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On e-positivity and e-unimodality of chromatic quasisymmetric functions

Soojin Cho*[1] and JiSun Huh†[1]

[1]*Department of Mathematics, Ajou University, Suwon 16499 Republic of Korea*

Abstract. The e-positivity conjecture and the e-unimodality conjecture of chromatic quasisymmetric functions are proved for some classes of natural unit interval orders. Recently, J. Shareshian and M. Wachs introduced chromatic *quasisymmetric* functions as a refinement of Stanley's chromatic symmetric functions and conjectured the e-positivity and the e-unimodality of these functions. Our work resolves the Stanley's conjecture on chromatic symmetric functions of $(3+1)$-free posets for two classes of natural unit interval orders.

Résumé. La conjecture d'e-positivité ainsi que celle d'e-unimodalité sur les fonctions chromatiques quasi-symétriques ont été démontrées pour quelques classes d'ordres naturels sur l'ensemble des intervalles unitaires. Récemment, J. Shareshian et M. Wachs ont introduit les fonctions chromatiques quasi-symétriques comme un raffinement des fonctions chromatiques symétriques, et ont conjecturé qu'elles sont e-positives et e-unimodales. L'e-positivité d'une fonction chromatique quasi-symétrique implique celle de la fonction chromatique symétrique correspondante, et ce travail résout donc la conjecture de Stanley pour les fonctions chromatiques symétriques des ensembles ordonnés sans $(3+1)$ pour deux ordres naturels sur les intervalles unitaires.

Keywords: chromatic quasisymmetric function, e-positivity, e-unimodality, $(3+1)$-free poset, natural unit interval order

1 Introduction

In 1995, R. Stanley [8] introduced the *chromatic symmetric function* $X_G(\mathbf{x})$ associated with any simple graph G, which generalizes the chromatic polynomial $\chi_G(n)$ of G. One of the long standing and well known conjecture due to Stanley on chromatic symmetric functions states that a chromatic symmetric function of any $(3+1)$-free poset is a linear sum of elementary symmetric function basis $\{e_\lambda\}$ with *nonnegative* coefficients. Recently, Shareshian and Wachs [7] introduced a chromatic *quasi*symmetric refinement $X_G(\mathbf{x}, t)$ of chromatic symmetric function $X_G(\mathbf{x})$ for a graph G. They conjectured the e-positivity

*chosj@ajou.ac.kr. This work is supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2015R1D1A1A01057476).
†hyunyjia@ajou.ac.kr

and the e-unimodality of the chromatic quasisymmetric functions of natural unit interval orders: That is, if $X_G(\mathbf{x},t) = \sum_{j=0}^{m} a_j(\mathbf{x})t^j$ for the incomparability graph G of a natural unit interval order, then $a_j(\mathbf{x})$, $0 \leqslant j \leqslant m$, is a nonnegative linear sum of e_λ's, which is a refinement of the Stanley's conjecture. Moreover, e-unimodality conjecture states that $a_{j+1}(\mathbf{x}) - a_j(\mathbf{x})$ is a sum of e_λ's with nonnegative coefficients for $0 \leqslant j < \frac{m-1}{2}$.

In this paper, we give combinatorial proofs of the conjectures on e-positivity and e-unimodality of chromatic quasisymmetric functions for certain classes of natural unit interval orders. We use the Schur function expansion of chromatic quasisymmetric functions in terms of Gasharov's P-tableaux ([2]), that was done by Shareshian and Wachs in [7]. We use the Jacobi–Trudi expansion of Schur functions into elementary symmetric functions to write chromatic quasisymmetric functions as a sum of elementary symmetric functions with coefficients of signed sum of positive t polynomials. We then define a sign reversing involution to cancel out negative terms and obtain only positive terms. We use an inductive argument on P-tableaux to find explicit e-expansion formulae of $X_G(\mathbf{x},t)$ and this shows the e-unimodality. We remark that one class of natural unit interval orders we consider (in Section 3.1) was considered by Stanley–Stembridge and recently by Harada–Precup and the e-positivity of the corresponding chromatic symmetric functions was proved (Remark 4.4 in [11], [6]).

2 Preliminaries

2.1 Chromatic quasisymmetric functions and the positivity conjecture.

We set $\mathbb{P} = \{1,2,\dots\}$ and $[n] = \{1,2,\dots,n\}$. For $n \in \mathbb{P}$, a *partition* $\lambda = (\lambda_1,\dots,\lambda_\ell)$ of n is a sequence of positive integers such that $\lambda_i \geqslant \lambda_{i+1}$ for all i and $\sum_i \lambda_i = n$. For a partition λ, the *conjugate* of λ is the partition $\lambda' = (\lambda'_1,\dots,\lambda'_{\lambda_1})$ with $\lambda'_j = |\{i \,|\, \lambda_i \geqslant j\}|$. For $n \in \mathbb{P}$, the n^{th} *elementary symmetric function* e_n is defined as $e_n = \sum_{i_1<\cdots<i_n} x_{i_1}\cdots x_{i_n}$, and the $\mathbb{Q}$-*algebra* $\Lambda_\mathbb{Q}$ *of symmetric functions* is the subalgebra of $\mathbb{Q}[[x_1,x_2,\dots]]$ generated by the e_n's; $\Lambda_\mathbb{Q} = \Lambda = \mathbb{Q}[e_1,e_2,\dots]$. Then $\Lambda = \bigoplus_{n=0}^{\infty}\Lambda^n$, where Λ^n is the subspace of symmetric functions of degree n, and $\{e_\lambda \,|\, \lambda \vdash n\}$ is a basis of Λ^n, where $e_\lambda = e_{\lambda_1}\cdots e_{\lambda_\ell}$. The set of *Schur functions* $\{s_\lambda \,|\, \lambda \vdash n\}$ forms another basis of Λ^n, where $s_\lambda = \det[e_{\lambda'_i-i+j}]$ is a determinant of a $\lambda_1 \times \lambda_1$ matrix, that is called *Jacobi–Trudi identity*. A *proper coloring* of a simple graph $G = (V,E)$ is a function $\kappa : V \to \mathbb{P}$ satisfying $\kappa(u) \neq \kappa(v)$ for any $u, v \in V$ such that $\{u,v\} \in E$. For a proper coloring κ, $\mathrm{asc}(\kappa) = \{\{i,j\} \in E \mid i < j \text{ and } \kappa(i) < \kappa(j)\}$.

Definition 2.1 ([7]). *For a simple graph $G = (V,E)$ which has a vertex set $V \subset \mathbb{P}$, the* chromatic quasisymmetric function *of G is a sum over all proper colorings of G:*

$$X_G(\mathbf{x},t) = \sum_{\kappa} t^{\mathrm{asc}(\kappa)}\mathbf{x}_\kappa.$$

Note that chromatic quasisymmetric function $X_G(\mathbf{x}, t)$ is a refinement of Stanley's *chromatic symmetric function* $X_G(\mathbf{x})$ introduced in [8]; $X_G(\mathbf{x}, t)|_{t=1} = X_G(\mathbf{x})$.

The *incomparability graph* $\mathrm{inc}(P)$ of a poset P is a graph which has as vertices the elements of P, with edges connecting pairs of incomparable elements. *Natural unit interval orders* are the posets we are interested in.

Definition 2.2 ([7]). *Let* $\mathbf{m} := (m_1, m_2, \ldots, m_{n-1})$ *be a list of integers satisfying* $i \leqslant m_i \leqslant m_{i+1} \leqslant n$ *for all* i. *The corresponding* natural unit interval order $P(\mathbf{m})$ *is the poset on* $[n]$ *with the order relation given by* $i <_{P(\mathbf{m})} j$ *if* $i < n$ *and* $j \in \{m_i + 1, m_i + 2, \ldots, n\}$.

Note that Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ counts the natural unit interval orders with n elements. Shareshian and Wachs showed that if G is the incomparability graph of a natural unit interval order then $X_G(\mathbf{x}, t)$ is a polynomial with very nice properties. They also made a conjecture on the e-positivity and the e-unimodality of $X_G(\mathbf{x}, t)$. Remember that a symmetric function $f(\mathbf{x}) \in \Lambda^n$ is *b-positive* if the expansion of $f(\mathbf{x})$ in the basis $\{b_\lambda\}$ has nonnegative coefficients when $\{b_\lambda \mid \lambda \vdash n\}$ is a basis of Λ^n.

Theorem 2.3 (Theorem 4.5 and Corollary 4.6 in [7]). *If* G *is the incomparability graph of a natural unit interval order then the coefficients of* t^i *in* $X_G(\mathbf{x}, t)$ *are symmetric functions and form a palindromic sequence in the sense that* $X_G(\mathbf{x}, t) = t^{|E|} X_G(\mathbf{x}, t^{-1})$.

Conjecture 2.4 ([7]). *If* G *is the incomparability graph of a natural unit interval order, then* $X_G(\mathbf{x}, t)$ *is e-positive and e-unimodal. That is, if* $X_G(\mathbf{x}, t) = \sum_{i=0}^{m} a_i(\mathbf{x}) t^i$ *then* $a_i(\mathbf{x})$ *is e-positive for all* i, *and* $a_{i+1}(\mathbf{x}) - a_i(\mathbf{x})$ *is e-positive whenever* $0 \leqslant i < \frac{m-1}{2}$.

A finite poset P is called $(r+s)$*-free* if P does not contain an induced subposet isomorphic to the direct sum of an r element chain and an s element chain. Due to the result by Guay-Paquet [4], Conjecture 2.4 specializes to the famous e-positivity conjecture on the chromatic symmetric functions of Stanley and Stembridge:

Conjecture 2.5 ([8, 11]). *If a poset* P *is* $(3+1)$*-free, then* $X_{\mathrm{inc}(P)}(\mathbf{x})$ *is e-positive.*

Since e_λ is s-positive, Conjecture 2.4 implies the s-positivity of $X_G(\mathbf{x}, t)$, that was proved using the notion of P-tableaux by Shareshian–Wachs and Gasharov ($t = 1$):

Definition 2.6 (Gasharov [2]). *Given a poset* P *with* n *elements and a partition* λ *of* n, *a* P-tableau of shape λ *is a filling* $T = [a_{i,j}]$ *of a Young diagram of shape* λ *in English notation*

$a_{1,1}$	$a_{1,2}$	$\cdots$
$a_{2,1}$	$a_{2,2}$	$\cdots$
$\vdots$	$\vdots$	

with all elements of P *so that* $a_{i,j} <_P a_{i,j+1}$ *and* $a_{i+1,j} \nless_P a_{i,j}$ *for all* i *and* j.

Definition 2.7 ([7]). *For a finite poset P on a subset of $\mathbb{P}$ and a P-tableau T, let G be the incomparability graph of P. An edge $\{i,j\} \in E(G)$ is a G*-inversion of T *if $i < j$ and i appears below j in T. We let $\mathrm{inv}_G(T)$ be the number of G-inversions of T.*

The following theorem is the s-positivity result for the chromatic quasisymmetric functions of natural unit interval orders, which specializes to Gasharov's s-positivity result for chromatic symmetric functions of $(3+1)$-free posets.

Theorem 2.8 ([7],[2]). *Let G be the incomparability graph of a natural unit interval order P. If we let $\lambda(T)$ be the shape of T, then $X_G(\mathbf{x},t) = \sum_T t^{\mathrm{inv}_G(T)} s_{\lambda(T)}$, where the sum is over all P-tableaux and therefore, $X_G(\mathbf{x},t)$ is s-positive.*

We state some related results that are useful for our arguments.

Lemma 2.9 ([10]). *If $A(t)$ and $B(t)$ are unimodal and palindromic polynomials with nonnegative coefficients and centers of symmetry m_A, m_B respectively, then $A(t)B(t)$ is unimodal and palindromic with nonnegative coefficients and center of symmetry $m_A + m_B$.*

From Lemma 2.9, if G is the incomparability graph of a natural unit interval order and $X_G(\mathbf{x},t) = \sum_{i=0}^{m} a_i(\mathbf{x})\, t^i = \sum_{\lambda \vdash n} C_\lambda(t)\, e_\lambda(\mathbf{x})$, then

- $X_G(\mathbf{x},t)$ is palindromic in t with center of symmetry $\frac{m}{2}$ if and only if $C_\lambda(t)$ is a palindromic polynomial with center of symmetry $\frac{m}{2}$,
- $X_G(\mathbf{x},t)$ is e-positive if and only if $C_\lambda(t)$ is a polynomial with nonnegative coefficients for all λ,
- $X_G(\mathbf{x},t)$ is e-unimodal with center of symmetry $\frac{m}{2}$ if every $C_\lambda(t)$ is unimodal with the same center of symmetry $\frac{m}{2}$.

Theorem 2.10 ([7]). *For $G = \mathrm{inc}(P)$ of a natural unit interval order P on $[n]$, if we let $X_G(\mathbf{x},t) = \sum C_\lambda(t)\, e_\lambda(\mathbf{x})$, then $C_{(n)}(t) = [n]_t \prod_{i=2}^{n} [b_i]_t$ and is therefore positive and unimodal with center of symmetry $\frac{|E(G)|}{2}$. Here, $[n]_t = 1 + t + \cdots + t^{n-1}$ and $b_i = |\{\{j,i\} \in E(G) \mid j < i\}|$.*

We summarize known results on e-positivity and e-unimodality of $X_{\mathrm{inc}(P)}(\mathbf{x},t)$ for natural unit interval orders $P = P(\mathbf{m})$ on $[n]$ with $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1})$.

1. For $\mathbf{m} = (2, 3, \ldots, n)$, Stanley obtained the formula for the generating function and Haiman [5] showed that $X_{\mathrm{inc}(P)}(\mathbf{x},t)$ is e-positive and e-unimodal.
2. For $\mathbf{m} = (m_1, m_2, n, \ldots, n)$, Shareshian and Wachs [7] obtained an e-positive and e-unimodal explicit formula of $X_{\mathrm{inc}(P)}(\mathbf{x},t)$.
3. For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of $n+k-1$, let $m_1 = \cdots = m_{\alpha_1 - 1} = \alpha_1$ and $m_i = \sum_{j=1}^{\ell}(\alpha_j - 1) + 1$ if $\sum_{j=1}^{\ell-1}(\alpha_j - 1) + 1 \leqslant i \leqslant \sum_{j=1}^{\ell}(\alpha_j - 1)$ for $\ell \geqslant 2$. The graph $\mathrm{inc}(P)$ is a K_α*-chain*. Gebhard and Sagan [3] proved that $X_{\mathrm{inc}(P)}(\mathbf{x})$ is e-positive.

4. For $\mathbf{m} = (2, 3, \ldots, m+1, n, \ldots, n)$, the graph $\mathrm{inc}(P)$ is a *lollipop graph* $L_{m,n-m}$. Dahlberg and van Willigenburg [1] computed an explicit e-positive formula for $X_{L_{m,n-m}}(\mathbf{x})$.

5. For $\mathbf{m} = (r, m_2, m_3 \ldots, m_r, n, \ldots, n)$, the graph $\mathrm{inc}(P)$ is a complement graph of a bipartite graph. Stanley and Stembridge [11] proved that $X_{\mathrm{inc}(P)}(\mathbf{x})$ is e-positive.

2.2 Basic setup

We often use Catalan paths from $(0, 0)$ to (n, n) to represent $P = P(m_1, m_2, \ldots, m_{n-1})$: The corresponding Catalan path of P has the ith horizontal step on the line $y = m_i$ for all $i \leqslant n-1$ and the last horizontal step on the line $y = n$. Figure 1 shows an example.

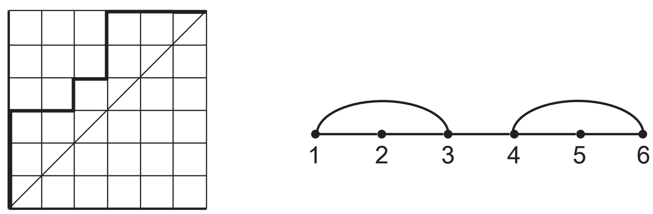

Figure 1: The corresponding Catalan path of $P = P(3, 3, 4, 6, 6)$ and $\mathrm{inc}(P)$.

If a graph G is a disjoint union of subgraphs $G_1, \ldots, G_\ell$, then $X_G(\mathbf{x}, t) = \prod_{i=1}^{\ell} X_{G_i}(\mathbf{x}, t)$. Moreover, it is easy to see that $G = \mathrm{inc}(P)$ for $P = P(m_1, \ldots, m_{n-1})$ is connected if and only if $m_i > i$ for all i or equivalently, C meets the line $y = x$ only at $(0, 0)$ and (n, n). Hence, we may restrict our attention to natural unit interval orders $P(m_1, \ldots, m_{n-1})$ with $m_i \neq i$ for all $i = 1, \ldots, n-1$. Note that there are C_{n-1} such Catalan paths of length n. Given a Catalan path C of length n, define the *bounce path* of C as follows: Starting at $(0, 0)$, and travel north until you encounter the beginning of an east step. Then turn east and travel straight until you hit the main diagonal $y = x$. Then turn north until you again encounter the beginning of an east step of C, then turn east and travel to the diagonal, etc. Continue in this way until you arrive at (n, n). The *bounce number* of C is the number of connected regions that the bounce path of C and the line $y = x$ enclose; that is one less than the number of times the bounce path of C meets the line $y = x$.

The following useful lemmas can be proved using Theorems 2.8 and 2.3, respectively.

Lemma 2.11. *Let C be the corresponding Catalan path of a natural unit interval order P and r be the bounce number of C. If we let $X_{\mathrm{inc}(P)}(\mathbf{x}, t) = \sum_\lambda B_\lambda(t) s_\lambda$, then $B_\lambda(t) = 0$ for partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 > r$.*

Lemma 2.12. *Given a natural unit interval order P of size n, let C be the corresponding Catalan path of P and $G = \mathrm{inc}(P)$. If $\widetilde{C}$ is the reflection of the Catalan path C about the line $y = n - x$, then $X_G(\mathbf{x}, t) = X_{\widetilde{G}}(\mathbf{x}, t)$, where $\widetilde{P}$ is the corresponding natural unit interval order of the Catalan path $\widetilde{C}$ and $\widetilde{G} = \mathrm{inc}(\widetilde{P})$.*

3 Two e-positive classes of natural unit interval orders

In this section we provide two classes of natural unit interval orders $P = P(\mathbf{m})$ satisfying that $X_{\text{inc}(P)}(\mathbf{x}, t)$ is e-positive.

3.1 The first e-positive class

We consider natural unit interval orders $P = P(r, m_2, \ldots, m_r, n, \ldots, n)$ on $[n]$ for $r < n$. Let $G = \text{inc}(P)$. Figure 2 shows a corresponding Catalan path (solid line) of P. The dotted line indicates the bounce path, and the bounce number of P's we consider is *two*.

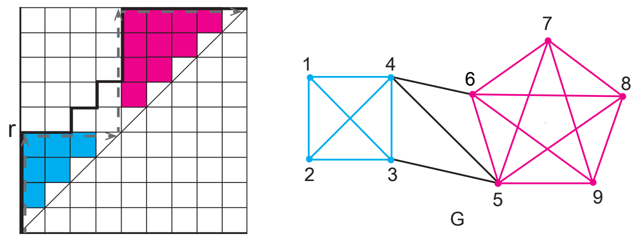

Figure 2: Catalan path, bounce path, and $G = \text{inc}(P)$ of $P(4,4,5,6,9,9,9,9)$

If we let $X_G(\mathbf{x}, t) = \sum_{\lambda \vdash n} B_\lambda(t) s_\lambda$, then by Lemma 2.11, $B_\lambda(t) \neq 0$ only for λ's whose conjugate has only two parts. By Theorem 2.8, $B_\lambda(t) = \sum_{T \in \mathcal{T}_{P,\lambda}} t^{\text{inv}_G(T)}$, where $\mathcal{T}_{P,\lambda}$ is the set of P-tableaux of shape λ. Let k be the largest positive integer satisfying $B_{2^k 1^{n-2k}}(t) \neq 0$. Due to Jacobi-Trudi identity, we have $s_{1^n} = e_n$ and $s_{2^\ell 1^{n-2\ell}} = e_{(n-\ell,\ell)} - e_{(n-\ell+1,\ell-1)}$ for $\ell \geqslant 2$. Hence, we can derive the e-basis expansion of $X_G(\mathbf{x}, t)$ as follows.

$$\begin{aligned} X_G(\mathbf{x}, t) &= B_{1^n}(t) s_{1^n} + \sum_{\ell=1}^{k} B_{2^\ell 1^{n-2\ell}}(t) s_{2^\ell 1^{n-2\ell}} \\ &= \sum_{\ell=0}^{k-1} (B_{2^\ell 1^{n-2\ell}}(t) - B_{2^{\ell+1} 1^{n-2\ell-2}}(t)) e_{(n-\ell,\ell)} + B_{2^k 1^{n-2k}}(t) e_{(n-k,k)} \end{aligned}$$

By Lemma 2.9, if we show that $B_{2^\ell 1^{n-2\ell}}(t) - B_{2^{\ell+1} 1^{n-2\ell-2}}(t)$ is a polynomial in t with nonnegative coefficients for each $\ell = 0, 1, \ldots, k-1$, then $X_G(\mathbf{x}, t)$ is e-positive.

For a fixed $\ell \in \{0, 1, \ldots, k-1\}$, let $\mathcal{T}'_{P,2^\ell 1^{n-2\ell}}$ be the subset of $\mathcal{T}_{P,2^\ell 1^{n-2\ell}}$, each of whose elements $T = [a_{i,j}]$ has some $s \geqslant \ell + 2$ satisfying $a_{i,1} <_P a_{s,1}$ for all $\ell + 1 \leqslant i \leqslant s - 1$.

Now we define a map $\psi_\ell : \mathcal{T}_{P,2^{\ell+1} 1^{n-2\ell-2}} \to \mathcal{T}'_{P,2^\ell 1^{n-2\ell}}$. Given $T = [b_{i,j}] \in \mathcal{T}_{P,2^{\ell+1} 1^{n-2\ell-2}}$, we let s be the smallest $i > \ell + 1$ such that $b_{i,1} \nless_P b_{\ell+1,2}$; otherwise $s = n - \ell$, then let $\psi_\ell(T)$ be the tableau obtained by inserting $b_{\ell+1,2}$ right below $b_{s-1,1}$. Then $\psi_\ell(T)$ is a P-tableau in $\mathcal{T}'_{P,2^\ell 1^{n-2\ell}}$. We can show that ψ_ℓ is a weight preserving bijection:

We now turn to the construction of the inverse map of ψ_ℓ. Given $T = [a_{i,j}] \in \mathcal{T}'_{P,2^\ell 1^{n-2\ell}}$, if we move $a_{s,1}$ to the right side of $a_{\ell+1,1}$, then we have the P-tableau $\psi_\ell^{-1}(T)$ in $\mathcal{T}'_{P,2^{\ell+1}1^{n-2\ell-2}}$.

Let $\overline{\mathcal{T}}'_{P,2^\ell 1^{n-2\ell}}$ be the set $\mathcal{T}_{P,2^\ell 1^{n-2\ell}} - \mathcal{T}'_{P,2^\ell 1^{n-2\ell}}$. Then

$$B_{2^\ell 1^{n-2\ell}}(t) - B_{2^{\ell+1}1^{n-2\ell-2}}(t) = \sum_{T \in \overline{\mathcal{T}}'_{P,2^\ell 1^{n-2\ell}}} t^{\mathrm{inv}_G(T)},$$

and this gives a combinatorial interpretation of the coefficient of $e_{(n-\ell,\ell)}$ in $X_G(\mathbf{x}, t)$.

We can now give our main result.

Theorem 3.1. *Let $P = P(r, m_2, \ldots, m_{n-1})$ be a natural unit interval order and let G be the incomparability graph of P. If $2 \leqslant r \leqslant n-1$ and $m_{r+1} = n$, then*

$$X_G(\mathbf{x}, t) = \sum_{\ell=0}^{k} C_{(n-\ell,\ell)}(t) e_{(n-\ell,\ell)},$$

where k is a positive integer and

$$C_{(n-\ell,\ell)}(t) = \sum_{T \in \overline{\mathcal{T}}'_{P,2^\ell 1^{n-2\ell}}} t^{\mathrm{inv}_G(T)}.$$

Here, $\overline{\mathcal{T}}'_{P,2^\ell 1^{n-2\ell}}$ is the set of elements $T = [a_{i,j}] \in \mathcal{T}_{P,2^\ell 1^{n-2\ell}}$ having no $s \geqslant \ell + 2$ such that $a_{i,1} <_P a_{s,1}$ for all $i \in \{\ell+1, \ell+2, \ldots, s-1\}$. Consequently $X_G(\mathbf{x}, t)$ is e-positive.

Example 3.2. *Let $P = P(2, 3, 4)$ be a natural unit interval order. There are 14 P-tableaux:*

$$T_1=\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \quad T_2=\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \quad T_3=\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \quad T_4=\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \quad T_5=\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \quad T_6=\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$$

$$T_7=\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \quad T_8=\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \quad T_9=\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline 4 \\ \hline \end{array} \quad T_{10}=\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \quad T_{11}=\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 3 \\ \hline \end{array} \quad T_{12}=\begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline 3 \\ \hline 2 \\ \hline \end{array} \quad T_{13}=\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} \quad T_{14}=\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 4 \\ \hline 3 \\ \hline \end{array}$$

By definition, $\mathcal{T}'_{P,1^4} = \{T_{11} = \psi_0(T_3), T_{12} = \psi_0(T_4), T_{13} = \psi_0(T_5), T_{14} = \psi_0(T_6)\}$ and $\mathcal{T}'_{P,21^2} = \{T_5 = \psi_1(T_1), T_6 = \psi_1(T_2)\}$. Therefore, $\overline{\mathcal{T}}'_{P,1^4} = \{T_7, T_8, T_9, T_{10}\}$, $\overline{\mathcal{T}}'_{P,21^2} = \{T_3, T_4\}$, and $\overline{\mathcal{T}}'_{P,2^2} = \{T_1, T_2\}$, by Theorem 3.1,

$$\begin{aligned} X_G(\mathbf{x}, t) &= \left(\sum_{i=1}^{2} t^{\mathrm{inv}_G(T_i)}\right) e_{(2,2)} + \left(\sum_{i=3}^{4} t^{\mathrm{inv}_G(T_i)}\right) e_{(3,1)} + \left(\sum_{i=7}^{10} t^{\mathrm{inv}_G(T_i)}\right) e_4 \\ &= (t + t^2) e_{(2,2)} + (t + t^2) e_{(3,1)} + (1 + t + t^2 + t^3) e_4. \end{aligned}$$

The next two corollaries follow easily from Theorem 3.1. Note that Corollary 3.3 is a quasisymmetric version of Remark 4.4 in [11].

Corollary 3.3. *For $G = \mathrm{inc}(P)$ of a natural unit interval order P on $[n]$, if induced subgraphs of G on $\{1,\dots,r\}$ and $\{r+1,\dots,n\}$ are both complete graphs for some r, then $X_G(\mathbf{x},t)$ is e-positive.*

Corollary 3.4. *Let $P_{n,r} = P(r, r+1, r+2, \dots, n)$ be a natural unit interval order on $[n]$ and let $G_{n,r}$ be the incomparability graph of $P_{n,r}$. If $r \geqslant \lceil \frac{n}{2} \rceil$, then $X_{G_{n,r}}(\mathbf{x},t)$ is e-positive.*

Remark 3.5. *Corollary 3.4 is proved for $r = 2, n-2, n-1$ in [7].*

3.2 The second e-positive class

In this section, we consider the natural unit interval orders $P = P(r, n-1, m_3, \dots, m_{n-2}, n)$ for some $r > 1$. If $m_{r+1} = n$, then $X_{\mathrm{inc}(P)}(\mathbf{x},t)$ is e-positive by Theorem 3.1. Hence, we consider the case when $m_{r+1} \neq n$. Figure 3 shows a corresponding Catalan path (solid line) of P with the bounce path (dotted line). In this case, the bounce number is 3 for P.

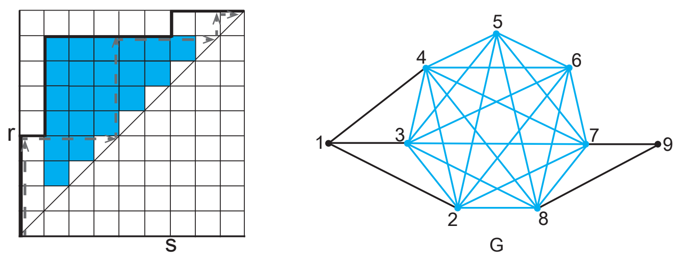

Figure 3: Catalan path, bounce path, and $G = \mathrm{inc}(P)$ of $P(4,8,8,8,8,8,9,9)$

Using the similar method in the Section 3.1, we have the following theorem.

Theorem 3.6. *Let G be the incomparability graph of a natural unit interval order $P = P(\mathbf{m})$. If $\mathbf{m} = (r, m_2, \dots, m_{n-1})$ with $m_2 = \cdots = m_s = n-1$ and m_{s+1}=n for $2 \leqslant r < s \leqslant n-2$, then*

$$X_G(\mathbf{x},t) = \sum_{\lambda \in A} C_\lambda(t) e_\lambda,$$

for $A = \{(n-2,1,1), (n-2,2), (n-1,1), (n)\}$. If we let

$$\begin{aligned}
G_1 &= \{[b_{i,j}] \in \mathcal{T}_{P,2^2 1^{n-4}} \mid b_{1,1} = 1,\ b_{1,2} <_P b_{2,2}\},\\
G_2 &= \{[b_{i,j}] \in \mathcal{T}_{P,2^2 1^{n-4}} \mid b_{1,1} = 1,\ b_{1,1} <_P b_{2,1},\ b_{1,2} \not<_P b_{2,2}\},\\
F_1 &= \{[b_{i,j}] \in \overline{\mathcal{T}}'_{P,2^1 1^{n-2}} \mid b_{1,1} <_P b_{2,1} <_P b_{1,2}\},\\
F_2 &= \{[b_{i,j}] \in \overline{\mathcal{T}}'_{P,2^1 1^{n-2}} \mid b_{1,1} <_P b_{1,2} <_P b_{2,1}\},\\
F_3 &= \{[b_{i,j}] \in \mathcal{T}'_{P,2^1 1^{n-2}} \mid b_{1,1} = 1,\ 1 \not<_P b_{2,1}\ \}\\
&\qquad \cup \{[b_{i,j}] \in \mathcal{T}'_{P,2^1 1^{n-2}} \mid b_{1,1} = 1,\ b_{1,2} <_P n\ \},\\
F_4 &= \{[b_{i,j}] \in \mathcal{T}'_{P,2^1 1^{n-2}} \mid b_{2,1} = 1,\ b_{2,1} <_P b_{3,1} \not<_P b_{1,2}\},
\end{aligned}$$

then, the coefficients of the e-basis expansion of $X_G(\mathbf{x},t)$ *are*

$$C_\lambda(t) = \sum_{T\in U_\lambda} t^{\mathrm{inv}_G(T)},$$

where

$$\begin{aligned} U_{(n-2,1,1)} &= \mathcal{T}_{P,3^1 1^{n-3}}, \\ U_{(n-2,2)} &= \mathcal{T}_{P,2^2 1^{n-4}} - G_1 - G_2, \\ U_{(n-1,1)} &= \mathcal{T}_{P,2^1 1^{n-2}} - F_1 - F_2 - F_3 - F_4, \\ U_{(n)} &= \overline{\mathcal{T}}'_{P,1^n}. \end{aligned}$$

Consequently $X_G(\mathbf{x},t)$ *is e-positive.*

Corollary 3.7. *Let G be the incomparability graph of a natural unit interval order on* $[n]$*. If G has the complete graph on* $\{2,3,\ldots,n-1\}$ *as a subgraph, then* $X_G(\mathbf{x},t)$ *is e-positive.*

Our proof for the e-positivity is to find sign reversing involutions on the signed sets of P-tableaux. Our hope is to extend it to cover all natural unit interval orders.

4 Some explicit formulae

In this section we give e-unimodal explicit formulae for some natural unit interval orders in the first e-positive class in Section 3.1. By Lemma 2.9, if we provide explicit formulae for all coefficients in the e-basis expansions of $X_{\mathrm{inc}(P)}(\mathbf{x},t)$ as sums of polynomials in t which are unimodal with center of symmetry $\frac{|E(\mathrm{inc}(P))|}{2}$, then $X_{\mathrm{inc}(P)}(\mathbf{x},t)$ is e-unimodal.

4.1 The first explicit formula

In this section we consider natural unit interval orders $P = P(m_1, m_2 \ldots, m_{n-1})$ with $m_1 = \cdots = m_s = r$ and $m_{s+1} = \cdots = m_{n-1} = n$ for any positive integers $s \leqslant r \leqslant n-1$. Figure 4 shows a corresponding Catalan path of P.

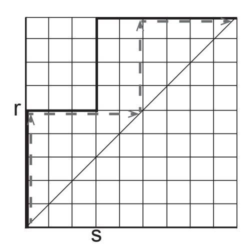

Figure 4: A corresponding Catalan path of $P(r,\ldots,r,n,\ldots,n)$

Lemma 4.1 (Stanley [9]). *Let $M = \{1^{c_1}, 2^{c_2}, \ldots, n^{c_n}\}$ be a multiset of cardinality $n = \sum_{i=1}^{n} c_i$ and $S(M)$ be the set of permutations on M. An* inversion *of $\pi = \pi_1 \cdots \pi_n \in S(M)$ is a pair (π_i, π_j) with $i < j$ and $\pi_i > \pi_j$. If we let $i(\pi)$ be the number of inversions of π, then*

$$\sum_{\pi \in S(M)} t^{i(\pi)} = \frac{[n]_t!}{[c_1]_t! \cdots [c_n]_t!}.$$

By using Theorem 2.10, Theorem 3.1, and Lemma 4.1, we have an explicit formula $X_G(\mathbf{x}, t)$ in the e-basis expansion.

Proposition 4.2. *Let $G = \mathrm{inc}(P)$ of a natural unit interval order $P = P(m_1, m_2 \ldots, m_{n-1})$. For r, s satisfying $1 \leqslant s \leqslant r \leqslant n-1$, if $m_1 = \cdots = m_s = r$ and $m_{s+1} = n$, then*

$$X_G(\mathbf{x}, t) = \sum_{\ell=0}^{\min\{n-r,s\}} t^{(r-s)\ell} \frac{[n-r]_t![s]_t![r-\ell-1]_t![n-s-\ell-1]_t![n-2\ell]_t}{[n-r-\ell]_t![s-\ell]_t![r-s-1]_t!} e_{(n-\ell,\ell)}.$$

Consequently, $X_G(\mathbf{x}, t)$ is e-positive and e-unimodal with center of symmetry $\frac{\binom{n}{2}-(n-r)s}{2}$.

Proof. Let $A_1 = \{1, \ldots, s\}$ and $A_2 = \{r+1, \ldots, n\}$. For any $B_1 \subset A_1$ and $B_2 \subset A_2$ with $|B_1| = |B_2| = \ell$, let $\overline{\mathcal{T}'}_P(B_1, B_2)$ be the set of $T = [a_{i,j}]$ in $\overline{\mathcal{T}'}_{P,2^{\ell}1^{n-2\ell}}$ satisfying $\{a_{i,1} \mid 1 \leqslant i \leqslant \ell\} = B_1$ and $\{a_{i,2} \mid 1 \leqslant i \leqslant \ell\} = B_2$, then by Theorem 3.1,

$$C_{(n-\ell,\ell)}(t) = \sum_{\substack{B_1 \subset A_1 \\ B_2 \subset A_2}} \sum_{T \in \overline{\mathcal{T}'}_P(B_1, B_2)} t^{\mathrm{inv}_G(T)}.$$

If we let $S(A)$ be the set of permutations for some set A, then $C_{(n-\ell,\ell)}(t)$ is equal to

$$t^{(r-s)\ell} \left(\sum_{\pi \in S(B_1)} t^{i(\pi)} \right)^2 \left(\sum_{T^b \in \overline{\mathcal{T}'}_{R_\ell, 1^{n-2\ell}}} t^{\mathrm{inv}_{\mathrm{inc}(R_\ell)}(T^b)} \right) \left(\sum_{\sigma_1 \in S(M_1)} t^{i(\sigma_1)} \right) \left(\sum_{\sigma_2 \in S(M_2)} t^{i(\sigma_2)} \right),$$

where $i(\pi)$ is the number of inversions of π, $R_\ell = P(m_1, \ldots, m_{n-2\ell-1})$ with $m_1 = \cdots = m_{s-\ell} = r - \ell$ and $m_{s-\ell+1} = n$, and $M_1 = \{1^{s-\ell} 2^{\ell}\}$, $M_2 = \{1^{n-r-\ell} 2^{\ell}\}$ are multisets.
By using Theorem 2.10 and Lemma 4.1, we can have $X_G(\mathbf{x}, t)$ in the e-basis expansion. □

The next two corollaries follow easily from Proposition 4.2.

Corollary 4.3. *For $r \geqslant 2$, let $P = P(r, n \ldots, n)$ be a natural unit interval order of size n then*

$$X_{\mathrm{inc}(P)}(\mathbf{x}, t) = [n-2]_t!([n]_t[r-1]_t e_n + t^{r-1}[n-r]_t e_{(n-1,1)}).$$

Corollary 4.4. *Let $P = P(m_1, m_2 \ldots, m_{n-1})$ be a natural unit interval order and G be the incomparability graph of P. If $m_1 = \cdots = m_{r-1} = r$ and $m_r = n$ for $r \leqslant n-1$, then*

$$X_G(\mathbf{x}, t) = \sum_{\ell=0}^{\min\{n-r,r-1\}} t^{\ell}[n-r]_t![r-1]_t![n-2\ell]_t e_{(n-\ell,\ell)}.$$

4.2 The second explicit formula

In this section we consider natural unit interval orders $P = P(m_1, m_2 \dots, m_{n-1})$ with $m_1 = r$, $m_2 = \cdots = m_s = n-1$, and $m_{s+1} = \cdots = m_{n-1} = n$ for some $0 < s \leqslant r \leqslant n-2$. Figure 5 shows a corresponding Catalan path of P with the bounce path in dotted line.

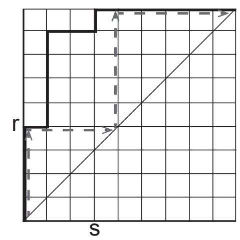

Figure 5: A corresponding Catalan path of $P(r, n-1, \dots, n-1, n, \dots, n)$

Proposition 4.5. *Let $P = P(m_1, m_2 \dots, m_{n-1})$ be a natural unit interval order and $G = \mathrm{inc}(P)$. If $m_1 = r$, $m_2 = \cdots = m_s = n-1$, and $m_{s+1} = n$ for some r, s with $2 \leqslant s \leqslant r \leqslant n-2$, then*

$$X_G(\mathbf{x}, t) = [n-4]_t!(\bar{C}_{(n)}(t)e_n + \bar{C}_{(n-1,1)}(t)e_{(n-1,1)} + \bar{C}_{(n-2,2)}(t)e_{(n-2,2)}),$$

where

$$\begin{aligned}
\bar{C}_{(n)}(t) &= [n]_t[n-3]_t[r-1]_t[n-s-1]_t \\
\bar{C}_{(n-2,2)}(t) &= t^{n+r-s-3}[2]_t[n-r-1]_t[s-1]_t \\
\bar{C}_{(n-1,1)}(t) &= \frac{1}{2}[n-2]_t\{t^{r-1}(1+t^{n-r-s})[n-3]_t + t^{n-s-1}[2]_t[r-2]_t[s-1]_t \\
&\qquad + t^{r-1}[2]_t[n-r-1]_t[n-s-2]_t\}.
\end{aligned}$$

Moreover, $X_G(\mathbf{x}, t)$ is e-positive and e-unimodal with center of the symmetry $\frac{\binom{n}{2}-(n-r+s-1)}{2}$.

Proof. By Theorem 2.10, we already know the formula of $C_{(n)}(t)$ which is e-unimodal with center of symmetry $\frac{|E(G)|}{2} = \frac{\binom{n}{2}-(n-r+s-1)}{2}$.
Let $\overline{\mathcal{T}'}_{P,(i,j)}$ be the set of all P-tableaux of $\overline{\mathcal{T}'}_{P,2^1 1^{n-2}} \cup \overline{\mathcal{T}'}_{P,2^2 1^{n-4}}$ such that the first row is occupied by i and j. If $G - \{i,j\}$ is the induced subgraph of G with vertex set $V(G) - \{i,j\}$, and $X_{G-\{i,j\}}(\mathbf{x}, t) = D_{(n-2)}(t)e_{(n-2)} + D_{(n-3,1)}(t)e_{(n-3,1)}$, then by Theorem 3.1,

$$\sum_{T \in \overline{\mathcal{T}'}_{P,(i,j)}} t^{\mathrm{inv}_G(T)} = t^{b_i+b_j}(D_{(n-2)}(t)e_{(n-1,1)} + D_{(n-3,1)}(t)e_{(n-2,2)}),$$

where $b_k = |\{\{j,k\} \in E(G) \mid j < k\}|$. By Corollary 4.3, we can evaluate $\sum_{T \in \overline{\mathcal{T}'}_{P,(i,j)}} t^{\mathrm{inv}_G(T)}$ and then have formulae of $C_{(n-2,2)}(t)$ and $C_{(n-1,1)}(t)$. By using Lemma 2.12 and Theorem 2.3, we can show that $C_{(n-1,1)}(t)$ is e-unimodal with center of symmetry $\frac{|E(G)|}{2}$. □

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