorem 1.2.

We now turn our discussion to defining ideals for ASM varieties. Recall the **ASM ideal** $I_A := \langle \text{minors of size } r_A(i,j)+1 \text{ in } Z_{[i],[j]} \rangle$. A matrix has rank at most r if and only if all of its minors of size $r+1$ vanish. As such, I_A set-theoretically cuts out X_A. Furthermore, I_A has generators which are homogeneous for the $\mathbb{Z}^n$ grading on $\Bbbk[Z]$. There is a smaller generating set for I_A. Write

$$\texttt{EssGen}(A) = \{\text{minors of size } r_A(i,j)+1 \text{ in } Z_{[i],[j]} : (i,j) \in \mathcal{E}ss(A)\}.$$

for **essential generators** of I_A.

We could have alternatively proved Proposition 4.2 by taking an explicit Gröbner degeneration of I_A with respect to an *antidiagonal term order*. In this case, the Stanley–Reisner complex of I_A is $\Delta(Q_{n\times n}, A)$. Furthermore, if $A = A_{\boldsymbol{\lambda},\mathbf{d}}$, the maximum dimensional facets of $\Delta(Q_{n\times n}, A)$ have a natural labeling by elements of $\texttt{Prism}(\boldsymbol{\lambda},\mathbf{d})$. This produces a specific connection between the Gröbner geometry of X_A and prism tableaux.

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