Séminaire Lotharingien de Combinatoire **80B** (2018)
Article #54, 12 pp.

Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover)

Reverse plane partitions of skew staircase shapes and q-Euler numbers

Byung-Hak Hwang[*1], Jang Soo Kim[†2], Meesue Yoo[‡3], and Sun-mi Yun[§4]

[1] *Department of Mathematics, Seoul National University, Seoul, South Korea*
[2] *Department of Mathematics, Sungkyunkwan University, Suwon, South Korea*
[3] *Applied Algebra and Optimization Research Center, Sungkyunkwan University, Suwon, South Korea*
[4] *Department of Mathematics, Sungkyunkwan University, Suwon, South Korea*

Abstract. Recently, Naruse discovered a hook length formula for the number of standard Young tableaux of a skew shape. Morales, Pak and Panova found two q-analogs of Naruse's hook length formula over semistandard Young tableaux (SSYTs) and reverse plane partitions (RPPs). As an application of their formula, they expressed certain q-Euler numbers, which are generating functions for SSYTs and RPPs of a zigzag border strip, in terms of weighted Dyck paths. They found a determinantal formula for the generating function for SSYTs of a skew staircase shape and proposed two conjectures related to RPPs of the same shape.

In this paper, we show that the results of Morales, Pak and Panova on the q-Euler numbers can be derived from previously known results due to Prodinger by manipulating continued fractions. These q-Euler numbers are naturally expressed as generating functions for alternating permutations with certain statistics involving *maj*. It has been proved by Huber and Yee that these q-Euler numbers are generating functions for alternating permutations with certain statistics involving *inv*. By modifying Foata's bijection we construct a bijection on alternating permutations which sends the statistics involving *maj* to the statistic involving *inv*. We also prove the aforementioned two conjectures of Morales, Pak and Panova.

Keywords: reverse plane partition, Euler number, alternating permutation, lattice path, continued fraction

1 q-Euler numbers and continued fractions

Morales, Pak and Panova [6, Corollaries 1.7 and 1.8] obtained that

$$\frac{E_{2n+1}(q)}{(q;q)_{2n+1}} = \sum_{D\in \mathrm{Dyck}_{2n}} \prod_{(a,b)\in D} \frac{q^b}{1-q^{2b+1}} \tag{1.1}$$

[*] xoda@snu.ac.kr
[†] jangsookim@skku.edu. This work was supported by NRF grants #2016R1D1A1A09917506 and #2016R1A5A1008055.
[‡] meesue.yoo@skku.edu. This work was supported by NRF grants #2016R1A5A1008055 and #2017R1C1B2005653.
[§] sera314@skku.edu

and

$$\frac{E^*_{2n+1}(q)}{(q;q)_{2n+1}} = \sum_{D\in\mathrm{Dyck}_{2n}} q^{H(D)} \prod_{(a,b)\in D} \frac{1}{1-q^{2b+1}}, \tag{1.2}$$

where Dyck_{2n} is the set of *Dyck paths* of length $2n$, $H(D) = \sum_{(a,b)\in\mathcal{HP}(D)}(2b+1)$, $\mathcal{HP}(D)$ is the set of *high peaks* in D,

$$E_n(q) = \sum_{\pi\in\mathrm{Alt}_n} q^{\mathrm{maj}(\pi^{-1})} \quad \text{and} \quad E^*_n(q) = \sum_{\pi\in\mathrm{Alt}_n} q^{\mathrm{maj}(\kappa_n\pi^{-1})}. \tag{1.3}$$

κ_n is the permutation $(1)(2,3)(4,5)\dots(2\lfloor(n-1)/2\rfloor, 2\lfloor(n-1)/2\rfloor+1)$ in cycle notation and $\mathrm{maj}(\pi)$ is the *major index* of π.

Prodinger [7] considered the probability $\tau_n^{\geq\leq}(q)$ that a random word $w_1\dots w_n$ of positive integers of length n satisfies the relations $w_1 \geq w_2 \leq w_3 \geq w_4 \leq \cdots$, where each w_i is chosen independently randomly with probability $\Pr(w_i = k) = q^{k-1}(1-q)$ for $0 < q < 1$. For other choices of inequalities, for example $\geq$ and $<$, the probability $\tau_n^{\geq<}(q)$ is defined similarly. From the definition, one can easily see that

$$\sum_{\pi\in\mathrm{SSYT}(\delta_{n+2}/\delta_n)} q^{|\pi|} = \frac{\tau^{\geq<}_{2n+1}(q)}{(1-q)^{2n+1}}, \tag{1.4}$$

$$\sum_{\pi\in\mathrm{RPP}(\delta_{n+2}/\delta_n)} q^{|\pi|} = \frac{\tau^{\geq\leq}_{2n+1}(q)}{(1-q)^{2n+1}} \tag{1.5}$$

and

$$\sum_{\pi\in\mathrm{ST}(\delta_{n+2}/\delta_n)} q^{|\pi|} = \frac{\tau^{><}_{2n+1}(q)}{(1-q)^{2n+1}}, \tag{1.6}$$

where $\mathrm{ST}(\lambda/\mu)$ is the set of *strict tableaux* of shape λ/μ and a strict tableau of shape λ/μ is a filling of λ/μ with nonnegative integers such that the integers are strictly increasing in each row and each column.

In this section we show (1.1) and (1.2) using Prodinger's results. Prodinger [7] found continued fraction expressions for the generating functions of $\tau^{\geq<}_{2n+1}(q)$ and $\tau^{\geq\leq}_{2n+1}(q)$. Using Flajolet's theory [1] of continued fractions we show that (1.1) is equivalent to Prodinger's continued fraction. We prove (1.2) in a similar fashion. However, unlike (1.1), the weight of a Dyck path in (1.2) is not a usual weight used in Flajolet's theory. To remedy this we first express $E^*_{2n+1}(q)$ as a generating function for weighted Schröder paths and change it to a generating function of weighted Dyck paths.

We recall Flajolet's theory[1] which gives a combinatorial interpretation for the continued fraction expansion as a generating function of weighted Dyck paths.

Let $u = (u_0, u_1, \dots)$, $d = (d_1, d_2, \dots)$ and $w = (w_0, w_1, \dots)$ be sequences satisfying $w_i = u_i d_{i+1}$ for $i \geq 0$. For a Dyck path $P \in \mathrm{Dyck}_{2n}$, we define the weight $\mathrm{wt}_w(P)$ with respect to w to be the product of the weight of each step in P, where the weight of an up step $\{(i,j),(i+1,j+1)\}$ is u_j and the weight of a down step $\{(i,j),(i+1,j-1)\}$

is d_j. Flajolet [1] showed that the generating function for weighted Dyck paths has a continued fraction expansion:

$$\sum_{n\geq 0}\sum_{P\in \mathrm{Dyck}_{2n}} \mathrm{wt}_w(P)x^{2n} = \cfrac{1}{1-\cfrac{w_0x^2}{1-\cfrac{w_1x^2}{1-\cfrac{w_2x^2}{1-\cdots}}}}. \tag{1.7}$$

1.1 The q-Euler numbers $E_{2n+1}(q)$

We give a new proof of (1.1) using (1.7).

Proposition 1.1 ([6, Corollary 1.7]). *We have*

$$\frac{E_{2n+1}(q)}{(q;q)_{2n+1}} = \sum_{P\in \mathrm{Dyck}_{2n}} \prod_{(a,b)\in P} \frac{q^b}{1-q^{2b+1}}. \tag{1.8}$$

Proof. By the result of Prodinger [7, Theorem 4.1] (with replacing z by $x/(1-q)$), we have the following continued fraction expansion:

$$\sum_{n\geq 0} E_{2n+1}(q)\frac{x^{2n+1}}{(q;q)_{2n+1}} = \frac{x}{1-q}\cdot \cfrac{1}{1-\cfrac{qx^2/(1-q)(1-q^3)}{1-\cfrac{q^3x^2/(1-q^3)(1-q^5)}{1-\cfrac{q^5x^2/(1-q^5)(1-q^7)}{1-\cdots}}}}. \tag{1.9}$$

By comparing (1.9) and (1.7) with $u_i = d_i = \frac{q^i}{1-q^{2i+1}}$ and $w_i = u_i d_{i+1}$, we deduce (1.1). □

1.2 The q-Euler numbers $E^*_{2n+1}(q)$

By using Prodinger's result on $E^*_{2n+1}(q)$, we give a new proof of (1.2).

Proposition 1.2 ([6, Corollary 1.8]). *We have*

$$\frac{E^*_{2n+1}(q)}{(q;q)_{2n+1}} = \sum_{P\in \mathrm{Dyck}_{2n}} q^{H(P)} \prod_{(a,b)\in P} \frac{1}{1-q^{2b+1}}.$$

Corollary 1.3. *We have*

$$\sum_{P\in \mathrm{Dyck}_{2n}} q^{H(P)} \prod_{(a,b)\in P} \frac{1}{1-q^{2b+1}} = \frac{1}{1-q}\sum_{P\in \mathrm{Dyck}_{2n}} \mathrm{wt}_w(P),$$

where $w = (w_0, w_1, \dots)$ is the suitable weight sequence.

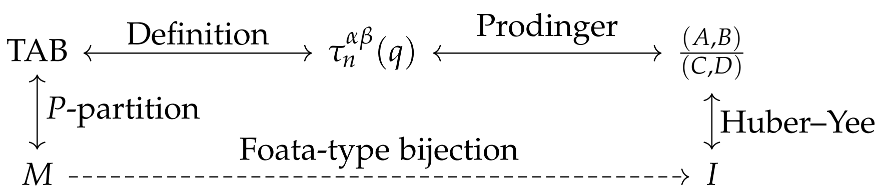


Figure 1: The connections in Theorems 2.1 and 2.2.

2 Prodinger's q-Euler numbers and Foata-type bijections

2.1 Prodinger's q-Euler numbers

Prodinger [7] showed that the generating function for $\tau_n^{\alpha\beta}(q)$ for any choice of alternating inequalities α and β, i.e.,

$$(\alpha,\beta) \in \{(\geq,\leq)\ (\geq,<),(>,\leq),(>,<),(\leq,\geq),(\leq,>),(<,\geq),(<,>)\},$$

has a nice expression as a quotient of series. Observe that we have $\tau_{2n+1}^{\geq<}(q) = \tau_{2n+1}^{>\leq}(q)$, $\tau_{2n+1}^{\leq>}(q) = \tau_{2n+1}^{<\geq}(q)$, $\tau_{2n}^{\geq\leq}(q) = \tau_{2n}^{\leq\geq}(q)$, $\tau_{2n}^{\geq<}(q) = \tau_{2n}^{\leq>}(q)$, $\tau_{2n}^{>\leq}(q) = \tau_{2n}^{<\geq}(q)$ and $\tau_{2n}^{><}(q) = \tau_{2n}^{<>}(q)$. Therefore, we only need to consider 6 q-tangent numbers $\tau_{2n+1}^{\alpha\beta}$ and 4 q-secant numbers $\tau_{2n}^{\alpha\beta}$.

Now we state a unifying theorem for Prodinger's q-tangent numbers combining some results of Huber and Yee [3].

Theorem 2.1. *For each row* $\tau_{2n+1}^{\alpha\beta}(q), \mathrm{TAB}, M, I, (A,B)/(C,D)$ *in Table 1, we have*

$$f_{2n+1} := \frac{\tau_{2n+1}^{\alpha\beta}(q)}{(1-q)^{2n+1}} = \sum_{\pi\in\mathrm{TAB}} q^{|\pi|} = \frac{M}{(q;q)_{2n+1}} = \frac{I}{(q;q)_{2n+1}},$$

whose generating function is

$$\sum_{n\geq 0} f_{2n+1}x^{2n+1} = \frac{\sum_{n\geq 0}(-1)^n q^{An^2+Bn}x^{2n+1}/(q;q)_{2n+1}}{\sum_{n\geq 0}(-1)^n q^{Cn^2+Dn}x^{2n}/(q;q)_{2n}}.$$

By the same arguments, we obtain a unifying theorem for Prodinger's q-secant numbers.

Theorem 2.2. *For each row* $\tau_{2n}^{\alpha\beta}(q), \mathrm{TAB}, M, I, 1/(C,D)$ *in Table 2, we have*

$$f_{2n} := \frac{\tau_{2n}^{\alpha\beta}(q)}{(1-q)^{2n}} = \sum_{\pi\in\mathrm{TAB}} q^{|\pi|} = \frac{M}{(q;q)_{2n}} = \frac{I}{(q;q)_{2n}},$$

$\tau_{2n+1}^{\alpha\beta}(q)$	TAB	M	I	$\frac{(A,B)}{(C,D)}$
$\tau_{2n+1}^{\geq<}(q)$	$\mathrm{SSYT}(\delta_{n+2}/\delta_n)$	$\sum_{\pi\in\mathrm{Alt}_{2n+1}} q^{\mathrm{maj}(\pi^{-1})}$	$\sum_{\pi\in\mathrm{Alt}^*_{2n+1}} q^{\mathrm{inv}(\pi)}$	$\frac{(0,0)}{(0,0)}$
$\tau_{2n+1}^{\geq\leq}(q)$	$\mathrm{RPP}(\delta_{n+2}/\delta_n)$	$\sum_{\pi\in\mathrm{Alt}_{2n+1}} q^{\mathrm{maj}(\kappa_{2n+1}\pi^{-1})}$	$\sum_{\pi\in\mathrm{Alt}^*_{2n+1}} q^{\mathrm{inv}(\pi)-\mathrm{ndes}(\pi_e)}$	$\frac{(1,1)}{(1,-1)}$
$\tau_{2n+1}^{><}(q)$	$\mathrm{ST}(\delta_{n+2}/\delta_n)$	$\sum_{\pi\in\mathrm{Alt}_{2n+1}} q^{\mathrm{maj}(\eta_{2n+1}\pi^{-1})}$	$\sum_{\pi\in\mathrm{Alt}^*_{2n+1}} q^{\mathrm{inv}(\pi)+\mathrm{nasc}(\pi_e)}$	$\frac{(1,0)}{(1,0)}$
$\tau_{2n+1}^{<\geq}(q)$	$\mathrm{SSYT}(\delta_{n+3}^{(1,1)}/\delta_{n+1})$	$\sum_{\pi\in\mathrm{Ralt}_{2n+1}} q^{\mathrm{maj}(\pi^{-1})}$	$\sum_{\pi\in\mathrm{Alt}^*_{2n+1}} q^{\mathrm{inv}(\pi)}$	$\frac{(0,0)}{(0,0)}$
$\tau_{2n+1}^{\leq\geq}(q)$	$\mathrm{RPP}(\delta_{n+3}^{(1,1)}/\delta_{n+1})$	$\sum_{\pi\in\mathrm{Ralt}_{2n+1}} q^{\mathrm{maj}(\eta_{2n+1}\pi^{-1})}$	$\sum_{\pi\in\mathrm{Alt}^*_{2n+1}} q^{\mathrm{inv}(\pi)-\mathrm{asc}(\pi_o)}$	$\frac{(1,0)}{(1,-1)}$
$\tau_{2n+1}^{<>}(q)$	$\mathrm{ST}(\delta_{n+3}^{(1,1)}/\delta_{n+1})$	$\sum_{\pi\in\mathrm{Ralt}_{2n+1}} q^{\mathrm{maj}(\kappa_{2n+1}\pi^{-1})}$	$\sum_{\pi\in\mathrm{Alt}^*_{2n+1}} q^{\mathrm{inv}(\pi)+\mathrm{des}(\pi_o)}$	$\frac{(1,1)}{(1,0)}$

Table 1: Interpretations for Prodinger's q-tangent numbers. The notation Alt^*_{2n+1} means it can be either Alt_{2n+1} or Ralt_{2n+1}.

$\tau_{2n}^{\alpha\beta}(q)$	TAB	M	I	$\frac{1}{(C,D)}$
$\tau_{2n}^{\geq<}(q)$	$\mathrm{SSYT}(\delta_{n+2}^{(0,1)}/\delta_n)$	$\sum_{\pi\in\mathrm{Alt}_{2n}} q^{\mathrm{maj}(\pi^{-1})}$	$\sum_{\pi\in\mathrm{Alt}_{2n}} q^{\mathrm{inv}(\pi)}$	$\frac{1}{(0,0)}$
$\tau_{2n}^{\geq\leq}(q)$	$\mathrm{RPP}(\delta_{n+2}^{(0,1)}/\delta_n)$	$\sum_{\pi\in\mathrm{Alt}_{2n}} q^{\mathrm{maj}(\kappa_{2n}\pi^{-1})}$	$\sum_{\pi\in\mathrm{Alt}_{2n}} q^{\mathrm{inv}(\pi)-\mathrm{asc}(\pi_*)}$	$\frac{1}{(1,-1)}$
$\tau_{2n}^{><}(q)$	$\mathrm{ST}(\delta_{n+2}^{(0,1)}/\delta_n)$	$\sum_{\pi\in\mathrm{Alt}_{2n}} q^{\mathrm{maj}(\eta_{2n}\pi^{-1})}$	$\sum_{\pi\in\mathrm{Alt}_{2n}} q^{\mathrm{inv}(\pi)+\mathrm{nasc}(\pi_*)}$	$\frac{1}{(1,0)}$
$\tau_{2n}^{<\geq}(q)$	$\mathrm{SSYT}(\delta_{n+2}^{(1,0)}/\delta_n)$	$\sum_{\pi\in\mathrm{Ralt}_{2n}} q^{\mathrm{maj}(\pi^{-1})}$	$\sum_{\pi\in\mathrm{Ralt}_{2n}} q^{\mathrm{inv}(\pi)}$	$\frac{1}{(2,-1)}$
$\tau_{2n}^{\leq\geq}(q)$	$\mathrm{RPP}(\delta_{n+2}^{(1,0)}/\delta_n)$	$\sum_{\pi\in\mathrm{Ralt}_{2n}} q^{\mathrm{maj}(\eta_{2n}\pi^{-1})}$	$\sum_{\pi\in\mathrm{Ralt}_{2n}} q^{\mathrm{inv}(\pi)-\mathrm{ndes}(\pi_*)}$	$\frac{1}{(1,-1)}$
$\tau_{2n}^{<>}(q)$	$\mathrm{ST}(\delta_{n+2}^{(1,0)}/\delta_n)$	$\sum_{\pi\in\mathrm{Ralt}_{2n}} q^{\mathrm{maj}(\kappa_{2n}\pi^{-1})}$	$\sum_{\pi\in\mathrm{Ralt}_{2n}} q^{\mathrm{inv}(\pi)+\mathrm{des}(\pi_*)}$	$\frac{1}{(1,0)}$

Table 2: Interpretations for Prodinger's q-secant numbers. The notation π_* means it can be either π_o or π_e.

whose generating function is

$$\sum_{n\geq 0} f_{2n}x^{2n} = \frac{1}{\sum_{n\geq 0}(-1)^n q^{Cn^2+Dn}x^{2n}/(q;q)_{2n}}.$$

2.2 Foata-type bijection for $E^*_{2n+1}(q)$.

We denote by Alt_n^{-1} the set of permutations $\pi \in \mathfrak{S}_n$ with $\pi^{-1} \in \mathrm{Alt}_n$.

Let $\prec$ be a total order on $\mathbb{N}$. For a word $w_1 \dots w_k$ consisting of distinct positive integers, we define $f(w_1 \dots w_k, \prec)$ as follows. Let $b_0, b_1, \dots, b_m$ be the integers such that

- $0 = b_0 < b_1 < \cdots < b_m = k-1$,
- if $w_{k-1} \prec w_k$, then $w_{b_1}, \dots, w_{b_m} \prec w_k \prec w_j$ for all $j \in [k-1] \setminus \{b_1, \dots, b_m\}$, and
- if $w_k \prec w_{k-1}$, then $w_j \prec w_k \prec w_{b_1}, \dots, w_{b_m}$ for all $j \in [k-1] \setminus \{b_1, \dots, b_m\}$.

For $1 \leq j \leq m$, let $B_j = w_{b_{j-1}+1} \dots w_{b_j}$. We denote

$$B(w_1 \dots w_k, \prec) = (B_1, B_2, \dots, B_m).$$

Note that $w_1 \dots w_{k-1}w_k$ is the concatenation $B_1B_2 \dots B_m w_k$. Let $B'_j = w_{b_j}w_{b_{j-1}+1} \dots w_{b_j-1}$. Then we define

$$f(w_1 \dots w_k, \prec) = B'_1B'_2 \dots B'_m w_k.$$

For a permutation $\pi = \pi_1 \dots \pi_n \in \mathfrak{S}_n$ and a total order $\prec$ on $\mathbb{N}$, we define $F(\pi, \prec)$ as follows. Let $w^{(1)} = \pi_1$. For $2 \leq k \leq n$, let $w^{(k)} = f(w^{(k-1)}\pi_k, \prec)$. Finally $F(\pi, \prec) = w^{(n)}$. Note that for the natural order $1 < 2 < \cdots$, the map $F(\pi, <)$ is the same as the Foata map.

For $i \geq 1$, we define $<_i$ to be the total order on $\mathbb{N}$ obtained from the natural ordering by reversing the order of i and $i+1$, i.e., for $a < b$ with $(a,b) \neq (i, i+1)$, we have $a <_i b$ and $i+1 <_i i$.

For $\pi \in \mathrm{Alt}^{-1}_{2n+1}$, we define $F_{\mathrm{alt}}(\pi)$ as follows. First, we set $w^{(1)} = \pi_1$. For $2 \leq k \leq 2n+1$, there are two cases:

- If $\pi_k = 2i$ and $\pi_1 \dots \pi_{k-1}$ does not have $2i+2$, then $w^{(k)} = f(w^{(k-1)}\pi_k, <_{2i})$.
- Otherwise, $w^{(k)} = f(w^{(k-1)}\pi_k, <)$.

Then $F_{\mathrm{alt}}(\pi)$ is defined to be $w^{(2n+1)}$. For example, if $\pi = 317295486 \in \mathrm{Alt}_9^{-1}$, then $w^{(4)} = 7312, w^{(8)} = 37912548$ and $F_{\mathrm{alt}}(\pi) = w^{(9)} = 739812546$.

Theorem 2.3. *The map F_{alt} induces a bijection $F_{\mathrm{alt}} : \mathrm{Alt}^{-1}_{2n+1} \to \mathrm{Alt}^{-1}_{2n+1}$. Moreover, if $\pi \in \mathrm{Alt}^{-1}_{2n+1}$ and $\sigma = F_{\mathrm{alt}}(\pi)$, then*

$$\mathrm{maj}(\kappa_{2n+1}\pi) = \mathrm{inv}(\sigma) - \mathrm{ndes}((\sigma^{-1})_e).$$

Corollary 2.4. *We have*

$$\sum_{\pi\in \mathrm{Alt}_{2n+1}} q^{\mathrm{maj}(\kappa_{2n+1}\pi^{-1})} = \sum_{\pi\in \mathrm{Alt}_{2n+1}} q^{\mathrm{inv}(\pi)-\mathrm{ndes}(\pi_e)}.$$

3 Proofs of two conjectures of Morales, Pak and Panova

In this section, we provide proofs of two conjectures of Morales et al. [6] via a modification of Lindström–Gessel–Viennot lemma. The two conjectures are of the form $A = Q\det(c_{ij})$. Let us briefly outline our proof. In Section 3.1 we interpret pleasant diagrams of δ_{n+2k}/δ_n as non-intersecting marked Dyck paths. This interpretation can be used to express A as a generating function for non-intersecting Dyck paths. In Section 3.2 we show a modification of Lindström–Gessel–Viennot lemma which allows us to express $\det(c_{ij})$ as a generating function for weakly non-intersecting Dyck paths. In Section 3.3 we find a connection between the generating function for weakly non-intersecting Dyck paths and the generating function for (strictly) non-intersecting Dyck paths. Using these results we prove Theorems 3.1 and 3.2 in Sections 3.4 and 3.5.

Let $p(\lambda/\mu)$ be the number of pleasant diagrams of λ/μ. Morales et al. [6] showed that $p(\delta_{n+2}/\delta_n) = \mathfrak{s}_n$, where $\mathfrak{s}_n = 2^{n+2}s_n$ for the *little Schröder number* s_n. They proposed the following conjectures on $p(\lambda/\mu)$ and the generating function for RPPs of shape λ/μ for $\lambda/\mu = \delta_{n+2k}/\delta_n$.

Theorem 3.1 ([6, Conjecture 9.3])**.** *We have*

$$p(\delta_{n+2k}/\delta_n) = 2^{\binom{k}{2}} \det(\mathfrak{s}_{n-2+i+j})_{i,j=1}^{k}. \tag{3.1}$$

Theorem 3.2 ([6, Conjecture 9.6])**.** *We have*

$$\sum_{\pi\in \mathrm{RPP}(\delta_{n+2k}/\delta_n)} q^{|\pi|} = q^{-\frac{k(k-1)(6n+8k-1)}{6}} \det\left(\frac{E^*_{2n+2i+2j-3}(q)}{(q;q)_{2n+2i+2j-3}}\right)_{i,j=1}^{k}. \tag{3.2}$$

Let Dyck_{2n} be the set of Dyck paths from $(-n,0)$ to $(n,0)$ and Dyck_{2n}^k the set of k-tuples $(D_1,\dots,D_k)$ of Dyck paths, where for $i\in[k]$,

$$D_i \in \mathrm{Dyck}_{2n+4i-4}.$$

For a Dyck path $D\in \mathrm{Dyck}_{2n}$, we denote by $\mathcal{V}(D)$ (resp. $\mathcal{HP}(D)$) the set of valleys (resp. high peaks) of D. For $D_1 \in \mathrm{Dyck}_{2n}$ and $D_2 \in \mathrm{Dyck}_{2n+4k}$, we write $D_1 \le D_2$ if $D_1(i) \le D_2(i)$ for all $-n \le i \le n$ and there is no i such that $D_1(i) = D_2(i)$ and $D_1(i+1) = D_2(i+1)$. Similarly, we write $D_1 < D_2$ if $D_1(i) < D_2(i)$ for all $-n \le i \le n$.

3.1 Pleasant diagrams of δ_{n+2k}/δ_n and non-intersecting marked Dyck paths

For a point $p = (i, j) \in \mathbb{Z} \times \mathbb{N}$, the *height* $\mathrm{ht}(p)$ of p is defined to be j. We identify the square $u = (i, j)$ in the ith row and jth column in δ_{n+2k} with the point $p = (j - i, n + 2k - i - j) \in \mathbb{Z} \times \mathbb{N}$. Under this identification one can easily check that if a square $u \in \delta_{n+2k}$ corresponds to a point $p \in \mathbb{Z} \times \mathbb{N}$ then the hook length $h(u)$ in δ_{n+2k} is equal to $2\,\mathrm{ht}(p) + 1$.

A *marked Dyck path* is a Dyck path in which each point that is not a valley may or may not be marked. Let

$$\mathcal{ND}^{*k}_{2n} = \left\{ (D_1, \dots, D_k, C) : (D_1 < D_2 < \cdots < D_k) \in \mathrm{Dyck}^k_{2n}, C \subset \bigcup_{i=1}^{k} (D_i \setminus \mathcal{V}(D_i)) \right\}.$$

The following proposition allows us to consider pleasant diagrams of δ_{n+2k}/δ_n as non-intersecting marked Dyck paths.

Proposition 3.3. *The map $\rho^* : \mathcal{ND}^{*k}_{2n} \to \mathcal{P}(\delta_{n+2k}/\delta_n)$ defined by*

$$\rho^*(D_1, \dots, D_k, C) = (D_1 \cup \cdots \cup D_k) \setminus C$$

is a bijection.

3.2 A modification of Lindström–Gessel–Viennot lemma

Let wt and $\mathrm{wt}_{\mathrm{ext}}$ be fixed weight functions defined on $\mathbb{Z} \times \mathbb{N}$. We define

$$\mathrm{wt}_{\mathcal{V}}(D) = \prod_{p \in D} \mathrm{wt}(p) \prod_{p \in \mathcal{V}(D)} \mathrm{wt}_{\mathrm{ext}}(p)$$

and

$$\mathrm{wt}_{\mathcal{HP}}(D) = \prod_{p \in D} \mathrm{wt}(p) \prod_{p \in \mathcal{HP}(D)} \mathrm{wt}_{\mathrm{ext}}(p).$$

One can regard $\mathrm{wt}_{\mathcal{V}}(D)$ as a weight of a Dyck path D in which every point p of D has the weight $\mathrm{wt}(p)$ and every valley p of D has the extra weight $\mathrm{wt}_{\mathrm{ext}}(p)$. For Dyck paths $D_1, \dots, D_k$, we define

$$\mathrm{wt}_{\mathcal{V}}(D_1, \dots, D_k) = \mathrm{wt}_{\mathcal{V}}(D_1) \cdots \mathrm{wt}_{\mathcal{V}}(D_k).$$

The next lemma is a modification of Lindström–Gessel–Viennot lemma.

Lemma 3.4. *For $1 \le i, j \le k$, let $A_i = (-n - 2i + 2, 0)$, $B_j = (n + 2j - 2, 0)$ and*

$$d^{i,j}_n(q) = \sum_{D \in \mathrm{Dyck}(A_i \to B_j)} \mathrm{wt}_{\mathcal{V}}(D).$$

Then

$$\det(d_n^{i,j}(q))_{i,j=1}^k = \sum_{(D_1\le\cdots\le D_k)\in \mathrm{Dyck}_{2n}^k} \mathrm{wt}_{\mathcal{V}}(D_1,\dots,D_k) \prod_{i=1}^{k-1} \prod_{p\in D_i\cap D_{i+1}} \left(1-\frac{1}{\mathrm{wt}_{\mathrm{ext}}(p)}\right). \tag{3.3}$$

Note that if wt and $\mathrm{wt}_{\mathrm{ext}}$ depend only on the y-coordinates, then $d_n^{i,j}(q)$ can be written as $d_{n+i+j-2}(q)$, where

$$d_n(q) = \sum_{D\in \mathrm{Dyck}_{2n}} \mathrm{wt}_{\mathcal{V}}(D).$$

Remark 3.5. Lindström–Gessel–Viennot lemma [2, 5] expresses a determinant as a sum over non-intersecting lattice paths. In our case, due to the extra weights on the valleys, the paths which have common points are not completely cancelled. Therefore the right-hand side of (3.3) is a sum over *weakly* non-intersecting lattice paths.

3.3 Weakly and strictly non-intersecting Dyck paths

The following proposition is the key ingredient for the proofs of Theorems 3.1 and 3.2.

Proposition 3.6. *Suppose that the weight functions* wt *and* $\mathrm{wt}_{\mathrm{ext}}$ *satisfy* $\mathrm{wt}(p)\,(\mathrm{wt}_{\mathrm{ext}}(p)-1) = c$ *for all* $p\in\mathbb{Z}\times\mathbb{N}$. *Let* $A\in \mathrm{Dyck}_{2n}$ *and* $B\in \mathrm{Dyck}_{2n+8}$ *be fixed Dyck paths with* $A<B$. *Then*

$$\sum_{(A\le D<B)\in \mathrm{Dyck}_{2n}^3} \mathrm{wt}_{\mathcal{V}}(D) \prod_{p\in A\cap D}\left(1-\frac{1}{\mathrm{wt}_{\mathrm{ext}}(p)}\right) = \sum_{(A<D\le B)\in \mathrm{Dyck}_{2n}^3} \mathrm{wt}_{\mathcal{HP}}(D) \prod_{p\in D\cap B}\left(1-\frac{1}{\mathrm{wt}_{\mathrm{ext}}(p)}\right).$$

Proposition 3.7. *Suppose that* wt *and* $\mathrm{wt}_{\mathrm{ext}}$ *satisfy the following conditions*

- $\mathrm{wt}(p)\,(\mathrm{wt}_{\mathrm{ext}}(p)-1) = c$ *for all* $p\in\mathbb{Z}\times\mathbb{N}$, *and*
- $\mathrm{wt}_{\mathcal{HP}}(D) = t_j\,\mathrm{wt}_{\mathcal{V}}(D)$ *for all* $D\in \mathrm{Dyck}_{2j}$ *such that every peak in* D *is a high peak.*

Then we have

$$\sum_{(D_1\le\cdots\le D_k)\in \mathrm{Dyck}_{2n}^k} \mathrm{wt}_{\mathcal{V}}(D_1,\dots,D_k) \prod_{i=1}^{k-1} \prod_{p\in D_i\cap D_{i+1}} \left(1-\frac{1}{\mathrm{wt}_{\mathrm{ext}}(p)}\right) = \prod_{i=1}^{k-1} t_{n+2i}^i \sum_{(D_1<\cdots<D_k)\in \mathrm{Dyck}_{2n}^k} \mathrm{wt}_{\mathcal{V}}(D_1,\dots,D_k).$$

3.4 Proof of Theorem 3.1

Let

$$d_n(q) = \sum_{D \in \mathrm{Dyck}_{2n}} q^{v(D)}$$

and

$$d_{n,k}(q) = \sum_{(D_1 < D_2 < \cdots < D_k) \in \mathrm{Dyck}_{2n}^k} q^{v(D_1)+\cdots+v(D_k)}.$$

Then by Proposition 3.3, (3.1) can be rewritten as

$$2^{-\binom{k}{2}} d_{n,k}(1/2) = \det(d_{n+i+j-2}(1/2))_{i,j=1}^k.$$

Thus Theorem 3.1 is obtained from the following theorem by substituting $q = 1/2$.

Theorem 3.8. *For $n, k \geq 1$, we have*

$$\det(d_{n+i+j-2}(q))_{i,j=1}^k = q^{\binom{k}{2}} d_{n,k}(q).$$

3.5 Proof of Theorem 3.2

By Morales, Pak and Panova's result [6]

$$\sum_{\pi \in \mathrm{RPP}(\lambda/\mu)} q^{|\pi|} = \sum_{P \in \mathcal{P}(\lambda/\mu)} \prod_{u \in P} \frac{q^{h(u)}}{1 - q^{h(u)}}$$

and Proposition 3.3, we have

$$\sum_{\pi \in \mathrm{RPP}(\delta_{n+2k}/\delta_n)} q^{|\pi|} = \sum_{(D_1 < \cdots < D_k) \in \mathrm{Dyck}_{2n}^k} \prod_{i=1}^k \left(\prod_{p \in \mathcal{V}(D_i)} q^{2\,\mathrm{ht}(p)+1} \prod_{p \in D_i} \frac{1}{1 - q^{2\,\mathrm{ht}(p)+1}} \right)$$

and

$$\frac{E^*_{2n+1}(q)}{(q;q)_{2n+1}} = \sum_{\pi \in \mathrm{RPP}(\delta_{n+2}/\delta_n)} q^{|\pi|} = \sum_{D \in \mathrm{Dyck}_{2n}} \prod_{p \in \mathcal{V}(D)} q^{2\,\mathrm{ht}(p)+1} \prod_{p \in D} \frac{1}{1 - q^{2\,\mathrm{ht}(p)+1}}.$$

Thus, by Lemma 3.4 with $\mathrm{wt}(p) = 1/(1 - q^{2\,\mathrm{ht}(p)+1})$ and $\mathrm{wt}_{\mathrm{ext}}(p) = q^{2\,\mathrm{ht}(p)+1}$, we can rewrite (3.2) as follows.

Theorem 3.9. *We have*

$$\sum_{(D_1 \leq \cdots \leq D_k) \in \mathrm{Dyck}_{2n}^k} \prod_{i=1}^k \left(\prod_{p \in \mathcal{V}(D_i)} q^{2\,\mathrm{ht}(p)+1} \prod_{p \in D_i} \frac{1}{1 - q^{2\,\mathrm{ht}(p)+1}} \right) \prod_{j=1}^{k-1} \prod_{p \in D_j \cap D_{j+1}} \left(1 - \frac{1}{q^{2\,\mathrm{ht}(p)+1}} \right)$$
$$= q^{\frac{k(k-1)(6n+8k-1)}{6}} \sum_{(D_1 < \cdots < D_k) \in \mathrm{Dyck}_{2n}^k} \prod_{i=1}^k \left(\prod_{p \in \mathcal{V}(D_i)} q^{2\,\mathrm{ht}(p)+1} \prod_{p \in D_i} \frac{1}{1 - q^{2\,\mathrm{ht}(p)+1}} \right).$$

4 A determinantal formula for a certain class of skew shapes

In this section, applying the same methods used in the previous section, we find a determinantal formula for $p(\lambda/\mu)$ and the generating function for the reverse plane partitions of shape λ/μ for a certain class including δ_{n+2k}/δ_n and δ_{n+2k+1}/δ_n.

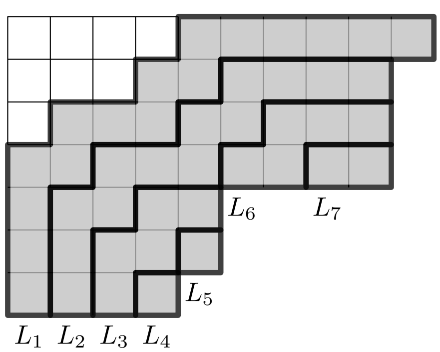
Consider a partition λ. Let $L = (u_0, u_1, \ldots, u_m)$ be a sequence of cells in λ. Each pair (u_{i-1}, u_i) is called a *step* of L. A step (u_{i-1}, u_i) is called an *up step* (resp. *down step*) if $u_i - u_{i-1}$ is equal to $(-1, 0)$ (resp. $(0, 1)$). We say that L is a λ*-Dyck path* if every step is either an up step or a down step. The set of λ-Dyck paths starting at a cell s and ending at a cell t is denoted by $\mathrm{Dyck}_\lambda(s, t)$. We denote by $L_\lambda(s, t)$ the lowest Dyck path in $\mathrm{Dyck}_\lambda(s, t)$.

Let $D = (u_0, u_1, \ldots, u_m)$ be a λ-Dyck path. A cell u_i, for $1 \le i \le m-1$, is called a *peak* (resp. *valley*) if (u_{i-1}, u_i) is an up step (resp. down step) and (u_i, u_{i+1}) is a down step (resp. up step). A peak u_i is called a λ*-high peak* if $u_i + (1, 1) \in \lambda$. The set of valleys in D is denoted by $\mathcal{V}(D)$. For two λ-Dyck paths D_1 and D_2, $D_1 \le D_2$ and $D_1 < D_2$ can be defined similar to Dyck paths cases.

The *Kreiman outer decomposition* [4] of λ/μ is a sequence $L_1, \ldots, L_k$ of mutually disjoint nonempty λ-Dyck paths satisfying the following conditions.

- Each L_i starts at the southmost cell of a column of λ and ends at the eastmost cell of a row of λ.
- $L_1 \cup \cdots \cup L_k = \lambda/\mu$.

And we can regard $\{L_1, \ldots, L_k\}$ as a poset. See Figure 2.



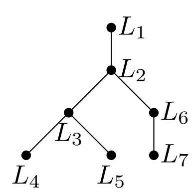


Figure 2: The left diagram shows the Kreiman outer decomposition $L_1, \ldots, L_7$ of λ/μ for $\lambda = (9, 8, 8, 8, 5, 5, 4)$ and $\mu = (4, 3, 1)$. The label L_i is written below the starting cell of it. The right diagram shows the poset of $L_1, \ldots, L_7$ with relation $<$.

Theorem 4.1. *Let $L_1, \ldots, L_k$ be the Kreiman outer decomposition of λ/μ. Let P be the poset of $L_1, \ldots, L_k$ with relation $<$. Suppose that the following conditions hold.*

- *P is a ranked poset.*
- *If $L_i < L_j$, then in L_j the first step is an up step, the last step is a down step and every peak is a λ-high peak.*

Let s_i (resp. t_i) be the first (resp. last) cell in L_i and r_i the rank of L_i in the poset P. Then we have

$$\sum_{\pi \in \mathrm{RPP}(\lambda/\mu)} q^{|\pi|} = q^{-\sum_{i=1}^{k} r_i |L_i|} \det \left(E_\lambda(s_i, t_j; q) \right)_{i,j=1}^{k},$$

where

$$E_\lambda(s_i, t_j; q) = \sum_{\pi \in \mathrm{RPP}(L_\lambda(s_i,t_j))} q^{|\pi|} = \sum_{D \in \mathrm{Dyck}_\lambda(s_i,t_j)} \prod_{u \in D} \frac{1}{1 - q^{h(u)}} \prod_{u \in \mathcal{V}(D)} q^{h(u)}.$$

Theorem 4.2. *Under the same conditions in Theorem 4.1, we have*

$$p(\lambda/\mu) = 2^{\sum_{i=1}^{k} r_i} \det \left(p(L_\lambda(s_i, t_j)) \right)_{i,j=1}^{k}.$$

Acknowledgements

The authors would like to thank Alejandro Morales for his helpful comments which motivated Section 4. The full version of this extended abstract appears in arxiv:1711.02337.

References

[1] P. Flajolet. "Combinatorial aspects of continued fractions". *Discrete Math.* **32**.2 (1980), pp. 125–161. DOI: 10.1016/0012-365X(80)90050-3.

[2] I. Gessel and G. Viennot. "Binomial determinants, paths, and hook length formulae". *Adv. Math.* **58**.3 (1985), pp. 300–321. DOI: 10.1016/0001-8708(85)90121-5.

[3] T. Huber and A.J. Yee. "Combinatorics of generalized q-Euler numbers". *J. Combin. Theory Ser. A* **117**.4 (2010), pp. 361–388. DOI: 10.1016/j.jcta.2009.07.012.

[4] V. Kreiman. "Schubert classes in the equivariant K-theory and equivariant cohomology of the Grassmannian" (2005). arXiv: math/0512204.

[5] B. Lindström. "On the vector representations of induced matroids". *Bull. London Math. Soc.* **5**.1 (1973), pp. 85–90. DOI: 10.1112/blms/5.1.85.

[6] A.H. Morales, I. Pak, and G. Panova. "Hook formulas for skew shapes II. Combinatorial proofs and enumerative applications". *SIAM J. Discrete Math.* **31**.3 (2017), pp. 1953–1989. DOI: 10.1137/16M1099625.

[7] H. Prodinger. "Combinatorics of geometrically distributed random variables: new q-tangent and q-secant numbers". *Int. J. Math. Math. Sci.* **24**.12 (2000), pp. 825–838. URL.