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Combinatorics of $\mathcal{X}$-variables in Finite Type Cluster Algebras

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Abstract. We compute the number of $\mathcal{X}$-variables (also called coefficients) of a cluster algebra of finite type when the underlying semifield is the universal semifield. For non-exceptional types, these numbers arise from a bijection between coefficients and quadrilaterals (with a choice of diagonal) appearing in triangulations of certain marked surfaces. We conjecture that similar results hold for cluster algebras from arbitrary marked surfaces, and obtain corollaries regarding the structure of finite type cluster algebras of geometric type.

Keywords: cluster algebra, coefficients, finite type

1 Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in the early 2000s [5], with the intent of establishing a general algebraic structure for studying dual canonical bases of semisimple groups and total positivity. A cluster algebra, or equivalently its seed pattern, is determined by an initial set of cluster variables (which we call $\mathcal{A}$-variables) and coefficients (which we call $\mathcal{X}$-variables), along with some additional data. As the terminology suggests, in the original definitions, $\mathcal{X}$-variables were somewhat in the background, and $\mathcal{A}$-variables were the object of importance. This is reflected in much of the research on cluster algebras to date, which focuses largely on $\mathcal{A}$-variables and their dynamics. However, $\mathcal{X}$-variables are important in total positivity and $\mathcal{X}$-variables over the universal semifield have recently appeared in the context of scattering amplitudes in $\mathcal{N} = 4$ Super Yang-Mills theory [8]. Moreover, in the setting of cluster varieties, introduced by Fock and Goncharov [2], the $\mathcal{A}$- and $\mathcal{X}$-varieties (associated with $\mathcal{A}$- and $\mathcal{X}$-variables, respectively) are on equal footing. Fock and Goncharov conjectured that a duality holds between the two varieties [2, Conjecture 4.3], which was later shown to be true under fairly general assumptions [9]. This duality suggests that studying $\mathcal{X}$-variables could be fruitful both in its own right and in furthering our understanding of cluster algebras.

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A study along these lines was undertaken by Speyer and Thomas in the case of acyclic cluster algebras with principal coefficients [13]. Using methods from quiver representation theory, they found that the $\mathcal{X}$-variables are in bijection with roots of an associated root system and give a combinatorial description of which roots can appear in the same $\mathcal{X}$-cluster. Seven found that in this context, mutation of $\mathcal{X}$-seeds roughly corresponds to reflection across hyperplanes orthogonal to roots [12]. However, the above results do not address $\mathcal{X}$-variables over the universal semifield, and the combinatorics obtained in the case of principal coefficients is quite different from what we obtain here.

We investigate the combinatorics of $\mathcal{X}$-variables for seed patterns of finite type, particularly in the case when the underlying semifield is the universal semifield. The combinatorics of $\mathcal{A}$-variables for finite type seed patterns is particularly rich, with connections to finite root systems [6] and triangulations of certain marked surfaces [4, Chapter 5]. Parker and Scherlis give a partial proof that in type A, $\mathcal{X}$-variables over the universal semifield are in bijection with the quadrilaterals of these triangulations [10, 11]. We prove this statement for all types.

Theorem 1.1. *Let $\mathcal{S}$ be an $\mathcal{X}$-seed pattern of non-exceptional type Z_n over the universal semifield such that one $\mathcal{X}$-cluster consists of algebraically independent elements. Let P be the marked polygon associated to type Z_n. Then the $\mathcal{X}$-variables of $\mathcal{S}$ are in bijection with the quadrilaterals (with a choice of diagonal) appearing in triangulations of P.*

As a corollary, we also compute the number of $\mathcal{X}$-variables in a finite type $\mathcal{X}$-seed pattern $\mathcal{S}_{sf}$ over the universal semifield, listed in the second row of the following table (the numbers for A_n for $n \leq 6$, D_4, and E_6 were also computed in [10]). For comparison, the third row gives the number of $\mathcal{X}$-variables in a finite type $\mathcal{X}$-seed pattern $\mathcal{S}_{pc}$ with principal coefficients, a corollary of the results in [13].

Type	A_n	B_n, C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\lvert\mathcal{X}(\mathcal{S}_{sf})\rvert$	$2\binom{n+3}{4}$	$\frac{1}{3}n(n+1)(n^2+2)$	$\frac{1}{3}n(n-1)(n^2+4n-6)$	770	2100	6240	196	16
$\lvert\mathcal{X}(\mathcal{S}_{pc})\rvert$	$n(n+1)$	$2n^2$	$2n(n-1)$	72	126	240	48	12

2 Seed Patterns

We largely follow the conventions of [7].

2.1 Seeds and Mutation

We begin by fixing a *semifield* $(\mathbb{P}, \cdot, \oplus)$, a multiplicative abelian group $(\mathbb{P}, \cdot)$ equipped with an (auxiliary) addition $\oplus$, a binary operation which is associative, commutative, and distributive with respect to multiplication.

Example 2.1. Let $t_1, \ldots, t_k$ be algebraically independent over $\mathbb{Q}$. The *universal semifield* $\mathbb{Q}_{sf}(t_1, \ldots, t_k)$ is the set of all rational functions in $t_1, \ldots, t_k$ that can be written as subtraction-free expressions in $t_1, \ldots, t_k$. This is a semifield with respect to the usual multiplication and addition of rational expressions. Note that any (subtraction-free) identity in $\mathbb{Q}_{sf}(t_1, \ldots, t_k)$ holds in an arbitrary semifield for any elements $u_1, \ldots, u_k$ [7].

Let $\mathbb{QP}$ denote the field of fractions of the group ring $\mathbb{ZP}$. We fix an *ambient field* $\mathcal{F}$, isomorphic to $\mathbb{QP}(t_1, \ldots, t_n)$.

Definition 2.2. A *labeled $\mathcal{X}$-seed* in $\mathbb{P}$ is a pair $(\mathbf{x}, B)$ where $\mathbf{x} = (x_1, \ldots, x_n)$ is a tuple of elements in $\mathbb{P}$ and $B = (b_{ij})$ is a *skew-symmetrizable* $n \times n$ integer matrix, that is there exists a diagonal integer matrix D with positive diagonal entries such that DB is skew-symmetric.

A *labeled $\mathcal{A}$-seed* in $\mathcal{F}$ is a triple $(\mathbf{a}, \mathbf{x}, B)$ where $(\mathbf{x}, B)$ is a labeled $\mathcal{X}$-seed in $\mathbb{P}$ and $\mathbf{a} = (a_1, \ldots, a_n)$ is a tuple of elements of $\mathcal{F}$ which are algebraically independent over $\mathbb{QP}$ and generate $\mathcal{F}$. $\mathbf{x}$ is the (labeled) *$\mathcal{X}$-cluster*, $\mathbf{a}$ the (labeled) *$\mathcal{A}$-cluster*, and B the *exchange matrix* of the labeled seed $(\mathbf{a}, \mathbf{x}, B)$.

The elements of an $\mathcal{X}$- (respectively $\mathcal{A}$-)cluster are called $\mathcal{X}$- (respectively $\mathcal{A}$-)variables. In the language of Fomin and Zelevinsky, the $\mathcal{X}$-cluster is the coefficient tuple, the $\mathcal{A}$-cluster is the cluster, and the $\mathcal{X}$- and $\mathcal{A}$-variables are coefficients and cluster variables, respectively. The notation here is chosen to parallel Fock and Goncharov's $\mathcal{A}$- and $\mathcal{X}$-cluster varieties. Note that an $\mathcal{X}$-seed consists only of an exchange matrix and an $\mathcal{X}$-cluster, but an $\mathcal{A}$-seed consists of an exchange matrix, an $\mathcal{A}$-cluster *and* an $\mathcal{X}$-cluster. For simplicity, we use "cluster", "seed", etc. without a prefix when a statement holds regardless of prefix.

One moves from labeled seed to labeled seed by a process called *mutation*.

Definition 2.3. Let $(\mathbf{a}, \mathbf{x}, B)$ be a labeled $\mathcal{A}$-seed in $\mathcal{F}$. *$\mathcal{A}$-seed mutation* in direction k, denoted μ_k, takes $(\mathbf{a}, \mathbf{x}, B)$ to the labeled $\mathcal{A}$-seed $(\mathbf{a}', \mathbf{x}', B')$ where

- The entries b'_{ij} of B' are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + b_{ik}|b_{kj}| & \text{if } b_{ik}b_{kj} > 0 \\ b_{ij} & \text{else.} \end{cases} \tag{2.1}$$

- The $\mathcal{A}$-cluster $\mathbf{a}' = (a'_1, \ldots, a'_n)$ is obtained from $\mathbf{a}$ by replacing the k^{th} entry a_k with an element $a'_k \in \mathcal{F}$ satisfying the *exchange relation*

$$a'_k a_k = \frac{x_k \prod_{b_{ik}>0} a_i^{b_{ik}} + \prod_{b_{ik}<0} a_i^{-b_{ik}}}{x_k \oplus 1}. \tag{2.2}$$

- The $\mathcal{X}$-cluster $\mathbf{x}' = (x'_1, \ldots, x'_n)$ is given by

$$x'_j = \begin{cases} x_j^{-1} & \text{if } j = k \\ x_j(x_k^{sgn(-b_{kj})} \oplus 1)^{-b_{kj}} & \text{else} \end{cases} \tag{2.3}$$

where $sgn(x) = 0$ for $x = 0$ and $sgn(x) = |x|/x$ otherwise.

Similarly, the *$\mathcal{X}$-seed mutation* μ_k in direction k takes the labeled $\mathcal{X}$-seed $(\mathbf{x}, B)$ to $\mathcal{X}$-seed $(\mathbf{x}', B')$ and *matrix mutation* takes B to B'. Two skew-symmetrizable integer matrices are *mutation equivalent* if some sequence of matrix mutations takes one to the other.

Note that $\mu_k(\mathbf{a}, \mathbf{x}, B)$ is indeed another labeled $\mathcal{A}$-seed, as B' is skew-symmetrizable and $\mathbf{a}'$ again consists of algebraically independent elements generating $\mathcal{F}$. One can check that μ_k is an involution.

2.2 Seed Patterns and Exchange Graphs

We organize all seeds obtainable from each other by a sequence of mutations in a seed pattern. Let $\mathbb{T}_n$ denote the (infinite) n-regular tree with edges labeled with $1, \ldots, n$ so that no vertex is in two edges with the same label.

Definition 2.4. A rank n *$\mathcal{A}$-seed pattern* (respectively, *$\mathcal{X}$-seed pattern*) $\mathcal{S}$ is an assignment of labeled $\mathcal{A}$-seeds (respectively $\mathcal{X}$-seeds) Σ_t to the vertices t of $\mathbb{T}_n$ so that if t and t' are connected by an edge labeled k, then $\Sigma_t = \mu_k(\Sigma_{t'})$.

Since mutation is involutive, a seed pattern $\mathcal{S}$ is completely determined by the choice of a single seed Σ; we write $\mathcal{S}(\Sigma)$ for the seed pattern containing Σ. Note that in the language of Fomin and Zelevinsky, an $\mathcal{A}$-seed pattern is a "seed pattern" and an $\mathcal{X}$-seed pattern is a "Y-pattern".

Given an $\mathcal{A}$-seed pattern $\mathcal{S}(\mathbf{a}, \mathbf{x}, B)$, one can obtain two $\mathcal{X}$-seed patterns. The first is $\mathcal{S}|_{\mathcal{X}} := \mathcal{S}(\mathbf{x}, B)$, the $\mathcal{X}$-seed pattern in $\mathbb{P}$ obtained by simply ignoring the $\mathcal{A}$-clusters of every seed. The second, which we denote $\hat{\mathcal{S}} = \mathcal{S}(\hat{\mathbf{x}}, B)$, is an $\mathcal{X}$-seed pattern in $\mathcal{F}$ constructed in [7, Proposition 3.9]. One can think of $\hat{\mathcal{S}}$ as recording the "exchange information" of $\mathcal{S}(\mathbf{a}, \mathbf{x}, B)$; indeed, the $\mathcal{X}$-variables of $\hat{\mathcal{S}}$ are rational expressions whose numerators and denominators are, up to multiplication by an element of $\mathbb{P}$, the two terms on the right hand side of an exchange relation of $\mathcal{S}$.

Two labeled seeds $\Sigma = (\mathbf{a}, \mathbf{x}, B)$ and $\Sigma' = (\mathbf{a}', \mathbf{x}', B')$ are equivalent (written as $\Sigma \sim \Sigma'$) if one can obtain Σ' by simultaneously reindexing $\mathbf{a}$, $\mathbf{x}$, and the rows and columns of B. We define an analogous equivalence relation for $\mathcal{X}$-seeds, also denoted $\sim$.

An *$\mathcal{A}$-seed* (respectively *$\mathcal{X}$-seed*) is an equivalence class of labeled $\mathcal{A}$-seeds (respectively labeled $\mathcal{X}$-seeds) with respect to $\sim$. The seed represented by the labeled seed Σ is denoted $[\Sigma]$. We mutate a seed $[\Sigma]$ by applying a mutation μ_k to Σ and taking its equivalence class.

Definition 2.5. The *exchange graph* of a seed pattern $\mathcal{S}$ is the (n-regular connected) graph whose vertices are the seeds in $\mathcal{S}$ and whose edges connect seeds related by a single mutation. Equivalently, the exchange graph is the graph one obtains by identifying the vertices t, t' of $\mathbb{T}_n$ such that $\Sigma_t \sim \Sigma_{t'}$.

Exchange graphs were defined for $\mathcal{A}$-seed patterns in [6, 7], but can equally be defined for $\mathcal{X}$-seed patterns. It is conjectured that the exchange graph of an $\mathcal{A}$-seed pattern $\mathcal{S} = \mathcal{S}(\mathbf{a}, \mathbf{x}, B)$ depends only on B [7, Conjecture 4.3], meaning that $\mathcal{S}|_{\mathcal{X}}$ does not influence the combinatorics of $\mathcal{S}$. The exchange graphs of $\mathcal{S}|_{\mathcal{X}}$ and $\hat{\mathcal{S}}$ can be obtained by identifying some vertices of the exchange graph of $\mathcal{S}$, as passing to either $\mathcal{X}$-seed pattern preserves mutation and the equivalence of labeled seeds. It is not known in general when any pair of these exchange graphs is equal.

3 Finite Type Seed Patterns

We now restrict our attention to seed patterns of finite type.

Definition 3.1. An $\mathcal{A}$-seed pattern is of *finite type* if it has finitely many seeds.

Finite type seed patterns were classified completely in [6]; they correspond exactly to finite (reduced crystallographic) root systems, or equivalently, finite type Cartan matrices (see for example [1, Chapter 5]).

For a skew-symmetrizable integer matrix $B = (b_{ij})$, its *Cartan counterpart* is the matrix $A(B) = (a_{ij})$ defined by $a_{ii} = 2$ and $a_{ij} = -|b_{ij}|$ for $i \neq j$.

Theorem 3.2 ([6, Theorems 1.5–1.7]).

i. An $\mathcal{A}$-seed pattern is of finite type if and only if the Cartan counterpart of one of its exchange matrices is a finite type Cartan matrix.

ii. Suppose B, B' are skew-symmetrizable integer matrices such that $A(B)$, $A(B')$ are finite type Cartan matrices. Then $A(B)$ and $A(B')$ are of the same Cartan–Killing type if and only if B and B' are mutation equivalent (modulo simultaneous relabeling of rows and columns.).

In light of this theorem, we refer to a finite type $\mathcal{A}$-seed pattern as type A_n, B_n, etc.

Definition 3.3. An $\mathcal{X}$-seed pattern $\mathcal{S}$ is of type Z_n if the Cartan companion of one of its exchange matrices is a type Z_n Cartan matrix.

We will call such $\mathcal{X}$-seed patterns *Dynkin type* (rather than finite type, since in fact not all $\mathcal{X}$-seed patterns with finitely many seeds are of this form).

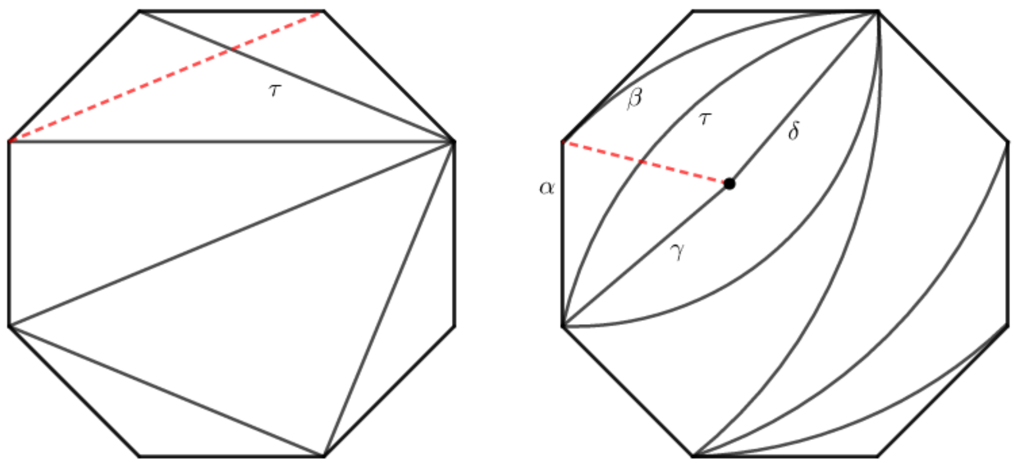

Figure 1: Triangulations of P_8 and $P_8^\bullet$ are shown in solid lines; the dashed arc is the flip of τ. On the right, $q_T(\tau) = \{\alpha, \beta, \gamma, \delta\}$.

3.1 Triangulations for types A and D

The material in this section is part of a more general theory of $\mathcal{A}$-seed patterns from surfaces, developed in [3].

Let P_n denote a convex n-gon and $P_n^\bullet$ denote a convex n-gon with a distinguished point p (a *puncture*) in the interior, with vertices labeled by $1, \dots, n$. For $P \in \{P_n, P_n^\bullet\}$, the vertices and puncture of P are called *marked points*. An *arc* of P is a non-self-intersecting curve γ in P such that the endpoints of γ are distinct marked points, the relative interior of γ is disjoint from $\partial P \cup \{p\}$, and γ does not cut out an unpunctured digon. An arc incident to the puncture p is a *radius*. Arcs are considered up to isotopy.

A *tagged arc* of P is either an ordinary arc between two vertices or a radius that is labeled either "notched" or "plain." Two tagged arcs γ, γ' are *compatible* if their untagged versions do not cross (or to be precise, there are two noncrossing arcs isotopic to γ and γ') with the following modification: if γ is a notched radius and γ' is plain, they are compatible if and only if their untagged versions coincide.

A *tagged triangulation* T is a maximal collection of pairwise compatible tagged arcs. All tagged triangulations of P consist of the same number of arcs. Given a tagged triangulation T, the *quadrilateral* $q_T(\gamma)$ of an arc γ in T consists of the arcs of T and boundary segments adjacent to γ (see Figure 1). γ is the *diagonal* of its quadrilateral.

Proposition 3.4 ([3]). *Let $T = \{\gamma_1, \dots, \gamma_n\}$ be a tagged triangulation of P. For all k, there exists a unique tagged arc $\gamma_k' \neq \gamma_k$ such that $\mu_k(T) := T \setminus \{\gamma_k\} \cup \{\gamma_k'\}$ is a tagged triangulation of P.*

The arc γ' in the above proposition is called the *flip* of γ with respect to T (or with respect to $q_T(\gamma)$, since γ and γ' are exactly the two diagonals of $q_T(\gamma)$).

We define the *flip graph* of P to be the graph whose vertices are tagged triangulations of P and whose edges connect triangulations that can be obtained from each other by flipping a single arc. The flip graph of P is connected.

We can encode a tagged triangulation T in a skew-symmetric $n \times n$ integer matrix $B(T)$. The nonzero entries in $B(T)$ correspond to pairs of adjacent arcs; the sign of these entries records the relative orientation of the arcs (see [3] for details).

Flips of arcs are related to matrix mutation in the following way: for a tagged triangulation $T = \{\gamma_1, \ldots, \gamma_n\}$ of P, $\mu_k(B(T)) = B(\mu_k(T))$, or, in words, flipping γ_k changes $B(T)$ by mutation in direction k.

As the following theorem shows, these triangulations entirely encode the combinatorics of type A and D $\mathcal{A}$-seed patterns.

Theorem 3.5 ([3]). *Let $P = P_{n+3}$ (resp. $P = P_n^\bullet$). Consider an $\mathcal{A}$-seed pattern $\mathcal{S}$ such that some exchange matrix is $B(T_0)$ for some triangulation T_0 of P. Then $\mathcal{S}$ is type A_n (resp. D_n) and there is a bijection $\gamma \mapsto a_\gamma$ between arcs of P and $\mathcal{A}$-variables of $\mathcal{S}$. Further, if $\Sigma = (\boldsymbol{a}, \boldsymbol{x}, B)$ is a seed of $\mathcal{S}$, there is a unique triangulation T such that $\boldsymbol{a} = \{a_\gamma\}_{\gamma \in T}$ and $B = B(T)$. Finally, mutation in direction k takes the seed corresponding to T to the seed corresponding to $\mu_k(T)$, implying that the exchange graph of $\mathcal{S}$ is isomorphic to the flip graph of P.*

3.2 Triangulations for types B and C

To obtain triangulations whose adjacency matrices are exchange matrices of type B_n and C_n $\mathcal{A}$-seed patterns, we "fold" triangulations of P_{2n+2} and $P_{n+1}^\bullet$. This is part of a larger theory of folded cluster algebras (see [4]).

Let $G = \mathbb{Z}/2\mathbb{Z}$. We write P_{2n}^G for P_{2n} equipped with the G-action taking vertex i to vertex $i' := i + n$ (with labels considered modulo $2n$). This induces an action of G on the arcs of P_{2n}. The triangulations of P_{2n}^G are the triangulations of P_{2n} fixed by the G-action, commonly called *centrally symmetric* triangulations.

We write $P_n^{\bullet\,G}$ for $P_n^\bullet$ equipped with the G-action switching the notched and plain version of a radius. Again, the triangulations of $P_n^{\bullet\,G}$ are the triangulations of $P_n^\bullet$ fixed under the G-action, which are exactly those triangulations containing both the notched and plain versions of the same radius.

Let $P \in \{P_{2n}, P_n^\bullet\}$, and let T be a triangulation of P^G. For $\gamma \in T$, let $[\gamma]$ denote the G-orbit of γ. The *quadrilateral* of $[\gamma]$, denoted $q_T([\gamma])$, is the G-orbit of $q_T(\gamma)$, or, in other words, all of the arcs and boundary segments adjacent to arcs in $[\gamma]$. Flipping the arcs in $[\gamma]$ results in another triangulation of P^G (which does not depend on the order of arc flips, since arcs in the same orbit are pairwise not adjacent). We define the flip graph of P^G in direct analogy to that of P; again, it is connected.

We associate to each triangulation T of P^G a skew-symmetrizable integer matrix $B^G(T)$, whose rows and columns are labeled by G-orbits of arcs of T. $B^G(T)$ is ob-

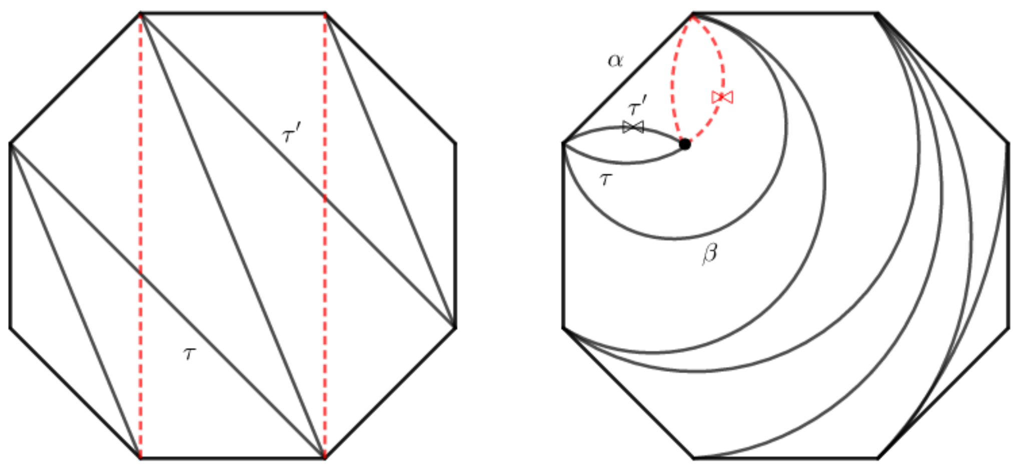

Figure 2: Triangulations of P_8^G and $P_8^{\bullet\,G}$ are shown in solid lines. The orbit $[\tau]$ of τ is $\{\tau, \tau'\}$ and the flip of $[\tau]$ is dashed. On the right, $q_T([\tau]) = \{\alpha, \beta, \tau, \tau'\}$.

tained from the usual signed adjacency matrix $B(T)$ of T (see [4, Section 4.4] for details); again, if two G-orbits consist of pairwise non-adjacent arcs, the corresponding matrix entry is 0.

Just as with usual triangulations, (orbit of) arc flips and matrix mutation interact nicely: if T is a triangulation containing arc γ, flipping the arcs in $[\gamma]$ corresponds to mutating $B^G(T)$ in the direction labeled by $[\gamma]$. Further, we have the following theorem.

Theorem 3.6 ([4])**.** *Let $P = P^\bullet_{n+1}$ (resp. $P = P_{2n+2}$). Consider an $\mathcal{A}$-seed pattern $\mathcal{S}$ such that some exchange matrix is $B^G(T_0)$ for some triangulation T_0 of P^G. Then $\mathcal{S}$ is type B_n (resp. C_n) and there is a bijection $[\gamma] \mapsto a_{[\gamma]}$ between arcs of P and $\mathcal{A}$-variables of $\mathcal{S}$. Further, if $\Sigma = (\boldsymbol{a}, \boldsymbol{x}, B)$ is a seed of $\mathcal{S}$, there is a unique triangulation T such that $\boldsymbol{a} = \{a_{[\gamma]}\}_{\gamma \in T}$ and $B = B^G(T)$. Finally, mutation in direction K takes the seed corresponding to T to the seed corresponding to $\mu_K(T)$, implying that the exchange graph of $\mathcal{S}$ is isomorphic to the flip graph of P^G.*

4 Dynkin type $\mathcal{X}$-seed patterns

Let $\mathcal{S}$ be an $\mathcal{X}$-seed pattern of type Z_n ($Z \in \{A, B, C, D\}$) over an arbitrary semifield $\mathbb{P}$, and let $\mathcal{X}(\mathcal{S})$ denote the set of $\mathcal{X}$-variables of $\mathcal{S}$. Let P be the surface whose triangulations encapsulate the combinatorics of type Z_n $\mathcal{A}$-seed patterns ($P = P_{n+3}$ for $Z = A$, $P = P^\bullet_n$ for $Z = D$, $P = P^G_{2n+2}$ for $Z = C$, $P = P^{\bullet\,G}_{n+1}$ for $Z = B$). In this section, we relate the $\mathcal{X}$-variables of $\mathcal{S}$ to the triangulations of P, and show a bijection between $\mathcal{X}\mathcal{S}$ and quadrilaterals (with a choice of diagonal) of P in the case when $\mathbb{P}$ is the universal semifield. Note that in what follows, "arc" should usually be understood to mean "orbit of arc" if $\mathcal{S}$ is type B or C.

First, notice that Theorems 3.5 and 3.6 imply that one can associate to each triangulation of P a seed of $\mathcal{S}$ such that mutation of seeds corresponds to flips of arcs. Indeed, consider any $\mathcal{A}$-seed pattern $\mathcal{R}$ with $\mathcal{R}|_{\mathcal{X}} = \mathcal{S}$; if the triangulation T corresponds to the $\mathcal{A}$-seed $(\mathbf{a}, \mathbf{x}, B)$ with arc γ_k corresponding to $\mathcal{A}$-variable a_k, then we associate to T the $\mathcal{X}$-seed $(\mathbf{x}, B)$ and to the arc γ_k the $\mathcal{X}$-variable x_k. We write Σ_T to indicate this association. Note that *a priori* two distinct triangulations may be associated to the same $\mathcal{X}$-seed, and an $\mathcal{X}$-variable may be associated to a number of different arcs. Further, an arc may be associated to different $\mathcal{X}$-variables in different triangulations.

The next observation follows immediately from the definition of seed mutation.

Remark 4.1. Consider a seed $(\mathbf{x}, B)$. For $j \neq k$, if $b_{jk} = 0$, then mutating at k will not change x_j. Further, $b'_{ji} = b_{ji}$ and $b'_{ij} = b_{ij}$ for all i, since the skew-symmetrizability of B implies $b_{kj} = 0$ as well. Thus, if $k_1, \ldots, k_t$ are indices such that $b_{k_s j} = 0$, then the mutation sequence $\mu_{k_t} \circ \cdots \circ \mu_{k_2} \circ \mu_{k_1}$ leaves x_j unchanged.

In other words, let x be an $\mathcal{X}$-variable of Σ_T corresponding to arc $\gamma \in T$. Any sequence of flips of arcs *not* in $q_T(\gamma) \cup \{\gamma\}$ will result in a triangulation T' such that $\gamma \in T'$ and the $\mathcal{X}$-variable of $\Sigma_{T'}$ corresponding to γ will be x. This, in combination with results on the connectivity of certain subgraphs of the flip graph of P [3], gives the following proposition.

Proposition 4.2. *Let $Q' = \{q_T(\gamma) \cup \{\gamma\} |\ T$ a triangulation of $P, \gamma \in T\}$ be the set of quadrilaterals (with a choice of diagonal) of P. Then there is a surjection $f : Q' \to \mathcal{X}(\mathcal{S})$.*

We remark that this proposition in fact holds in the generality of $\mathcal{X}$-seed patterns from surfaces, though we do not need that here.

We have the immediate corollary:

Corollary 4.3. *Let $\mathfrak{q}(P) = |\{q_T(\gamma) |\ T$ a triangulation of $P, \gamma \in T\}|$ denote the number of quadrilaterals of P. Then $|\mathcal{X}(\mathcal{S})| \leq 2\mathfrak{q}(P)$.*

The above statements hold regardless of the choice of $\mathbb{P}$ and initial $\mathcal{X}$-cluster. Because $|\mathcal{X}(\mathcal{S})|$ can easily be 1, it is clear that we must make additional assumptions to determine anything further about $|\mathcal{X}(\mathcal{S})|$. If we fix an exchange matrix B and allow $\mathbb{P}$ and the $\mathcal{X}$-cluster of the seed $(\mathbf{x}, B)$ to vary, $\mathcal{S}(\mathbf{x}, B)$ will have the largest number of $\mathcal{X}$-variables when $\mathbb{P} = \mathbb{Q}_{sf}(t_1, \ldots, t_n)$ and $\mathbf{x}$ consists of elements that are algebraically independent over $\mathbb{Q}(t_1, \ldots, t_n)$. Indeed, let $\mathcal{S}_{sf}$ be such a seed pattern, and $\mathcal{S}$ an arbitrary seed pattern containing the exchange matrix B. The $\mathcal{X}$-variables of $\mathcal{S}$ can be obtained from the $\mathcal{X}$-variables of $\mathcal{S}_{sf}$ by replacing "$+$" with "$\oplus$" and evaluating at the appropriate elements of $\mathbb{P}$, so we have $|\mathcal{X}(\mathcal{S})| \leq |\mathcal{X}(\mathcal{S}_{sf})|$.

In light of this observation, we now focus on $\mathcal{X}(\mathcal{S}_{sf})$. To show that the surjection f of Proposition 4.2 is a bijection for $\mathcal{S}_{sf}$, it suffices to show that f is injective for some $\mathcal{X}$-seed pattern of type Z_n.

To do this, we use the "geometric" $\mathcal{A}$-seed pattern of type Z_n given in [6, 4], which we denote by $\mathcal{R}(Z_n)$. Briefly, the $\mathcal{A}$-variables are certain elements of $\mathbb{C}[\mathrm{Mat}_{2,m}]$, the algebra of polynomial functions on the space of $2 \times m$ complex matrices, where m depends on Z and n. In $\mathcal{R}(A_n)$ for example, $m = n + 3$ and the $\mathcal{A}$-variables are exactly the Plücker coordinates; the $\mathcal{X}$-variables are quotients of Plücker coordinates.

Proposition 4.4. *Consider the $\mathcal{A}$-seed pattern $\mathcal{R} = \mathcal{R}(Z_n)$. Then the surjection $f : Q' \to \mathcal{X}(\hat{\mathcal{R}})$ of Proposition 4.2 is injective, where $\hat{\mathcal{R}}$ is given in [7, Proposition 3.9].*

This proposition is proved by showing that the $\mathcal{X}$-variables in $\hat{\mathcal{R}}$ associated to different quadrilaterals have different values on specific elements of $\mathrm{Mat}_{2,m}$.

Theorem 1.1 follows as a corollary, as does the number of $\mathcal{X}$-variables in $\mathcal{S}_{sf}$.

Corollary 4.5. $|\mathcal{X}(\mathcal{S}_{sf})| = 2\mathfrak{q}(P)$.

4.1 Quadrilateral counts

Proposition 4.6. *The number of quadrilaterals of each surface P is*

P	P_{n+3}	$P^G_{2n+2}, P^{\bullet G}_{n+1}$	$P^{\bullet}_n$
$\mathfrak{q}(P)$	$\binom{n+3}{4}$	$\frac{1}{6}n(n+1)(n^2+2)$	$\frac{1}{6}n(n-1)(n^2+4n-6)$

For $P \neq P^G_{2n+2}$, $\mathfrak{q}(P)$ follow from a fairly straightforward inspection of the triangulations of the appropriate surfaces. For $P = P^G_{2n+2}$, $\mathfrak{q}(P)$ follows from a bijection between quadrilaterals not appearing in centrally symmetric triangulations of P_{2n+2} and quadrilaterals of P_{2n+2} that appear in centrally symmetric triangulations and are not fixed by G-action.

4.2 Exceptional types

Let $Z \in \{E_6, E_7, E_8, F_4, G_2\}$ and let $\mathcal{S}_{sf}$ be a type Z $\mathcal{X}$-seed pattern over $\mathbb{Q}_{sf}$ with one (equivalently every) $\mathcal{X}$-cluster consisting of algebraically independent elements. $\mathcal{X}(\mathcal{S}_{sf})$ was computed using a computer algebra system (Mathematica), by generating all possible $\mathcal{X}$-seeds via mutation.

5 Corollaries and Conjectures

We make a few remarks regarding the implications of these results to the larger theory of $\mathcal{X}$-seed patterns and cluster algebras, and conjectural extensions.

The bijection between quadrilaterals and $\mathcal{X}$-variables for $\mathcal{S}_{sf}$ implies

Corollary 5.1. *The exchange graph of $\mathcal{S}_{sf}$ in types A, B, C, and D coincides with the exchange graph of any $\mathcal{A}$-seed pattern of the same type.*

Recall that the diagonals of a quadrilateral can be uniquely associated to a pair of $\mathcal{A}$-variables. These pairs are precisely those variables that appear together on the left hand side of an exchange relation; such $\mathcal{A}$-variables are called *exchangeable*. Composing the bijection from ordered pairs of exchangeable $\mathcal{A}$-variables to quadrilaterals with a choice of diagonal with the bijection of Theorem 1.1 gives the following corollary for classical types. It was checked by computer for exceptional types.

Corollary 5.2. *Let $\mathcal{R}$ be an $\mathcal{A}$-seed pattern of finite type. There is a bijection between ordered pairs of exchangeable $\mathcal{A}$-variables in $\mathcal{R}$ and $\mathcal{X}(\mathcal{S}_{sf})$.*

If $\mathcal{R}$ is a finite type $\mathcal{A}$-seed pattern over the tropical semifield, the numbers computed here are the number of $\mathcal{X}$-variables in $\hat{\mathcal{R}}$ if one (equivalently, every) extended exchange matrix of $\mathcal{R}$ is full rank. Recall that in this setting, the $\mathcal{X}$-variables of $\hat{\mathcal{R}}$ exactly record the two terms on the right hand side of an exchange relation. In the bijection of Corollary 5.2, the pairs of exchangeable $\mathcal{A}$-variables are mapped exactly to the $\mathcal{X}$-variables recording the exchange relation that the $\mathcal{A}$-variables satisfy. This implies

Corollary 5.3. *Let $\mathcal{R}$ be an $\mathcal{A}$-seed pattern of type A, B, C, or D over the tropical semifield with one full rank extended exchange matrix. Then the two terms of the right hand side of an exchange relation* (2.2) *uniquely determines the left.*

Conjecture 5.4. *Let $\mathcal{S}$ be an $\mathcal{X}$-seed pattern from a marked surface (S, M). Then the map from $\{q_T(\gamma) \cup \{\gamma\} |\ T$ a triangulation of $(S, M), \gamma \in T\}$ and $\mathcal{X}(\mathcal{S}_{sf})$, which sends $q \cup \{\gamma\}$ to $x_{q,\gamma}$, is a bijection.*

Lastly, in the original development of finite type $\mathcal{A}$-seed patterns, seed patterns were connected to root systems of the same type. In particular, there is a bijection between $\mathcal{A}$-variables and almost positive roots (positive roots and negative simple roots). Two variables are exchangeable if and only if the corresponding roots α, β have $(\alpha||\beta) = (\beta||\alpha) = 1$, where $(-||-)$ is the *compatibility degree* [6]. This, combined with Corollary 5.2, give a root theoretic interpretation of the number of $\mathcal{X}$-coordinates of $\mathcal{S}_{sf}$.

Corollary 5.5. *For $\mathcal{S}_{sf}$ of finite type, $|\mathcal{X}(\mathcal{S}_{sf})|$ is the number of pairs of roots (α, β) such that $(\alpha||\beta) = (\beta||\alpha) = 1$ in the root system of the same type.*

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