*Séminaire Lotharingien de Combinatoire* **80B** (2018)
Article #56, 12 pp.
*Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover)*

# Connectivity Properties of Factorization Posets in Generated Groups

Henri Mühle*[1] and Vivien Ripoll†[2]

[1]*Institut für Algebra, Technische Universität Dresden, Zellescher Weg 12–14, 01069 Dresden, Germany*
[2]*Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria*

**Abstract.** We consider three notions of connectivity and their interactions in partially ordered sets coming from reduced factorizations of elements in generated groups. While one form of connectivity essentially reflects the connectivity of the poset diagram, the other two are a bit more involved: Hurwitz-connectivity has its origins in algebraic geometry, and shellability in topology. We propose a framework to study these connectivity properties in a uniform way. Our main tool is a certain total order of the generators that is compatible with the chosen element.

**Résumé.** Nous considérons trois notions de connectivité et leurs interactions pour les posets construits à partir des factorisations réduites d'éléments d'un groupe engendré. Une forme de connectivité reflète essentiellement la connectivité du diagramme du poset, tandis que les deux autres sont un peu plus complexes: la connectivité d'Hurwitz a ses origines en géométrie algébrique, et l'épluchabilité est une notion topologique. Nous proposons un cadre pour étudier ces propriétés de connectivité de manière uniforme. L'outil principal est un certain ordre total sur les générateurs, qui est compatible avec l'élément choisi.

**Keywords:** generated group, braid group, Hurwitz action, factorization poset, shellability, compatible order, well covered poset, noncrossing partition lattice

# 1 Introduction

The main objects in this work are factorization posets $\mathcal{P}_c(G, A)$ coming from a group $G$ with a fixed generating set $A$, and some group element $c \in G$. When $A$ is closed under $G$-conjugation, there is a natural action of the braid group on the set of maximal chains of $\mathcal{P}_c(G, A)$, and the number of orbits under this action can be interpreted as a "connectivity coefficient" of $\mathcal{P}_c(G, A)$.

*henri.muehle@tu-dresden.de
†vivien.ripoll@univie.ac.at

This article revolves around the relation of the previously described *Hurwitz-connectivity* to two other forms of connectivity of a poset: *chain-connectivity* (motivated by graph theory) and *shellability* (motivated by topology). The main result of this article is the following uniform approach to proving Hurwitz-connectivity, chain-connectivity and shellability of the factorization poset $\mathcal{P}_c(G, A)$. The statement of this result uses two notions that will be formally defined later in the article, namely a total order of the generators that is compatible with $c$ (Definition 4.2), and a certain "well-covered" property (Definition 4.10). The latter property asserts that for every generator that is not minimal with respect to a given total order we can find a smaller generator such that both have a common upper cover in $\mathcal{P}_c(G, A)$.

Let us fix the following notation for the upcoming three statements. Let $G$ denote a group that is generated by $A \subseteq G$ as a monoid; we assume that $A$ is closed under $G$-conjugation. For $c \in G$ let $A_c \subseteq A$ denote the set of all generators that appear in at least one $A$-reduced factorization of $c$.

**Theorem 1.1.** *If the factorization poset $\mathcal{P}_c(G, A)$ is finite, admits a $c$-compatible order $\prec$ of $A_c$ and is totally well covered with respect to $\prec$, then $\mathcal{P}_c(G, A)$ is chain-connected, Hurwitz-connected and shellable.*

We want to emphasize that Theorem 1.1 uniformly and simultaneously approaches the question whether a factorization poset is chain-connected, Hurwitz-connected or shellable. Note that it is far from trivial in full generality to establish that a factorization poset is well covered and admits a compatible order. However, for some special groups the framework presented here may provide a convenient method to reach uniform insights about the connectivity of the respective factorization posets.

By definition, to be well covered is a property of the factorization poset with respect to a given total order of the generators. We conjecture that we can weaken the assumptions of Theorem 1.1 a bit, and we prove this conjecture for the Hurwitz-connectivity part.

**Theorem 1.2.** *If the factorization poset $\mathcal{P}_c(G, A)$ is finite, chain-connected and admits a $c$-compatible order of $A_c$, then the Hurwitz action is transitive on $\mathrm{Red}_A(c)$.*

**Conjecture 1.3.** *If the factorization poset $\mathcal{P}_c(G, A)$ is finite, totally chain-connected and admits a $c$-compatible order of $A_c$, then $\mathcal{P}_c(G, A)$ is shellable.*

This manuscript is an extended abstract of [9]. We have omitted here the proofs of statements which are either straightforward to prove or which can be proven analogously to statements that have appeared (as special cases of our construction) before. We refer the reader to the full version for those missing proofs and illustrating examples.

In Section 2 we formally define chain-connectivity, Hurwitz-connectivity and shellability. In the process we recall the necessary background and define the needed concepts. In Section 3 we investigate relations between our three connectivity properties without any further assumptions. The heart of this work is Section 4 in which we define the notion of a compatible order of the generators and the "well-covered" property. We prove Theorem 1.2 and provide an equivalent formulation of Conjecture 1.3. This section culminates in the proof of Theorem 1.1. We refer the interested reader to [9, Figure 1] for an overview of implications, non-implications and conjectures between the different types of connectivity considered in this abstract.

# 2 Three Notions of Connectivity

## 2.1 Poset Terminology

Let us start by recalling the basic concepts from the theory of partially ordered sets, and by introducing a first notion of connectivity.

A partially ordered set (*poset* for short) is a set $P$ equipped with a partial order $\leq$, and we usually write $\mathcal{P} = (P, \leq)$. If $\mathcal{P}$ has a least element $\hat{0}$ and a greatest element $\hat{1}$, then it is *bounded*, and the *proper part* of $\mathcal{P}$ is the subposet $\overline{\mathcal{P}} = \bigl(P \setminus \{\hat{0}, \hat{1}\}, \leq\bigr)$. An *interval* of $\mathcal{P}$ is a set of the form $[x, y] = \{z \in P \mid x \leq z \leq y\}$ for $x, y \in P$ with $x \leq y$.

Two elements $x, y \in P$ form a *covering pair* if $x < y$ and there is no $z \in P$ with $x < z < y$. We then write $x \lessdot y$, and equivalently say that $x$ *is covered by* $y$ or that $y$ *covers* $x$. We denote the set of covering pairs of $\mathcal{P}$ by $\mathscr{E}(\mathcal{P})$.

From now on we will only consider finite posets. A *chain* of $\mathcal{P}$ is a totally ordered subset $C \subseteq P$, meaning that for any $x, y \in C$ we have $x < y$ or $y < x$. If $C = \{x_1, x_2, \ldots, x_k\}$ with $x_i < x_j$ whenever $i < j$, we occasionally use the notation $C : x_1 < x_2 < \cdots < x_k$ to emphasize the order of the elements. Moreover, a chain is *maximal* if it is not properly contained in some other chain. Let $\mathscr{M}(\mathcal{P})$ denote the set of maximal chains of $\mathcal{P}$. A poset is *graded* if all maximal chains have the same cardinality; this common cardinality minus one is called the *rank* of $\mathcal{P}$, denoted by $\mathrm{rk}(\mathcal{P})$.

The first notion of connectivity of a poset that springs to mind is the connectivity of its poset diagram, namely the graph $\bigl(P, \mathscr{E}(\mathcal{P})\bigr)$. Observe that this graph is trivially connected whenever $\mathcal{P}$ is bounded. However, the poset diagram of the proper part of a bounded poset need not be connected. We are in fact interested in the following stronger version of connectivity.

**Definition 2.1.** *Let $\mathcal{P}$ be a graded, bounded poset, and define*

$$\mathscr{I}_{chain} = \bigl\{\{C, C'\} \mid |C \cap C'| = \mathrm{rk}(\mathcal{P})\bigr\}. \tag{2.1}$$

*The chain graph of $\mathcal{P}$ is the graph $\mathscr{C}(\mathcal{P}) = \bigl(\mathscr{M}(\mathcal{P}), \mathscr{I}_{chain}\bigr)$.*

In other words two maximal chains of $\mathcal{P}$ are adjacent in the chain graph if they differ in exactly one element. We call $\mathcal{P}$ *chain-connected* if $\mathscr{C}(\mathcal{P})$ is connected. Observe that the poset diagram of the proper part of a chain-connected poset is again connected as soon as the rank of $\mathcal{P}$ is at least three. Moreover, if every interval of $\mathcal{P}$ is chain-connected, then we call $\mathcal{P}$ *totally chain-connected*.

## 2.2 Factorization Posets in Generated Groups

In this section we introduce the main construction that associates a bounded graded poset with each triple $(G, A, c)$, where $G$ is a group generated as a monoid by the set $A \subseteq G$, and where $c$ is some element of $G$.

Fix a group $G$ and a subset $A \subseteq G$ that generates $G$ as a monoid. We then call the pair $(G, A)$ a *generated group*, and we define the *A-length* of $x \in G$ by

$$\ell_A(x) = \min\bigl\{k \in \mathbb{N} \mid x = a_1a_2\cdots a_k, \text{ where } a_i \in A \text{ for } i \in [k]\bigr\}, \tag{2.2}$$

where $[k] = \{1, 2, \ldots, k\}$. If $k = \ell_A(x)$, then any factorization $x = a_1a_2\cdots a_k$ is called *A-reduced*. Let $\mathrm{Red}_A(x)$ denote the set of all $A$-reduced factorizations of $x \in G$. We consider the following partial order on $G$; the *A-prefix order*:

$$x \leq_A y \quad \text{if and only if} \quad \ell_A(y) = \ell_A(x) + \ell_A(x^{-1}y). \tag{2.3}$$

In other words, $x \leq_A y$ if and only if $x$ lies on a geodesic from $\mathbb{1}$ to $y$ in the right Cayley graph of $(G, A)$, where $\mathbb{1}$ denotes the identity of $G$. The definition of the $A$-prefix order as given in (2.3) has perhaps first appeared explicitly in [6] in the case of the symmetric group.

Now fix some $c \in G$ and consider the interval $[\mathbb{1}, c]$ in $(G, \leq_A)$, i.e. the poset

$$\mathcal{P}_c(G, A) = \bigl(\{g \in G \mid g \leq_A c\}, \leq_A\bigr), \tag{2.4}$$

which we call the *factorization poset* of $c$ in $(G, A)$. Whenever it is clear from the context, we omit the group and the generating set. Observe that a poset is completely determined by its set of maximal chains; and in the given setting, the maximal chains of $\mathcal{P}_c$ correspond bijectively to the $A$-reduced factorizations of $c$ via the map

$$\begin{aligned} \lambda_c : \mathscr{M}(\mathcal{P}_c) &\to \mathrm{Red}_A(c) \\ \mathbb{1} = x_0 \lessdot_A x_1 \lessdot_A \cdots \lessdot_A x_n = c &\mapsto \bigl(x_0^{-1}x_1, x_1^{-1}x_2, \ldots, x_{n-1}^{-1}x_n\bigr), \end{aligned} \tag{2.5}$$

where $n = \ell_A(c)$.

**Example 2.2.** *Let $G = \mathfrak{S}_4$ be the symmetric group of permutations of $[4]$. It is well known that $\mathfrak{S}_4$ is generated by its set of transpositions $T = \bigl\{(1\,2), (1\,3), (1\,4), (2\,3), (2\,4), (3\,4)\bigr\}$. Since any transposition is an involution, $T$ generates $\mathfrak{S}_4$ as a monoid. It is moreover easy to check that $T$ is closed under $\mathfrak{S}_4$-conjugation. Let $c = (1\ 2\ 3\ 4)$ be a long cycle in $\mathfrak{S}_4$. The factorization poset $\mathcal{P}_c(\mathfrak{S}_4, T)$ is shown in Figure 1.*

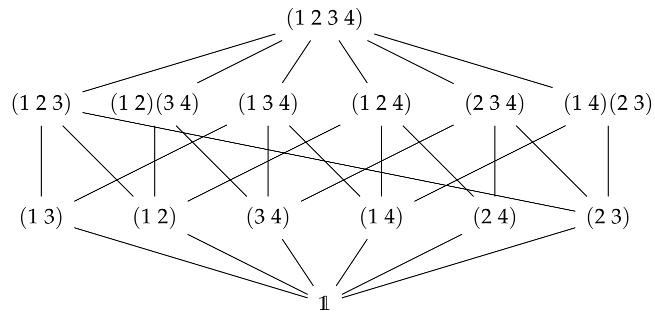



**Figure 1:** The factorization poset $\mathcal{P}_c$ of the long cycle $c = (1\ 2\ 3\ 4)$ in the symmetric group $\mathfrak{S}_4$ generated by its transpositions.

Perhaps the most important consequence of the assumption that $A$ is closed under $G$-conjugation is the existence of a braid group action on $\mathrm{Red}_A(x)$ (and thus in view of (2.5) also on $\mathscr{M}(\mathcal{P}_c)$.) Recall that the *braid group* on $n$ strands can be defined via the group presentation

$$\mathfrak{B}_n = \big\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i \in [n-2], \\ \text{and } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } i,j \in [n-1] \text{ with } |i-j| > 1\big\rangle. \tag{2.6}$$

Now fix $x \in G$ with $\ell_A(x) = n$. For $i \in [n-1]$, we define an action of the braid group generator $\sigma_i$ on $\mathrm{Red}_A(x)$ by

$$\begin{aligned} &\sigma_i \cdot (a_1, \ldots, a_{i-1}, \quad a_i, \quad a_{i+1}, \quad a_{i+2}, \ldots, a_n) \\ = \;&\quad (a_1, \ldots, a_{i-1}, \quad a_{i+1}, \quad a_{i+1}^{-1}a_ia_{i+1}, \quad a_{i+2}, \ldots, a_n). \end{aligned} \tag{2.7}$$

In other words, the generators of $\mathfrak{B}_n$ swap two consecutive factors of an $A$-reduced factorization of $x$ and conjugate one by the other, so that the product stays the same. Since $A$ is closed under $G$-conjugation, $\sigma_i$ is indeed a map on $\mathrm{Red}_A(x)$, and it is straightforward to verify that this action respects the relations of (2.6), and therefore extends to a group action of $\mathfrak{B}_n$ on $\mathrm{Red}_A(x)$: the *Hurwitz action*.

We can now define the second notion of connectivity used in this abstract.

**Definition 2.3.** *Let $c \in G$, and define*

$$\mathscr{I}_{hurwitz} = \big\{ \{\mathbf{x}, \mathbf{x}'\} \mid \mathbf{x}, \mathbf{x}' \in Red_A(c) \text{ and } \mathbf{x}' = \sigma_i\mathbf{x} \text{ for some } i \in [\ell_A(c) - 1] \big\}. \tag{2.8}$$

*The* Hurwitz graph *of $c$ is the graph $\mathscr{H}(c) = \big(Red_A(c), \mathscr{I}_{hurwitz}\big)$.*

In view of (2.5) we may as well define the Hurwitz graph of $c$ as a graph on the maximal chains of $\mathcal{P}_c$, and from this point of view it is clearly (isomorphic to) a subgraph of $\mathscr{C}(\mathcal{P}_c)$. We call $\mathcal{P}_c$ *Hurwitz-connected* if $\mathscr{H}(c)$ is connected. This is the case if and only

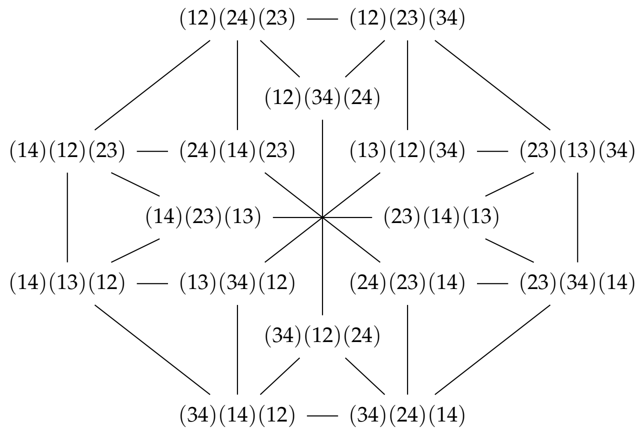



**Figure 2:** The Hurwitz graph of the long cycle $c = (1\ 2\ 3\ 4)$ in the symmetric group $\mathfrak{S}_4$ generated by its transpositions.

if the braid group $\mathfrak{B}_{\ell_A(c)}$ acts transitively on $\mathrm{Red}_A(c)$. By abuse of notation we sometimes write $\mathscr{H}(\mathcal{P}_c)$ instead of $\mathscr{H}(c)$. Figure 2 shows the Hurwitz graph of the factorization poset from Figure 1.

## 2.3 Shellability of Posets

The last notion of connectivity that will be important for this article has its origins in algebraic topology. Recall that the set of chains of a graded poset $\mathcal{P}$ forms a pure simplicial complex; the *order complex* of $\mathcal{P}$, denoted by $\Delta(\mathcal{P})$.

In this section we want to outline how a simple combinatorial tool, an edge-labeling of a graded bounded poset $\mathcal{P}$, may serve to learn about the homotopy type of $\Delta(\overline{\mathcal{P}})$. A class of pure simplicial complexes with a particularly nice homotopy type are the *shellable* simplicial complexes: their homotopy type is in fact that of a wedge of spheres [5, Theorem 4.1], the corresponding (co-)homology groups are torsion-free, and the Stanley–Reisner ring of such complexes is Cohen–Macaulay [3, Appendix].

We phrase the definition of shellability directly in terms of a graded bounded poset $\mathcal{P}$. It can be transferred to pure simplicial complexes via the correspondence between maximal chains of $\mathcal{P}$ and facets of $\Delta(\overline{\mathcal{P}})$.

**Definition 2.4.** *Let $\mathcal{P}$ be a graded bounded poset. A* shelling *of $\mathcal{P}$ is a linear order $\prec$ on $\mathscr{M}(\mathcal{P})$ such that whenever two maximal chains $M, M' \in \mathscr{M}(\mathcal{P})$ satisfy $M \prec M'$, then there exists $N \in \mathscr{M}(\mathcal{P})$ with $N \prec M'$ and $x \in M'$ with the property that $M \cap M' \subseteq N \cap M' = M' \setminus \{x\}$.*

A poset that admits a shelling is *shellable*. There is a nice combinatorial way to establish shellability, by exhibiting a particular edge-labeling of the poset. An *edge-labeling* of $\mathcal{P}$ is a map $\lambda : \mathscr{E}(\mathcal{P}) \to \Lambda$, where $\Lambda$ is an arbitrary partially ordered set.

An edge-labeling of $\mathcal{P}$ naturally extends to a labeling of $\mathscr{M}(\mathcal{P})$, where for $C : \hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ we set $\lambda(C) = \big(\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k)\big)$.

A maximal chain $C \in \mathscr{M}(\mathcal{P})$ is *rising* if $\lambda(C)$ is weakly increasing with respect to the partial order on $\Lambda$, and it is *falling* if $\lambda(C)$ is strictly decreasing. A chain $C \in \mathscr{M}(\mathcal{P})$ *precedes* a chain $C' \in \mathscr{M}(\mathcal{P})$ if $\lambda(C)$ is lexicographically smaller than $\lambda(C')$ with respect to the order on $\Lambda$. An edge-labeling $\lambda$ of $\mathcal{P}$ is an *EL-labeling* if in every interval of $\mathcal{P}$ there exists a unique rising maximal chain, and this chain precedes every other maximal chain in that interval. A poset that admits an EL-labeling is *EL-shellable*. A. Björner proved the following fundamental property.

**Theorem 2.5** ([3, Theorem 2.3])**.** *Every EL-shellable poset is shellable.*

The converse of Theorem 2.5 is not true, see for instance [11, 12]. We observe that in the case of factorization posets coming from a generated group $(G, A)$ the map (2.5) induces an edge-labeling of $\mathcal{P}_c$ whose labels are elements of $A$. One of the main motivations for this work was the question whether there is a local criterion to guarantee that a total order on $A$ turns this labeling into an EL-labeling.

**Remark 2.6.** *The motivating example for the work presented here are the lattices of c-noncrossing W-partitions, where W is an irreducible, well-generated complex reflection group, and $c \in W$ is a Coxeter element. See [2, 10] for more background on these posets. It is well known that these posets are Hurwitz-connected [2, 7], and EL-shellable [1, 8].*

# 3 Interaction of Different Types of Connectivity

In this section we investigate the implications between the three types of connectivity.

**Proposition 3.1.** *Every Hurwitz-connected bounded graded poset is chain-connected. Every shellable bounded graded poset is chain-connected.*

None of the converse statements in Proposition 3.1 is true without further assumptions, see for instance [9, Examples 4.2 and 4.3]. We are not aware of a factorization poset that is totally chain-connected, but not shellable.

**Problem 3.2.** *Find a generated group $(G, A)$ and some $c \in G$ such that $Red_A(c)$ is finite and $\mathcal{P}_c(G, A)$ is totally chain-connected but not shellable, or show that this cannot exist.*

A solution to Problem 3.2 would be of great importance within the framework presented here: we could either reduce the difficulty to prove that a factorization poset is shellable, or the group structure of such an example would exhibit a new obstruction to shellability.

Example 4.2 in [9] implies that chain-connectivity does not necessarily imply Hurwitz-connectivity. However, we may add the following local criterion to make things work.

**Definition 3.3.** *Let $\mathcal{P}_c(G,A)$ be factorization poset. If $\mathfrak{B}_2$ acts transitively on $\mathrm{Red}_A(g)$ for every $g \leq_A c$ with $\ell_A(g) = 2$, then we call $\mathcal{P}_c(G,A)$ locally Hurwitz-connected.*

**Theorem 3.4.** *Assume $A$ is closed under $G$-conjugation, and fix $c \in G$ with $\ell_A(c) = n$. If $\mathcal{P}_c$ is chain-connected and locally Hurwitz-connected, then $\mathfrak{B}_n$ acts transitively on $\mathrm{Red}_A(c)$, i.e., $\mathcal{P}_c$ is Hurwitz-connected.*

Example 4.7 in [9] illustrates that being locally Hurwitz-connected is actually not a necessary condition for the Hurwitz-connectivity of $\mathcal{P}_c$.

# 4 Compatible $A$-Orders

In this section we introduce our main tool, a total order of $A$ that is compatible with the top element $c$. This concept is an algebraic generalization of the compatible reflection order introduced in [1], and it also appeared in [8] in the context of reflection groups. In order to make this definition work, we assume from now on that $\mathrm{Red}_A(c)$ is finite (which is equivalent to the requirement that $\mathcal{P}_c$ is finite).

## 4.1 Definition and Properties of Compatible Orders

Let $A_c = \{a \in A \mid a \leq_A c\}$. Observe that we trivially have $\mathrm{Red}_A(c) = \mathrm{Red}_{A_c}(c)$. Let $\prec$ be any total order on $A_c$. We say that a factorization $(a_1, a_2, \ldots, a_{\ell_A(c)}) \in \mathrm{Red}_A(c)$ is *$\prec$-rising* if $a_i \preceq a_{i+1}$ for $i \in [\ell_A(c) - 1]$. We denote by $\mathrm{Rise}(c;\prec)$ the number of $\prec$-rising $A$-reduced factorizations of $c$ for a given total order $\prec$ on $A_c$.

The next statement relates these rising factorizations to the Hurwitz orbits of $\mathrm{Red}_A(c)$, in the specific case when $\ell_A(c) = 2$.

**Proposition 4.1.** *Let $c \in G$ have $\ell_A(c) = 2$. For any total order $\prec$ on $A_c$, the number* $\mathrm{Rise}(c;\prec)$ *is at least as large as the number of Hurwitz orbits of $\mathrm{Red}_A(c)$, and there exists a total order on $A_c$ such that these numbers are equal.*

If $\ell_A(c) > 2$, then it is not guaranteed that we can find a total order $\prec$ on $A_c$ such that $\mathrm{Rise}(c;\prec)$ equals the number of Hurwitz orbits of $\mathrm{Red}_A(c)$, see for instance [9, Example 5.2]. This brings us to the main definition of this section.

**Definition 4.2.** *A total order $\prec$ on $A_c$ is $c$-compatible if for any $g \leq_A c$ with $\ell_A(g) = 2$ there exists a unique $\prec$-rising $A$-reduced factorization of $g$.*

Proposition 4.1 has the following immediate consequences.

**Corollary 4.3.** *If $\ell_A(c) = 2$, then there exists a $c$-compatible order of $A_c$ if and only if $\mathcal{P}_c(G,A)$ is Hurwitz-connected.*

**Corollary 4.4.** *If there exists a $c$-compatible order of $A_c$, then $\mathcal{P}_c(G, A)$ is locally Hurwitz-connected.*

We see immediately that if $G$ is abelian, then for any $c \in G$ every total order of $A$ is $c$-compatible. Moreover, [9, Example 5.2] illustrates that there are factorization posets which do not admit a compatible order. However, there are also non-trivial examples.

**Example 4.5.** *Let us continue Example 2.2. Let $\prec$ be the lexicographic order on the set $T$ of transpositions of $\mathfrak{S}_4$, i.e. $(1\,2) \prec (1\,3) \prec (1\,4) \prec (2\,3) \prec (2\,4) \prec (3\,4)$. It is straightforward to check that this order is $c$-compatible for $c = (1\,2\,3\,4)$.*

*On the other hand, if we consider the following total order $(1\,3) \prec' (1\,2) \prec' (1\,4) \prec' (2\,3) \prec' (2\,4) \prec' (3\,4)$, then we observe that $(1\,2\,3)$ has two $\prec'$-rising $T$-reduced factorizations, namely $\big((1\,2), (2\,3)\big)$ and $(1\,3), (1\,2)\big)$.*

We are now in the position to prove Theorem 1.2.

*Proof of Theorem 1.2.* Suppose that $\mathcal{P}_c$ is chain-connected and admits a $c$-compatible order $\prec$ of $A_c$. Then Corollary 4.4 implies that $\mathcal{P}_c$ is locally Hurwitz-connected. Theorem 3.4 now implies that $\mathfrak{B}_{\ell_A}(c)$ acts transitively on $\mathrm{Red}_A(c)$. $\square$

However, [9, Example 5.2] once again shows that there are cases where $\mathrm{Red}_A(c)$ is Hurwitz-connected, but there does not exist a $c$-compatible order of $A_c$.

In the remainder of this section we provide some evidence that $c$-compatible orders are also closely related to the shellability of the factorization poset $\mathcal{P}_c$. Recall that for any total order on $A_c$ we can consider the lexicographic order on $\mathrm{Red}_A(c)$, which is itself a total order.

**Lemma 4.6.** *For any total order $\prec$ of $A_c$ and any $g, h \in G$ with $g \leq_A h \leq_A c$ the lexicographically smallest $A$-reduced factorization of $g^{-1}h$ is $\prec$-rising.*

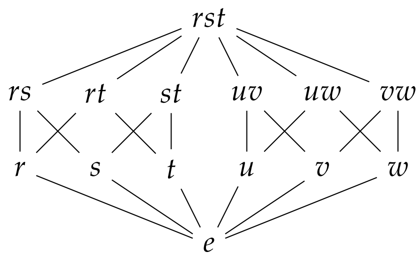
The following conjecture implies Conjecture 1.3.

**Conjecture 4.7.** *The natural labeling $\lambda_c$ from (2.5) induces an EL-labeling of $\mathcal{P}_c$ with respect to some total order $\prec$ of $A_c$ if and only if $\mathcal{P}_c$ is totally chain-connected and $\prec$ is $c$-compatible.*

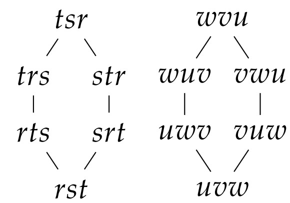
Observe that one direction of Conjecture 4.7 is trivially true. If $\lambda_c$ is an EL-labeling of $\mathcal{P}_c$ with respect to $\prec$, then every interval of $\mathcal{P}_c$ is shellable, and by Proposition 3.1 chain-connected. Since every rank-2 interval of $\mathcal{P}_c$ has a unique rising chain, it follows that $\prec$ is $c$-compatible.

Conjecture 4.7, however, does *not* suggest that $\mathcal{P}_c$ can only be EL-shellable if there exists a $c$-compatible order of $A_c$. If there is no $c$-compatible order of $A_c$, we may only conclude that $\lambda_c$ is not an EL-labeling (there may exist others, though).

The next example shows that the assumption of (total) chain-connectivity of $\mathcal{P}_c$ cannot be left out in Conjecture 4.7.




(a) An interval in the prefix order on the quotient of the free abelian group on six generators $r, s, t, u, v, w$ given by the relation $rst = uvw$.




(b) The Hurwitz graph of the poset in Figure 3a.

**Figure 3:** A factorization poset which admits a compatible generator order, but is neither chain-connected, nor EL-shellable.

**Example 4.8.** *Let $G$ be the quotient of the free abelian group on six generators $r, s, t, u, v, w$ given by the relation $rst = uvw$. The factorization poset $\mathcal{P}_{rst}$ is shown in Figure 3a; its chain graph, depicted in Figure 3b, has two connected components. Since the generators all commute, any total order on $\{r, s, t, u, v, w\}$ is $rst$-compatible, but we always find exactly two rising maximal chains.*

## 4.2 The Well-Covered Property and EL-Labelings

Let us first state some further properties of factorization posets admitting compatible orders. Fix a total order $\prec$ of $A_c$, and for $a \in A_c$ define

$$F_{\prec}(a;c) = \{g \in G \mid a \lessdot_A g \leq_A c \text{ and there is } a' \in A_c \text{ with } a' \prec a \text{ and } a' \lessdot_A g\}.$$

In other words, $F_{\prec}(a;c)$ consists of all upper covers of $a$ in $\mathcal{P}_c$ that also cover some $a' \prec a$.

**Proposition 4.9.** *If $\prec$ is a $c$-compatible order of $A_c$, then the natural labeling $\lambda_c$ satisfies*

*(i) if $g \in F_{\prec}(a;c)$, then $\lambda_c(a,g) \prec \lambda_c(\mathbb{1},a)$, and*

*(ii) if $g \notin F_{\prec}(a;c)$, then $\lambda_c(a,g) \succeq \lambda_c(\mathbb{1},a)$*

*for any $a \in A_c$ and any $g \in G$ with $a \lessdot_A g$.*

By definition for any total order $\prec$ of $A_c$ the set $F_{\prec}(a;c)$ is empty for $a = \min A_c$. Factorization posets in which this is the only case when $F_{\prec}(a;c)$ is empty will be awarded a special name. The purpose of this definition is that it provides a different perspective on Conjecture 1.3.

**Definition 4.10.** *A factorization poset $\mathcal{P}_c$ is well covered with respect to a total order $\prec$ of $A_c$ if $F_{\prec}(a;c)$ is empty if and only if $a = \min A_c$. Moreover, $\mathcal{P}_c$ is totally well covered with respect to $\prec$ if for all $g \in P_c$ the factorization poset $\mathcal{P}_g$ is well covered with respect to the appropriate restriction of $\prec$.*

In other words, $\mathcal{P}_c$ is well covered if and only if for every atom $a$ (except the smallest one with respect to $\prec$), we can find an upper cover $g$ of $a$ such that $a \neq \min A_g$.

**Example 4.11.** *Let us continue Example 2.2 once more, and fix the lexicographic order $\prec$ on $T$ from Example 4.5 again. It is straightforward to verify that $\mathcal{P}_c$ is well covered with respect to $\prec$.*

**Example 4.12.** *Let us continue Example 4.8. If we fix the total order $r \prec s \prec t \prec u \prec v \prec w$, then we observe that $F_{\prec}(u;rst)$ is empty even though $u$ is not minimal with respect to $\prec$. By definition $\mathcal{P}_{rst}$ is not well covered with respect to $\prec$. In fact, it is not well covered with respect to any total order on $\{r,s,t,u,v,w\}$.*

**Proposition 4.13.** *Let $\prec$ be a total order of $A_c$. If $\mathcal{P}_c$ is totally well covered with respect to $\prec$, then $\mathcal{P}_c$ is totally chain-connected.*

We proceed with the announced observation that $\mathcal{P}_c$ is totally well covered with respect to a $c$-compatible order $\prec$ if and only if $\lambda_c$ is an EL-labeling with respect to $\prec$.

**Theorem 4.14.** *Let $\prec$ be a total order of $A_c$. Then $\lambda_c$ is an EL-labeling of $\mathcal{P}_c$ if and only if $\prec$ is $c$-compatible and $\mathcal{P}_c$ is totally well covered with respect to $\prec$.*

We thus obtain the following equivalent statement of Conjecture 4.7.

**Conjecture 4.15.** *If $\mathcal{P}_c$ is totally chain-connected, then it is totally well covered with respect to any $c$-compatible order of $A_c$.*

**Remark 4.16.** *The well-covered property is modeled after Condition (ii) in [4, Definition 3.1], which introduces the concept of a recursive atom order of a bounded graded poset. Most of the statements in this section can be proven analogously to the corresponding statements in [4, Section 3] and are therefore omitted.*

*In particular, if Conjecture 4.15 were true, any $c$-compatible order of $A_c$ in a totally chain-connected factorization poset would be a recursive atom order.*

We conclude this extended abstract with the proof of Theorem 1.1 and a remark.

*Proof of Theorem 1.1.* Suppose that $\mathcal{P}_c$ admits a $c$-compatible order of $A_c$ and that it is totally well-covered. Proposition 4.13 implies that $\mathcal{P}_c$ is (totally) chain-connected. Moreover, Theorem 1.2 implies that it is Hurwitz-connected, and Theorem 4.14 implies that it is shellable. $\square$

**Remark 4.17.** *In the full article, we prove a special case of Conjecture 1.3, see [9, Theorem 6.8]. To do so, we introduce a directed, labeled graph whose vertex set is $A_c$; the cycle graph. This graph essentially encodes the Hurwitz orbits of the elements of length 2 in $\mathcal{P}_c(G, A)$. We then rephrase the existence of a compatible order of $A_c$ and the well-covered property of $\mathcal{P}_c(G, A)$ in graph-theoretical terms, and use it to verify Conjecture 1.3 under particular assumptions. See [9, Section 6] for the details.*

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