

# Eulerian polynomials on segmented permutations

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**Abstract.** We define a generalization of the Eulerian polynomials and the Eulerian numbers by considering a descent statistic on segmented permutations coming from the study of 2-species exclusion processes and a change of basis in a Hopf algebra. We give some properties satisfied by these generalized Eulerian numbers. We also define a  $q$ -analog of these Eulerian polynomials which gives back usual Eulerian polynomials and ordered Bell polynomials for specific values of its variables. We also define a noncommutative analog living in the algebra of segmented compositions. It gives us an explicit generating function and some identities satisfied by the generalized Eulerian polynomials such as a Worpitzky-type relation.

**Keywords:** Eulerian polynomials, ordered Bell numbers, segmented permutations, segmented compositions, Hopf algebras, generating functions

## Introduction

The *Eulerian numbers*  $A(n, k)$  are involved in many combinatorial problems. They count the number of permutations  $\sigma \in \mathfrak{S}_n$  with  $k$  descents, *i.e.*, positions of the values followed by a smaller value in  $\sigma$  written as a word. These numbers define a natural  $t$ -analog of  $n!$ , the *Eulerian polynomials* whose coefficient are the  $A(n, k)$ :

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)}.$$

These polynomials and numbers have been extensively studied over the years, see [7] for an overview.

Another approach is to consider these polynomials as a formal sum of permutations living in the algebra of free quasi-symmetric functions (**FQSym**) whose bases are indexed by permutations [3]. In this case, the polynomials are in fact living in a subalgebra of **FQSym**: the noncommutative symmetric functions (**Sym**) defined in [4]. Studying this noncommutative analogs easily gives back many properties of the Eulerian numbers, see [6].

More recently, in order to refine the probabilities arising from the study of the 2-species exclusion processes, the authors of [2] defined a recoil statistic on partially signed permutations. With a natural notion of inverse on these objects, this statistic corresponds

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to a descent statistic on *segmented permutations*  $\mathfrak{P}_n$ , permutations where values can be separated by vertical bars. Moreover, the authors defined a generalization of **FQSym** and its relation with a known generalization of **Sym**: the segmented composition quasi-symmetric functions algebra (**SCQSym**), defined in [5], using this descent statistic.

In this paper we study this descent statistic on segmented permutations through a generalization of the Eulerian numbers  $T(n, k)$  alongside with a natural refinement  $K(n, i, j)$  counting the number of segmented permutations having  $i$  descents and  $j$  bars. They define natural  $t$ -analogs and  $(q, t)$ -analogs of  $2^{n-1}n!$ , the number of segmented permutations of size  $n$ :

$$P_n(t) = \sum_{\sigma \in \mathfrak{P}_n} t^{\text{des}(\sigma)}, \quad (0.1)$$

$$\alpha_n(t, q) = \sum_{\sigma \in \mathfrak{P}_n} t^{\text{des}(\sigma)} q^{\text{seg}(\sigma)}. \quad (0.2)$$

In addition to generalizing the Eulerian polynomials in their definition and some of their properties, these polynomials involve the ordered Bell numbers and the Stirling numbers of the second kind that play an important role in combinatorics. Moreover, the  $\alpha_n$  gives back the usual Eulerian polynomials at  $q = 0$  and the ordered Bell polynomials at  $t = 0$ . Then we define a noncommutative analog  $\mathcal{A}_n(t, q)$  of the polynomials  $\alpha_n$  using the ribbon basis of **SCQSym**.

$$\mathcal{A}_n(t, q) = \sum_{I \Vdash n} t^{\text{des}(I)} q^{\text{seg}(I)} R_I. \quad (0.3)$$

Considering the expression of the  $\mathcal{A}_n$  on the complete basis we define below, we obtain in Theorem 3.3 an explicit expression of the generating function of the  $\alpha_n$ . As a consequence, we obtain in Proposition 3.5 an identity satisfied by our generalized Eulerian polynomials which is very similar to the one satisfied by the usual Eulerian polynomials. This identity can be seen as a  $t$ -analog of the Dobiński-type relation satisfied by the ordered Bell numbers in the sense of [1].

Finally, we use the generating function of the polynomials  $\alpha_n$  to prove Theorem 3.7, a generalization of Worpitzky's identities expressing the discrete derivatives of the  $n$ -th power of an integer in terms of the coefficients of  $\alpha_n$ .

We give all detailed definitions in Section 1. In Section 2, we define the generalized Eulerian numbers and polynomials with some elementary propositions. In section 3 we use a noncommutative analog of the generalized Eulerian polynomials to prove the main results of the paper.

# 1 Definitions and background

## 1.1 Segmented compositions and permutations

A *segmented composition* of an integer  $n$  (denoted by  $I \Vdash n$ ) is a finite sequence  $I = (i_1, \dots, i_r)$  of positive integers separated by commas or bars that sum to  $n$ . In the following examples, when there is no ambiguity we do not represent the commas. The integer  $r$  is called the *length* of the segmented composition, denoted by  $\ell(I)$ . We denote by  $\text{seg}(I)$  the number of bars in  $I$  and by  $\text{des}(I) = \ell(I) - \text{seg}(I)$  the number of values that are not followed by a bar. The *descent set* of a segmented composition  $I$ , denoted by  $\text{Des}(I)$ , is the set of values  $i_1 + i_2 + \dots + i_k$  for  $k < r$  where  $i_k$  is not followed by a bar in  $I$ . Similarly, the *segmentation set* of  $I$ , denoted by  $\text{Seg}(I)$ , is the set of values  $i_1 + i_2 + \dots + i_k$  for  $k < r$  where  $i_k$  is followed by a bar in  $I$ . Note that  $|\text{Des}(I)| = \text{des}(I) - 1$  and  $|\text{Seg}(I)| = \text{seg}(I)$ . For example, with  $I = 21|2|31$ ,  $(\text{Des}(I), \text{Seg}(I)) = (\{2, 8\}, \{3, 5\})$ . We shall use that when  $n$  is fixed, any segmented composition  $I$  is in bijection with the pair of sets  $(\text{Des}(I), \text{Seg}(I))$ . We define the *reverse refinement order* on segmented compositions by  $I \succeq K$  if and only if  $I$  and  $K$  are two segmented composition of the same integer and

$$\begin{aligned} \text{Des}(I) &\supseteq \text{Des}(K), \\ \text{Seg}(I) &\subseteq \text{Seg}(K) \subseteq \left( \text{Des}(I) \cup \text{Seg}(I) \right). \end{aligned} \tag{1.1}$$

In this case we say that  $I$  is finer than  $K$ . For example,  $213|22|121 \succeq 24|22|1|3$ .

Let  $I$  and  $K$  be two segmented compositions. Then  $I \cdot K$  represent their concatenation with a comma between the last value of  $I$  and the first value of  $K$ ,  $I \triangleright K$  represent their concatenation where the last value of  $I$  and the first value of  $K$  are added together, and  $I|K$  represent their concatenation where the last value of  $I$  is followed by a bar. For example, with  $I = 21|1$  and  $K = 1|15$ ,  $I \cdot K = 21|11|15$ ,  $I \triangleright K = 21|2|15$ , and  $I|K = 21|1|1|15$ .

A *segmented permutation* is a permutation where the values can be separated by bars. We denote by  $\mathfrak{P}_n$  the set of segmented permutations of size  $n$ . There are  $2^{n-1}n!$  segmented permutations of size  $n$ .

In this section,  $\sigma$  denotes a segmented permutation of size  $n$ . A position  $i < n$  is called a *segmentation* if there is a bar between  $\sigma_i$  and  $\sigma_{i+1}$ . A position  $i$  is a *descent* if it is not a segmentation and  $\sigma_i > \sigma_{i+1}$ . We denote by  $\text{des}(\sigma)$  (resp.  $\text{seg}(\sigma)$ ) the number of descents (resp. bars) of  $\sigma$ . For example, with  $\sigma = 3|7156|24$ , we have  $(\text{des}(\sigma), \text{seg}(\sigma)) = (1, 2)$ . We define the *segmented composition of descents*  $\sigma$ , denoted by  $\text{SDes}(\sigma)$ , as the segmented composition of  $n$  whose descent set corresponds to the descents of  $\sigma$  and whose segmentation set corresponds to the segmentations of  $\sigma$ . For example, with  $\sigma = 3|7156|24$  and  $I = (1|13|2)$ , we have  $\text{SDes}(\sigma) = I$  as  $\text{Des}(I) = \{2\}$  is the only position of descent in  $\sigma$  and  $\text{Seg}(I) = \{1, 5\}$  are the two positions of segmentation of  $\sigma$ . Note that with  $I = \text{SDes}(\sigma)$ , these definitions imply  $\text{des}(\sigma) = \text{des}(I) - 1$  and  $\text{seg}(\sigma) = \text{seg}(I)$ .

In [2], the authors defined a recoil statistic on *partially signed permutations*. There is a

bijection between segmented permutations and partially signed permutations that sends the recoil statistic to the segmented composition of descents of a segmented permutation.

## 1.2 Segmented composition quasi-symmetric functions: SCQSym

In [5], the authors defined an algebra over segmented compositions (**SCQSym**) and a so called ribbon basis ( $R_I$ ). This basis satisfies the following product rule:

$$R_I \cdot R_K = R_{I \cdot K} + R_{I|K} + R_{I \triangleright K}. \quad (1.2)$$

For example,  $R_{21|1} \cdot R_{2|15} = R_{21|12|15} + R_{21|1|2|15} + R_{21|3|15}$ .

We define a multiplicative basis of **SCQSym** as an analogue of the complete functions of the noncommutative symmetric functions [4]

$$S^I = \sum_{I \succeq K} R_K \quad (1.3)$$

The inverse relation is given by

$$R_I = \sum_{I \succeq K} (-1)^{\text{des}(I) - \text{des}(K)} S^K. \quad (1.4)$$

We have for example  $S^{2|13} = R_{2|13} + R_{2|4} + R_{2|1|3}$  and  $R_{2|13} = S^{2|13} - S^{2|4} - S^{2|1|3}$ .

The multiplication rule for this basis is the following.

**Proposition 1.1.** *Let  $I$  and  $K$  be two segmented compositions.*

$$S^I \cdot S^K = S^{I \cdot K} \quad (1.5)$$

## 1.3 Segmented permutation quasi-symmetric functions: SPQSym

Recall that the *standardization* of a word  $w$  is the permutation obtained by iteratively scanning  $w$  from left to right, and labeling  $1, 2, \dots$  the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. We define the standardization for segmented word by standardizing the underlying word and keeping the bars in the same positions. For example,  $\text{std}(41|2116|4) = 51|4237|6$ .

Given two segmented permutations,  $\sigma \in \mathfrak{P}_n$  and  $\tau \in \mathfrak{P}_r$ , define the *convolution* of  $\sigma$  and  $\tau$  (denoted by  $\sigma * \tau$ ) as the set of all segmented permutations in  $\mathfrak{P}_{n+r}$  such that the standardization of the  $n$  first letters is equal to  $\sigma$  and the standardization of the  $r$  last letters gives  $\tau$ . For example,  $2|13 * 12 = \{2|1345, 2|13|45, 2|1435, 2|14|35, \dots, 4|3512, 4|35|12\}$ . The number of segmented permutations in  $\sigma * \tau$  is  $2^{\binom{n+r}{n}}$ .

In [2] we defined an algebra over *partially signed permutations* which can be seen as an algebra over segmented permutations that we call **SPQSym**. This identification defines a basis  $\mathbf{G}_\sigma$  in **SPQSym** with the following product rule:

$$\mathbf{G}_\sigma \cdot \mathbf{G}_\tau = \sum_{\mu \in \sigma * \tau} \mathbf{G}_\mu. \quad (1.6)$$

The algebra **SCQSym** can be seen as a subalgebra of **SPQSym** with the following identification.

$$R_I = \sum_{\text{SDes}(\sigma)=I} \mathbf{G}_\sigma. \tag{1.7}$$

For example,  $R_{2|11} = \mathbf{G}_{12|43} + \mathbf{G}_{13|42} + \mathbf{G}_{14|32} + \mathbf{G}_{23|41} + \mathbf{G}_{24|31} + \mathbf{G}_{34|21}$ .

## 2 Generalized Eulerian numbers

### 2.1 Generalized Eulerian triangle

Let us now consider the triangle of numbers  $T(n, k)$  corresponding to the number of segmented permutations of size  $n$  having  $k$  descents. In other words

$$T(n, k) = \#\{\sigma \in \mathfrak{P}_n \mid \text{des}(\sigma) = k\}. \tag{2.1}$$

The first values of the triangle corresponding to the numbers  $T(n, k)$  are presented in Figure 1 alongside with the usual Eulerian numbers.

$n \setminus r$	0	1	2	3	4	5	$n \setminus r$	0	1	2	3	4	5
0	1						0	1					
1	1						1	1					
2	3	1					2	1	1				
3	13	10	1				3	1	4	1			
4	75	91	25	1			4	1	11	11	1		
5	541	896	426	56	1		5	1	26	66	26	1	
6	4683	9829	6734	1674	119	1	6	1	57	302	302	57	1

**Figure 1:** Triangles of generalized Eulerian numbers and usual Eulerian numbers.

The numbers appearing on the first column of this triangle are known as the ordered Bell numbers, sequence [A000670](#) of [8]. Among other things, they count the number of ordered set partitions of size  $n$ . They are in bijection with segmented permutations with no descents by considering the sets of values delimited by the bars in a segmented permutation. These numbers are also equal to the Eulerian polynomials evaluated at  $t = 2$ ,  $A_n(2)$ .

We can also give a combinatorial interpretation of the rows of the triangle read from right to left.

**Proposition 2.1.** *Let  $m < n$  be two positive integers, we have*

$$T(n, n - m - 1) = \#\{\sigma \in \mathfrak{P}_n \mid \text{seg}(\sigma) + \text{des}(\sigma) = m\}. \tag{2.2}$$

*Proof.* Let  $\sigma \in \mathfrak{P}_n$  having  $\text{seg}(\sigma) + \text{des}(\sigma) = m$  descents and bars. In the mirror image of  $\sigma$ , there are exactly  $n - 1 - m$  descents which correspond to the positions that are neither a descent nor a segmentation in  $\sigma$ .  $\square$

We can refine the triangle in Figure 1 by considering the 3-dimensional tetrahedron consisting in the numbers

$$K(n, i, j) = \#\{\sigma \in \mathfrak{P}_n \mid \text{des}(\sigma) = i, \text{seg}(\sigma) = j\}, \quad (2.3)$$

so that

$$T(n, k) = \sum_{j=0}^{n-1-k} K(n, k, j). \quad (2.4)$$

In Figure 2, we represent some values for the numbers  $K(n, i, j)$  where we fix the size of the segmented permutations.

$$\begin{array}{c}
 n = 2: \quad \begin{array}{c|cc} j \setminus i & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 2 & \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 n = 3: \quad \begin{array}{c|ccc} j \setminus i & 0 & 1 & 2 \\ \hline 0 & 1 & 4 & 1 \\ 1 & 6 & 6 & \\ 2 & 6 & & \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 n = 4: \quad \begin{array}{c|cccc} j \setminus i & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 11 & 11 & 1 \\ 1 & 14 & 44 & 14 & \\ 2 & 36 & 36 & & \\ 3 & 24 & & & \end{array}
 \end{array}$$

**Figure 2:** Slices of the tetrahedron of refined generalized Eulerian numbers.

We represent these triangles with the parameter  $i$  for the columns and the parameter  $j$  for the rows such that the first row of the  $n$ -th triangle corresponds to the  $n$ -th row of the Eulerian triangle. Moreover, by considering the mirror image of a segmented permutation we have a straightforward proof of the following property corresponding to the symmetry of the rows of the triangles.

**Proposition 2.2.** *Let  $n > 0$  and  $0 \leq i + j < n$ , we have*

$$K(n, i, j) = K(n, n - j - i, j). \quad (2.5)$$

The numbers appearing in the first column of the triangles are known as sequence [A019538](#) of [8].

$$K(n, 0, j) = (j + 1)! S(n, j + 1), \quad (2.6)$$

where  $S(n, k)$  are the Stirling numbers of the second kind, sequence [A008277](#) of [8], which count the number of ways to partition the set  $\{1, 2, \dots, n\}$  into  $k$  subsets. In fact for all  $n$  and  $j < n$ , the  $j$ -th row of the  $n$ -th triangle can be divided by  $(j + 1)!$  as the order of the blocks of numbers between the bars of a segmented permutation does not change the number of descents.

The numbers  $K(n, i, j)$  can be described recursively, by decreasing either  $n$  or  $j$ .

**Proposition 2.3.** For  $n > 0$  and  $0 \leq i + j < n$ ,

$$\begin{aligned} K(n, i, j) = & (i + j + 1) \left[ K(n - 1, i, j) + K(n - 1, i, j - 1) \right] \\ & + (n - i - j) \left[ K(n - 1, i - 1, j) + K(n - 1, i - 1, j - 1) \right]. \end{aligned} \quad (2.7)$$

For  $j > 0$ ,

$$jK(n, i, j) = (n - i - j)K(n, i, j - 1) + (i + 1)K(n, i + 1, j - 1). \quad (2.8)$$

*Proof.* The first equation of the proposition is proved bijectively based on the different possibilities we have to add an  $n$  in a segmented permutation of  $n - 1$ . The second one is obtained by considering what happens when we add a bar in a segmented permutation.  $\square$

Using (2.7), one obtains a similar recurrence for the numbers  $T(n, k)$ :

**Corollary 2.4.** Let  $n > 0$  and  $0 \leq k < n$ .

$$T(n, k) = (n - k)T(n - 1, k - 1) + (n + 1)T(n - 1, k) + (k + 1)T(n - 1, k + 1). \quad (2.9)$$

With an induction on (2.8), we obtain the following corollary expressing  $K(n, i, j)$  in terms of Eulerian numbers.

**Corollary 2.5.** Let  $n > 0$  and  $0 \leq i + j < n$ .

$$K(n, i, j) = \sum_{k=0}^{n-1} \binom{k}{i} \binom{n-1-k}{i+j-k} A(n, k). \quad (2.10)$$

Note that this result can also be proved directly using the combinatorial interpretation.

## 2.2 Generalized Eulerian polynomials

A different approach to study the numbers  $K(n, i, j)$  is to study the associated polynomials:

$$\alpha_n(t, q) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} q^{\text{seg}(\sigma)}. \quad (2.11)$$

Let us denote the generating polynomials of the triangle in Figure 1 by  $P_n(t)$ . We have

$$P_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)}. \quad (2.12)$$

Different values of the parameters  $t$  and  $q$  in  $\alpha_n$  give known polynomials or values:

**Proposition 2.6.** Let  $n \geq 0$ .

1.  $\alpha_n(t, 0) = A_n(t)$ ,
2.  $\alpha_n(0, q) = B_n(q)$ ,
3.  $\alpha_n(t, 1) = P_n(t)$ ,
4.  $\alpha_n(t, t) = t^{n-1}P_n\left(\frac{1}{t}\right)$ ,
5.  $\alpha_n(-1, 1) = 2^{n-1}$ ,

where  $A_n(t)$  are the Eulerian polynomials and  $B_n(t)$  are the ordered Bell polynomials.

*Proof.* Items 1 and 3 come from the combinatorial interpretation. By definition,  $B_n(q) = \sum_{r=0}^{n-1} S(n, r+1)(r+1)!q^r$  so item 2 is proved using (2.6). Item 4 is a consequence of Proposition 2.1. Finally, the last item can be proved bijectively but we shall only consider the proof given by substituting  $t$  to  $-1$  and  $q$  to  $1$  in the generating function of the  $\alpha_n$  that we make explicit in Theorem 3.3.  $\square$

The symmetry described by Proposition 2.2 implies the following relation on the polynomials.

**Proposition 2.7.** *Let  $n \geq 0$ .*

$$\alpha_n(t, q) = t^{n-1}\alpha_n\left(\frac{1}{t}, \frac{q}{t}\right) \quad (2.13)$$

Using (2.8), we obtain the following differential recurrence relation satisfied by the  $\alpha_n$ .

**Proposition 2.8.** *Let  $n \geq 2$ .*

$$\begin{aligned} \alpha_n(t, q) = & \left( (n-2)tq + (n-1)t + 2q + 1 \right) \alpha_{n-1}(t, q) \\ & + (t-t^2)(q+1) \frac{\partial}{\partial t} \alpha_{n-1}(t, q) \\ & + (1-t)(q^2+q) \frac{\partial}{\partial q} \alpha_{n-1}(t, q) \end{aligned} \quad (2.14)$$

Let  $G(t, q, x)$  be the exponential generating function of the polynomials  $\alpha_n$ :

$$G(t, q, x) = \sum_{n \geq 0} \alpha_n(t, q) \frac{x^n}{n!}. \quad (2.15)$$

Using (2.14) it is possible to obtain a differential equation satisfied by  $G$  but using it to obtain an explicit form of  $G$  seems at least as difficult as solving the equivalent problem on the usual Eulerian polynomials. We shall obtain it easily and directly by means of a noncommutative analog.

### 3 Algebraic study

The results in this section are inspired from a similar approach to Eulerian polynomials presented in [4].



### 3.1 Definition and generating function

Let  $t$  and  $q$  be indeterminates that commute with the  $\mathbf{G}_\sigma$  of  $\mathbf{SPQSym}$ . We define the *generalized noncommutative Eulerian polynomials* as follows.

$$\mathcal{A}_n(t, q) = \sum_{\sigma \in \mathfrak{P}_n} t^{\text{des}(\sigma)+1} q^{\text{seg}(\sigma)} \mathbf{G}_\sigma = \sum_{I \Vdash n} t^{\text{des}(I)} q^{\text{seg}(I)} R_I. \quad (3.1)$$

Let  $n > 0$  and  $\sigma \in \mathfrak{P}_n$ , consider the following morphism of algebras  $\varphi$  defined by

$$\varphi(\mathbf{G}_\sigma) = \frac{x^n}{2^{n-1}n!}. \quad (3.2)$$

For  $n > 0$ , we have

$$\varphi(\mathcal{A}_n(t, q)) = t\alpha_n(t, q) \frac{x^n}{2^{n-1}n!}. \quad (3.3)$$

To obtain a more suitable expression we expand the  $\mathcal{A}_n$  on the complete functions.

**Proposition 3.1.** *Let  $n \geq 0$ .*

$$\mathcal{A}_n(t, q) = \sum_{I \Vdash n} t^{\text{des}(I)} (1-t)^{n-\ell(I)} (q-t)^{\text{seg}(I)} S^I. \quad (3.4)$$

*Proof.* In (3.1), we expand the ribbon basis on the complete one using (1.4) which gives us:

$$\begin{aligned} \mathcal{A}_n(t, q) &= \sum_{I \Vdash n} \sum_{I \succeq K} t^{\text{des}(K)} q^{\text{seg}(K)} (-t)^{\text{des}(I)-\text{des}(K)} q^{\text{seg}(I)-\text{seg}(K)} S^K \\ &= \sum_{K \Vdash n} t^{\text{des}(K)} q^{\text{seg}(K)} \left[ \sum_{u=0}^{n-\ell(K)} \sum_{v=0}^{\text{seg}(K)} \binom{n-\ell(K)}{u} (-t)^u \binom{\text{seg}(K)}{v} \left(\frac{-t}{q}\right)^v \right] S^K. \end{aligned} \quad (3.5)$$

To obtain the last equality, one needs to count how many segmented compositions of  $n$  are finer than  $K$  with a length  $\ell(K) + u$  and  $v$  bars less than  $K$ . The statement is obtained after applying twice the binomial theorem.  $\square$

Equation (3.4) can be rewritten as:

$$\mathcal{A}_n(t, q) = (1-t)^n \sum_{I \Vdash n} \left(\frac{t}{1-t}\right)^{\text{des}(I)} \left(\frac{q-t}{1-t}\right)^{\text{seg}(I)} S^I. \quad (3.6)$$

Before applying  $\varphi$  to this relation, we need to define the following series of complete functions.

$$\Pi_{i,j} = \sum_{n \geq 1} \sum_{\substack{I \Vdash n \\ \text{des}(I)=i \\ \text{seg}(I)=j}} S^I. \quad (3.7)$$

The image by  $\varphi$  of the  $\Pi_{1,u}$  is given by the following lemma.

**Lemma 3.2.** *Let  $u \geq 0$ .*

$$\phi(\Pi_{1,u}) = 2 \left( e^{x/2} - 1 \right)^{u+1} \quad (3.8)$$

*Proof.* We expand  $\Pi_{1,u}$  in **SPQSym**,

$$\Pi_{1,u} = \sum_{n \geq 1} \sum_{\substack{\sigma \in \mathfrak{P}_n \\ \text{des}(\sigma)=0 \\ \text{seg}(\sigma)=u}} \mathbf{G}_\sigma. \quad (3.9)$$

By applying  $\phi$  and then (2.6) we obtain

$$\begin{aligned} \phi(\Pi_{1,u}) &= \sum_{n \geq 1} K(n, 0, u) \frac{x^n}{2^{n-1} n!} \\ &= 2 \sum_{n \geq 1} S(n, u+1) (u+1)! \frac{(x/2)^n}{n!} \end{aligned} \quad (3.10)$$

Using the fact that  $\sum_{n \geq 1} S(n, u) u! \frac{x^n}{n!} = (e^x - 1)^u$ , we obtain the statement.  $\square$

We can now obtain the generating function for the polynomials  $\alpha_n$ .

**Theorem 3.3.** *We have the following generating function:*

$$G(t, q, x) = 1 + \frac{e^{x(1-t)} - 1}{1 + q - (t+q)e^{x(1-t)}}. \quad (3.11)$$

The proof of the theorem is straightforward from the following lemma.

**Lemma 3.4.** *We have*

$$1 + \sum_{n \geq 1} \frac{t \alpha_n(t, q)}{(1-t)^n} \frac{x^n}{2^{n-1} n!} = \frac{(1-t) - (q-t)(e^{x/2} - 1)}{(1-t) - (q+t)(e^{x/2} - 1)}. \quad (3.12)$$

*Proof.* Let us consider the series:

$$\begin{aligned} \sum_{n \geq 0} \frac{\mathcal{A}_n(t, q)}{(1-t)^n} &= \sum_{n \geq 0} \sum_{I \Vdash n} \left( \frac{t}{1-t} \right)^{\text{des}(I)} \left( \frac{q-t}{1-t} \right)^{\text{seg}(I)} S^I \\ &= \sum_{v \geq 0} \left( \frac{t}{1-t} \right)^v \left[ \sum_{u \geq 0} \left( \frac{q-t}{1-t} \right)^u \Pi_u \right]^v \end{aligned} \quad (3.13)$$

Then, by applying  $\phi$  and using Lemma 3.2 we obtain

$$1 + \sum_{n \geq 1} \frac{t \alpha_n(t, q)}{(1-t)^n} \frac{x^n}{2^{n-1} n!} = \sum_{v \geq 0} \left( \frac{2t}{1-t} \right)^v \left[ \sum_{u \geq 0} \left( \frac{q-t}{1-t} \right)^u \left( e^{x/2} - 1 \right)^u \right]^v \quad (3.14)$$

This equation can be rewritten to obtain the lemma as both series on the right are geometric.  $\square$

As a corollary of Theorem 3.3, one describes the generating functions of the polynomials  $P_n(t)$ .

### 3.2 Applications

By substituting  $q$  to 1 and  $x$  to  $2x$  in Lemma 3.4 and dividing each side by  $(1-t)$ , we obtain

$$\frac{1}{1-t} + 2 \sum_{n \geq 1} \frac{t \alpha_n(t, 1)}{(1-t)^{n+1}} \frac{x^n}{n!} = 1 + t \frac{e^x}{2 - (1+t)e^x}. \quad (3.15)$$

As  $\alpha_n(t, 1) = P_n(t)$ , this equation leads to the following.

**Proposition 3.5.** *Let  $n > 0$ , we have*

$$\frac{P_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (1+t)^{k-1} \frac{k^n}{2^{k+1}}. \quad (3.16)$$

This equation gives a generalization of the similar result for the Eulerian numbers. Moreover, by specializing  $t$  to 0, we recover a known expression of the ordered Bell numbers as a series.

Another result can be obtained by considering the image under  $\phi$  of the component of degree  $n$  of the  $\Pi_{i,j}$  and applying  $\phi$  to (3.4):

**Proposition 3.6.** *Let  $n > 0$ ,*

$$\alpha_n(t, q) = \sum_{0 \leq i+j \leq n-1} t^i (q-t)^j (1-t)^{n-i-j-1} 2^i (i+j+1)! \binom{i+j}{j} S(n, i+j+1). \quad (3.17)$$

### 3.3 Identities of Worpitzky

The identities of Worpitzky are well-known identities involving Eulerian numbers.

$$k^n = \sum_{i=0}^{k-1} \binom{k+n-i-1}{n} A(n, i).$$

Consider the discrete derivation of polynomials defined on monomials as

$$\Delta(X^n) = (X+1)^n - X^n. \quad (3.18)$$

Applying this derivation to Worpitzky's identities gives

$$\Delta^r(k^n) = \sum_{i=0}^{k-1} \binom{k+n-i-1}{n-r} A(n, i). \quad (3.19)$$

Worpitzky's identities are usually proved by explicitly describing the coefficient of the series  $\sum_{n \geq 0} \frac{A_n(t)}{(1-t)^n} \frac{x^n}{n!}$ . By doing so with the polynomials  $\alpha_n(t, q)$ , we obtain the following

identity.

**Theorem 3.7.** *Let  $n$ ,  $k$ , and  $r$  be three positive integers.*

$$\binom{k+r-1}{r} \Delta^{r+1}((k-1)^n) = \sum_{i=0}^{k-1} \binom{n+k-i}{n-1} K(n, i, r). \quad (3.20)$$

## Conclusion

In this paper we defined generalized Eulerian numbers and generalized Eulerian polynomials which appear to interact with several known sequences of integers. We presented here some of them, the specialization of the polynomials  $\alpha_n(t, q)$  at small values ( $t, q = -2, -1, 1, 2, \dots$ ) gives other sequences of [8].

We also noticed that the rows and columns of the triangles and tetrahedron of the  $T(n, k)$  and  $K(n, i, j)$  are unimodal sequences. We verified these conjectures using *sage* up to  $n = 1000$ .

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