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On cyclic descents for tableaux

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Abstract. The notion of descent set, for permutations as well as for standard Young tableaux (SYT), is classical. Cellini introduced a natural notion of *cyclic descent set* for permutations, and Rhoades introduced such a notion for SYT — but only for rectangular shapes. In this work we define *cyclic extensions* of descent sets in a general context, and prove existence and essential uniqueness for SYT of almost all shapes. The proof applies nonnegativity properties of Postnikov's toric Schur polynomials, providing a new interpretation of certain Gromov–Witten invariants.

Keywords: descent, cyclic descent, standard Young tableau, ribbon Schur function, Gromov–Witten invariant

1 Introduction

For a permutation $\pi = [\pi_1, \ldots, \pi_n]$ in the symmetric group $\mathfrak{S}_n$ on n letters, one defines its *descent set* as

$$\mathrm{Des}(\pi) := \{1 \le i \le n-1 \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n-1],$$

where $[m] := \{1, 2, \ldots, m\}$. For example, $\mathrm{Des}([2,1,4,5,3]) = \{1,4\}$. On the other hand, its *cyclic descent set* was defined by Cellini [6] as

$$\mathrm{cDes}(\pi) := \{1 \le i \le n \,:\, \pi_i > \pi_{i+1}\} \quad \subseteq [n], \tag{1.1}$$

with the convention $\pi_{n+1} := \pi_1$. For example, $\mathrm{cDes}([2,1,4,5,3]) = \{1,4,5\}$. This cyclic descent set was further studied by Dilks, Petersen, Stembridge [8] and others. It has several important properties. Consider the two $\mathbb{Z}$-actions, on $\mathfrak{S}_n$ and on the power set of $[n]$, in which the generator p of $\mathbb{Z}$ acts by

$$\begin{aligned} [\pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n] &\overset{p}{\longmapsto} [\pi_n, \pi_1, \pi_2, \ldots, \pi_{n-1}], \\ \{i_1, \ldots, i_k\} &\overset{p}{\longmapsto} \{i_1+1, \ldots, i_k+1\} \bmod n. \end{aligned}$$

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Then, for every permutation π, one has the following three properties:

$$\mathrm{cDes}(\pi) \cap [n-1] = \mathrm{Des}(\pi) \qquad \text{(extension)} \tag{1.2}$$

$$\mathrm{cDes}(p(\pi)) = p(\mathrm{cDes}(\pi)) \qquad \text{(equivariance)} \tag{1.3}$$

$$\varnothing \subsetneq \mathrm{cDes}(\pi) \subsetneq [n] \qquad \text{(non-Escher)} \tag{1.4}$$

The term *non-Escher* refers to M. C. Escher's drawing "Ascending and Descending", which paradoxically depicts the impossible cases $\mathrm{cDes}(\pi) = \varnothing$ and $\mathrm{cDes}(\pi) = [n]$.

There is also an established notion of descent set for a *standard Young tableau* T of a skew shape λ/μ:

$$\mathrm{Des}(T) := \{1 \le i \le n-1 \,:\, i+1 \text{ appears in a lower row of } T \text{ than } i\} \quad \subseteq [n-1].$$

For example, the following standard Young tableau T of shape $\lambda/\mu = (4,3,2)/(1,1)$ has $\mathrm{Des}(T) = \{2,3,5\}$:

	1	2	7
	3	5	
4	6		

For the special case of standard Young tableaux T of *rectangular* shapes, Rhoades [17, Lemma 3.3] introduced a notion of *cyclic descent set* $\mathrm{cDes}(T)$, having the same properties (1.2), (1.3) and (1.4) with respect to the $\mathbb{Z}$-action in which the generator p acts on tableaux via Schützenberger's *jeu-de-taquin promotion* operator. A similar concept of $\mathrm{cDes}(T)$ and accompanying action p was introduced for two-row shapes and certain other skew shapes in [1, 9], and used there to answer Schur positivity questions.

Our first main result is a necessary and sufficient condition for the existence of a cyclic extension cDes of the descent map Des on the set $\mathrm{SYT}(\lambda/\mu)$ of standard Young tableaux of shape λ/μ, with an accompanying $\mathbb{Z}$-action on $\mathrm{SYT}(\lambda/\mu)$ via an operator p, satisfying properties (1.2), (1.3) and (1.4). In this story, a special role is played by the skew shapes known as *ribbons* (connected skew shapes containing no 2×2 rectangle), and in particular *hooks* (straight ribbon shapes, namely $\lambda = (n-k, 1^k)$ for $k = 0, 1, \ldots, n-1$). Early versions of [1] and [9] conjectured the following result.

Theorem 1.1. *Let λ/μ be a skew shape. The descent map* Des *on* $\mathrm{SYT}(\lambda/\mu)$ *has a cyclic extension* (cDes, p) *if and only if λ/μ is not a connected ribbon. Furthermore, for all $J \subseteq [n]$, all such cyclic extensions share the same cardinalities* $\#\mathrm{cDes}^{-1}(J)$.

Our strategy for proving Theorem 1.1 is inspired by a result of Gessel [10, Theorem 7] that we recall here. For a subset $J = \{j_1 < \ldots < j_t\} \subseteq [n-1]$, the composition (of n)

$$\alpha(J, n) := (j_1, j_2 - j_1, j_3 - j_2, \ldots, j_t - j_{t-1}, n - j_t) \tag{1.5}$$

defines a *connected ribbon* having the entries of $\alpha(J,n)$ as row lengths, and thus an associated *(skew) ribbon Schur function*

$$s_{\alpha(J,n)} := \sum_{\varnothing \subseteq I \subseteq J} (-1)^{\#(J \setminus I)} h_{\alpha(I,n)} \tag{1.6}$$

with the following property: for any skew shape λ/μ, the map $\mathrm{Des} : \mathrm{SYT}(\lambda/\mu) \to 2^{[n-1]}$ has fiber sizes given by

$$\#\mathrm{Des}^{-1}(J) = \langle s_{\lambda/\mu}, s_{\alpha(J,n)} \rangle \qquad (\forall J \subseteq [n-1]), \tag{1.7}$$

where $\langle -, - \rangle$ is the usual inner product on symmetric functions.

By analogy, for a subset $\varnothing \neq J = \{j_1 < j_2 < \ldots < j_t\} \subseteq [n]$ we define the corresponding *cyclic composition* of n as

$$\alpha^{\mathrm{cyc}}(J,n) := (j_2 - j_1, \ldots, j_t - j_{t-1}, j_1 + n - j_t), \tag{1.8}$$

with $\alpha^{\mathrm{cyc}}(J,n) := (n)$ when $J = \{j_1\}$; note that $\alpha^{\mathrm{cyc}}(\varnothing, n)$ is not defined. The corresponding *affine (or cyclic) ribbon Schur function* is then defined as

$$\tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} := \sum_{\varnothing \neq I \subseteq J} (-1)^{\#(J \setminus I)} h_{\alpha^{\mathrm{cyc}}(I,n)}. \tag{1.9}$$

We then collect enough properties of this function to show that there must exist a map $\mathrm{cDes} : \mathrm{SYT}(\lambda/\mu) \to 2^{[n]}$ and a $\mathbb{Z}$-action p on $\mathrm{SYT}(\lambda/\mu)$, as in Theorem 1.1, such that fiber sizes are given by

$$\#\mathrm{cDes}^{-1}(J) = \langle s_{\lambda/\mu}, \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} \rangle \qquad (\forall\, \varnothing \subsetneq J \subsetneq [n]). \tag{1.10}$$

See Corollary 4.3 below. The nonnegativity of this inner product when λ/μ is not a connected ribbon ultimately relies on relating $\tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)}$ to a special case of Postnikov's *toric Schur polynomials*, with their interpretation in terms of *Gromov–Witten invariants* for Grassmannians [16].

We also compare the distribution of cDes on $\mathrm{SYT}(\lambda)$ to the distribution of cDes on $\mathfrak{S}_n$. Recall [18, Theorem 3.1.1 and Section 5.6 Ex. 22(a)] that the *Robinson–Schensted correspondence* is a bijection between $\mathfrak{S}_n$ and the set of all pairs of standard Young tableaux of the same shape (and size n), having the property that if $w \mapsto (P,Q)$ then $\mathrm{Des}(w) = \mathrm{Des}(Q)$. Consequently

$$\sum_{w \in \mathfrak{S}_n} \mathbf{t}^{\mathrm{Des}(w)} = \sum_{\lambda \vdash n} f^{\lambda} \sum_{T \in \mathrm{SYT}(\lambda)} \mathbf{t}^{\mathrm{Des}(T)}.$$

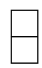
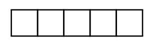
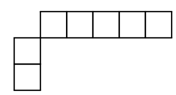
Here $\mathbf{t}^S := \prod_{i \in S} t_i$ for $S \subseteq \{1,2,\ldots\}$, $\lambda \vdash n$ means that λ is a partition of n, and $f^{\lambda} := \#\mathrm{SYT}(\lambda)$. Note that Theorem 1.1 implies that any *non-hook* shape λ, as well as any disconnected skew shape λ/μ, will have $\sum_{T \in \mathrm{SYT}(\lambda/\mu)} \mathbf{t}^{\mathrm{cDes}(T)}$ well-defined and independent of the choice of cyclic extension (cDes, p) for Des on $\mathrm{SYT}(\lambda/\mu)$. We then have the following second main result.

Theorem 1.2. *For any $n \geq 2$*

$$\sum_{w \in \mathfrak{S}_n} \mathbf{t}^{\mathrm{cDes}(w)} = \sum_{\substack{\text{non-hook} \\ \lambda \vdash n}} f^\lambda \sum_{T \in \mathrm{SYT}(\lambda)} \mathbf{t}^{\mathrm{cDes}(T)} \quad + \quad \sum_{k=1}^{n-1} \binom{n-2}{k-1} \sum_{T \in \mathrm{SYT}((1^k) \oplus (n-k))} \mathbf{t}^{\mathrm{cDes}(T)},$$

where cDes *is defined on $\mathfrak{S}_n$ by Cellini's formula* (1.1) *and on standard Young tableaux (of the relevant shapes) as in Theorem 1.1.*

The *direct sum* operation $\lambda \oplus \mu$ in the last summation denotes a skew shape having the diagram of λ strictly southwest of the diagram of μ, with no rows or columns in common. For example, when $(n, k) = (7, 2)$,

$(1^k) \oplus (n-k) \quad = \quad (1^2) \oplus (5) \quad =$  $\oplus$  $=$ 

The current work is an extended abstract, stating the definitions and main results together with some indication of the proofs. For a complete exposition, including all the relevant proofs and additional material, see the full paper version [2].

2 Definition, examples, and basic properties

Let us begin by formalizing the concept of a cyclic extension. Recall the bijection $p : 2^{[n]} \longrightarrow 2^{[n]}$ induced by the cyclic shift $i \mapsto i + 1 \pmod{n}$, for all $i \in [n]$.

Definition 2.1. Let $\mathcal{T}$ be a finite set. A *descent map* is any map $\mathrm{Des} : \mathcal{T} \longrightarrow 2^{[n-1]}$. A *cyclic extension* of Des is a pair (cDes, p), where $\mathrm{cDes} : \mathcal{T} \longrightarrow 2^{[n]}$ is a map and $p : \mathcal{T} \longrightarrow \mathcal{T}$ is a bijection, satisfying the following axioms: for all T in $\mathcal{T}$,

$$\begin{aligned} &\text{(extension)} && \mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T), \\ &\text{(equivariance)} && \mathrm{cDes}(p(T)) = p(\mathrm{cDes}(T)), \\ &\text{(non-Escher)} && \varnothing \subsetneq \mathrm{cDes}(T) \subsetneq [n]. \end{aligned}$$

The non-Escher axiom will be important for the uniqueness of the cyclic extension.

Example 2.2. Given any (strict) *composition* α of n, that is, an ordered sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_t)$ with $\sum_i \alpha_i = n$, define the associated *horizontal strip* skew shape

$$\alpha^\oplus := (\alpha_1) \oplus (\alpha_2) \oplus \cdots \oplus (\alpha_t)$$

whose rows, from southwest to northeast, have sizes $\alpha_1, \ldots, \alpha_t$. For T in $\mathcal{T} = \mathrm{SYT}(\alpha^\oplus)$ we define

$$\mathrm{cDes}(T) := \{1 \leq i \leq n \,:\, i+1 \text{ is in a lower row than } i\},$$

where $n+1$ is interpreted as 1, as well as a bijection $p : \mathrm{SYT}(\alpha^{\oplus}) \to \mathrm{SYT}(\alpha^{\oplus})$ which first replaces each entry j of T by $j+1 \pmod{n}$ and then re-orders each row to make it left-to-right increasing. One can check that this (cDes, p) is a cyclic extension of Des, with $t \geq 2$ required for the non-Escher property.

This generalizes the case of (cDes, p) on $\mathfrak{S}_n$, since for $\alpha = (1^n) = (1, 1, \ldots, 1)$ one has a bijection $\mathfrak{S}_n \to \mathrm{SYT}(\alpha^{\oplus})$ which sends a permutation w to the tableau whose entries are $w^{-1}(1), \ldots, w^{-1}(n)$ read from southwest to northeast. This bijection maps Cellini's cyclic extension (cDes, p) on $\mathfrak{S}_n$ to the one on $\mathrm{SYT}(\alpha^{\oplus})$ for $\alpha = (1, 1, \ldots, 1)$ defined above[1].

Cyclic extensions of the descent map on $\mathcal{T} = \mathrm{SYT}(\lambda)$ have been given previously for several families of shapes: for *rectangular shapes* see [17, Lemma 3.3]; for *flag shapes*, namely $(k, k, 1^{n-2k})$ with $1 \leq k \leq n/2$, see [15, 7]; for *two-row shapes* (k, ell) and *hook plus internal corner* shapes, namely $(n-2-k, 2, 1^k)$ with $0 \leq k \leq n-4$, see [1]; for skew shapes of type $\lambda \oplus (1)$, where $\lambda \vdash n-1$ is any nonempty partition, see [9]. The last case was, in fact, our original motivation, and the question of existence of a cyclic extension of Des on $\mathrm{SYT}(\lambda/\mu)$ appears there as [9, Problem 5.5].

Lemma 2.3. *Fix a set $\mathcal{T}$ and a map* $\mathrm{Des} : \mathcal{T} \to 2^{[n-1]}$.

- (i) *Any cyclic extension (cDes, p) of* Des *has fiber sizes $m(J) := \#\mathrm{cDes}^{-1}(J)$ (for $J \subseteq [n]$) satisfying*
  - (a) *$m(J) \geq 0$ for all J, with $m(\varnothing) = m([n]) = 0$;*
  - (b) *$m(J) = m(p(J))$ for all J; and*
  - (c) *$m(J) + m(J \sqcup \{n\}) = \mathrm{Des}^{-1}(J)$ for all $J \subseteq [n-1]$.*
- (ii) *Conversely, if $(m(J))_{J \subseteq [n]}$ are integers satisfying conditions (a),(b) and (c) above, then there exists at least one cyclic extension (cDes, p) of* Des *satisfying $\#\mathrm{cDes}^{-1}(J) = m(J)$ for all $J \subseteq [n]$.*

The definition of a cyclic extension (cDes, p) of Des on $\mathcal{T}$ leads immediately to a relation between the ordinary generating functions

$$\mathcal{T}^{\mathrm{cdes}}(t) := \sum_{T \in \mathcal{T}} t^{\#\mathrm{cDes}(T)}, \qquad \mathcal{T}^{\mathrm{des}}(t) := \sum_{T \in \mathcal{T}} t^{\#\mathrm{Des}(T)}.$$

Lemma 2.4. *For any cyclic extension (cDes, p) of* Des *on $\mathcal{T}$ one has*

$$\frac{n\mathcal{T}^{\mathrm{des}}(t)}{(1-t)^{n+1}} = \frac{d}{dt}\left[\frac{\mathcal{T}^{\mathrm{cdes}}(t)}{(1-t)^n}\right].$$

This lemma completely determines $\mathcal{T}^{\mathrm{cdes}}(q)$ in terms of $\mathcal{T}^{\mathrm{des}}(q)$, since the non-Escher condition implies that the polynomial $\mathcal{T}^{\mathrm{cdes}}(t)$ has no constant term in t, and similarly for the formal power series $\mathcal{T}^{\mathrm{cdes}}(t)/(1-t)^n$.

[1]This cyclic descent map can further be generalized to *strips*, which are the disconnected shapes with each connected component consisting of either one row or one column; see [1].

3 Ribbon Schur functions: affine, cylindric and toric

Recall from the introduction that our proof strategy for Theorem 1.1 involves the introduction of a family of new symmetric functions, which we call *affine* (or *cyclic*) *ribbon Schur functions*. We recall here their definition and develop some of their properties, using standard terminology and properties of symmetric functions.

3.1 Affine ribbon Schur functions

Definition 3.1. To each nonempty subset $\varnothing \neq J = \{j_1 < j_2 < \ldots < j_t\} \subseteq [n]$ associate (as in (1.8) above) a *cyclic composition* of n,

$$\alpha^{\mathrm{cyc}}(J,n) := (j_2 - j_1, \ldots, j_t - j_{t-1}, j_1 + n - j_t).$$

In particular, $\alpha^{\mathrm{cyc}}(\{j_1\},n) := (n)$ while $\alpha^{\mathrm{cyc}}(\varnothing,n)$ is undefined. The corresponding *affine (or cyclic) ribbon Schur function* is defined (as in (1.9) above) by

$$\tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} := \sum_{\varnothing \neq I \subseteq J} (-1)^{\#(J \setminus I)} h_{\alpha^{\mathrm{cyc}}(I,n)}.$$

Example 3.2. For $n = 9$ and $J = \{2,6\}$ the ordinary ribbon Schur function is

$$\begin{aligned} s_{\alpha(\{2,6\},9)} &= h_{\alpha(\{2,6\},9)} - h_{\alpha(\{6\},9)} - h_{\alpha(\{2\},9)} + h_{\alpha(\varnothing,9)} \\ &= h_{(2,4,3)} - h_{(6,3)} - h_{(2,7)} + h_{(9)}. \end{aligned}$$

By contrast, the corresponding *affine* ribbon Schur function is

$$\begin{aligned} \tilde{s}_{\alpha^{\mathrm{cyc}}(\{2,6\},9)} &= h_{\alpha^{\mathrm{cyc}}(\{2,6\},9)} - h_{\alpha^{\mathrm{cyc}}(\{6\},9)} - h_{\alpha^{\mathrm{cyc}}(\{2\},9)} \\ &= h_{(4,5)} - h_{(9)} - h_{(9)} = h_{(4,5)} - 2h_{(9)}. \end{aligned}$$

Our proof strategy for Theorem 1.1 involves showing that, whenever λ/μ is a skew shape of size n which is not a connected ribbon, the integers $m(J) := \langle s_{\lambda/\mu}, \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} \rangle$ satisfy conditions (a),(b) and (c) of Lemma 2.3. In fact, the nonnegativity condition (a) is the most subtle; conditions (b) and (c) follow from the following two propositions.

Proposition 3.3 (Equivariance)**.** *For each nonempty subset $\varnothing \neq J \subseteq [n]$, the cyclic composition $\alpha^{\mathrm{cyc}}(p(J),n)$ is a cyclic shift of $\alpha^{\mathrm{cyc}}(J,n)$, and consequently*

$$\tilde{s}_{\alpha^{\mathrm{cyc}}(p(J),n)} = \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)}.$$

Proposition 3.4 (Extension)**.** *For each nonempty $\varnothing \neq J \subseteq [n-1]$,*

$$\tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} + \tilde{s}_{\alpha^{\mathrm{cyc}}(J \sqcup \{n\},n)} = s_{\alpha(J,n)}.$$

The subtle nonnegativity condition (a) in Lemma 2.3 will be derived from the following result, which is the main goal of this section.

Theorem 3.5. *For every $J \subseteq [n]$ and non-hook partition $\lambda \vdash n$,*

$$\langle \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)}, s_\lambda \rangle \geq 0.$$

3.2 Cylindric ribbon shapes and cylindric Schur functions

The key to Theorem 3.5 is a relation between the affine ribbon symmetric function $\tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)}$, defined above, and a special case of Postnikov's cylindric Schur functions [16] which was introduced implicitly already by Gessel and Krattenthaler [11] and studied further by McNamara [12].

Definition 3.6 (Cf. [16, Section 3] and [12, Definition 3.4])**.** For each subset $\varnothing \neq J = \{j_1 < \ldots < j_t\} \subseteq [n]$ define a *cylindric ribbon shape* $C_J = \lambda/1/\lambda$ in one of the following equivalent ways:

- (Postnikov) The partition

  $$\lambda := (n-t, j_t - j_1 - (t-1), \ldots, j_3 - j_1 - 2, j_2 - j_1 - 1)$$

  fits inside a $t \times (n-t)$ rectangle, and may be viewed as a lattice path connecting the southwestern and northeastern corners of this rectangle. Repeat this path, periodically, to obtain an infinite path. The cylindric ribbon shape $\lambda/1/\lambda$ is the infinite ribbon contained between this infinite path and its shift by one step eastward (equivalently, southward).

- Consider the cyclic composition

  $$\alpha^{\mathrm{cyc}}(J,n) = (j_2 - j_1, j_3 - j_2, \ldots, j_1 + n - j_t).$$

  The cylindric ribbon shape C_J is the infinite ribbon whose sequence of row lengths (from southwest to northeast) is the sequence of parts of this composition, repeated periodically.

- Consider the cyclic composition $\alpha^{\mathrm{cyc}}(J,n)$ above, and let R_J be the corresponding (finite) ribbon shape. Its northwest boundary is given by the partition λ above, except that the first part should be $n-t+1$ rather than $n-t$. Denote by a (respectively, b) the extreme southwestern (respectively, northeastern) square of this ribbon shape. The cylindric ribbon shape $C_J = \lambda/1/\lambda$ is an infinite ribbon made up of copies $(R_i)_{i\in\mathbb{Z}}$ of the ribbon R_J, placed in the plane such that square a of R_{i+1} is immediately north of square b of R_i, for all $i \in \mathbb{Z}$.

Definition 3.7 ([16, Definition 3.2 and Lemma 3.3])**.** A cylindric ribbon shape $C_J = \lambda/1/\lambda$ is called *toric* if (each) one of the following equivalent conditions holds:

1. Each row of C_J has length at most $n-t$.
2. Each column of C_J has length at most t.
3. The first and last columns of R_J have no squares in the same row.

Definition 3.8 ([16, Section 5])**.** Let $\lambda/1/\lambda$ be a cylindric ribbon shape. Define the corresponding *cylindric Schur function* by

$$s_{\lambda/1/\lambda}(x_1,\ldots) := \sum_T \mathbf{x}^T,$$

where summation is over all *semistandard cylindric tableaux* T filling the shape $\lambda/1/\lambda$ with entries in $\{1,2,\ldots\}$. This means that the entries of T are "(t,n)-periodic" (see [16, Figure 4]), weakly increasing from left to right in rows and strictly increasing from top to bottom in columns.

3.3 Affine vs. cylindric ribbon Schur functions

Our next result shows that our affine ribbon Schur functions (Definition 3.1) are almost the same as Postnikov's cylindric ribbon Schur functions (Definition 3.8). To state it, recall the *power sum* symmetric function defined by $p_n := x_1^n + x_2^n + \ldots$.

Proposition 3.9. *For $\varnothing \neq J \subseteq [n]$, with associated cylindric ribbon shape $C_J = \lambda/1/\lambda$, one has*

$$s_{\lambda/1/\lambda} = \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} + (-1)^{\#J} p_n.$$

For an algebraic consequence of the toric property of a shape, consider the specialization $x_{t+1} = x_{t+2} = \ldots = 0$.

Proposition 3.10 (Cf. McNamara [12, Lemma 5.3 and Proposition 5.5])**.** *Fix $1 \le t \le n$. Then*

$$\tilde{s}_{\alpha^{\mathrm{cyc}}([t],n)} = \sum_{k=0}^{t-1} (-1)^{t-1-k} s_{(n-k,1^k)} \tag{3.1}$$

and, for each nonempty subset $\varnothing \neq J \subseteq [n]$ with $\#J = t$, there exist nonnegative integers $c_{J,\nu}$ such that

$$\tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} = \tilde{s}_{\alpha^{\mathrm{cyc}}([t],n)} + \sum_{\text{non-hook } \nu \vdash n} c_{J,\nu} s_\nu. \tag{3.2}$$

Theorem 3.5 is clearly a consequence of Proposition 3.10, since for any nonempty subset $J \subseteq [n]$ and any non-hook partition $\nu \vdash n$

$$\langle \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)}, s_\nu \rangle = \langle s_{\lambda/1/\lambda}, s_\nu \rangle = C_{\lambda,\nu}^{\lambda,1} \ge 0.$$

It is worth noting that the Gromov–Witten invariants $C_{\mu,\nu}^{\lambda,d}$ have several interpretations, in addition to the one given in the proof of Proposition 3.10:

- They count certain *puzzles*, as conjectured by Knutson and proved by Buch, Kresch, Purbhoo, and Tamvakis [5].

- They have algebraic interpretations involving Morse's *k-Schur functions*, and in the *Verlinde fusion algebra*; see, e.g., the discussion by Morse and Schilling [13, Section 1.4].
- Pawlowski has proved [14, Theorem 7.8] a conjecture of Postnikov [16, Conjecture 9.1], asserting that that $s_{\lambda/d/\mu}$ is the Frobenius characteristic for the Specht module of the toric shape $\lambda/d/\mu$, so that $C^{\lambda,d}_{\mu,\nu}$ are its irreducible expansion coefficients.

In the special case where $d = 1$ and $\lambda = \mu$, the shape $\lambda/1/\lambda = C_J$ corresponds to some $J \subseteq [n]$, and then, for a (non-hook) shape ν, one can regard Theorem 1.1 as yielding another interpretation:

$$C^{\lambda,1}_{\lambda,\nu} = \langle \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)}, s_\nu \rangle = \#\{T \in \mathrm{SYT}(\nu) : \mathrm{cDes}(T) = J\},$$

where (cDes, p) is any cyclic extension of Des on $\mathrm{SYT}(\nu)$.

4 Remarks and questions

We close with several remarks and questions raised by this work.

4.1. Bijective proofs and cyclic sieving. Detailed proofs of Theorems 1.1 and 1.2 may be found in the full version paper [2]. Our proof of the existence of (cDes, p) in Theorem 1.1, whose strategy was sketched above, is indirect and involves arbitrary choices.

Problem 4.1. *Find a natural, explicit map* cDes *and cyclic action p on* $\mathrm{SYT}(\lambda/\mu)$ *as in Theorem 1.1.*

Problem 4.2. *Find a Robinson–Schensted style bijective proof of Theorem 1.2.*

Let us state explicitly a consequence of the proof of Theorem 1.1.

Corollary 4.3. *Let λ/μ be a skew shape of size n which is not a connected ribbon. For any $J \subseteq [n]$ and every cyclic extension* cDes *of the usual descent map on* $\mathrm{SYT}(\lambda/\mu)$, *the fiber size*

$$\#\mathrm{cDes}^{-1}(J) = \langle s_{\lambda/\mu}, \tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} \rangle.$$

For $J = \{j_1, \cdots, j_t\} \subseteq [n]$ let $-J := \{-j_1, \ldots, -j_t\}$ (interpreted modulo n).

Corollary 4.4. *Let λ/μ be a skew shape of size n which is not a connected ribbon. For any $J \subseteq [n]$ and any cyclic extension* cDes *of the usual descent map on* $\mathrm{SYT}(\lambda/\mu)$, *the fiber size*

$$\#\mathrm{cDes}^{-1}(J) = \#\mathrm{cDes}^{-1}(-J).$$

Problem 4.5. *For a solution of Problem 4.1, find an involution on* $\mathrm{SYT}(\lambda/\mu)$ *which sends the cyclic descent set to its negative.*

Problem 4.6. *For non-ribbon shapes λ/μ, can one choose the operator p in Theorem 1.1 and a polynomial $X(q)$ to obtain a cyclic sieving phenomenon (CSP) ?*

This problem was solved by Rhoades [17] for rectangular shapes and by Pechenik [15] for shapes $(k, k, 1^{n-2k})$. Recalling from [9] the cyclic descent extension for $\mathrm{SYT}(\lambda \oplus (1))$, Corollary 4.3 in the current paper has been applied in [3] to obtain a refined CSP on SYT of these skew shapes.

4.2. Topological interpretation of affine ribbon Schur functions. The alternating sum definition (1.9) of the affine ribbon Schur function $\tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} := \sum_{\varnothing \neq I \subseteq J} (-1)^{\#(J \setminus I)} h_{\alpha^{\mathrm{cyc}}(I,n)}$ has a topological interpretation, as (the Frobenius image of) a certain virtual Euler characteristic representation of $\mathfrak{S}_n$. In particular, the special case $t = n$ of (3.1),

$$\sum_{\varnothing \neq I \subseteq [n]} (-1)^{n-\#I} h_{\alpha^{\mathrm{cyc}}(I,n)} = \tilde{s}_{\alpha^{\mathrm{cyc}}([n],n)} = \sum_{i=0}^{n-1} (-1)^{n-1-i} s_{(n-i,1^i)}, \tag{4.1}$$

is the *Euler–Poincaré relation* for the (ordinary, non-reduced) homology of the (type A_{n-1}) *Steinberg torus* considered in [8].

Recall the well-known homological interpretation [4, Section 6], [19], [20, Section 4] of $s_{\alpha(J,n)}$. The *type A_{n-1} Coxeter complex* is a simplicial complex Δ triangulating an $(n-2)$-dimensional sphere. It has a *balanced* coloring of its vertices by $[n-1]$: each maximal simplex has exactly one vertex of each color. Furthermore, for each $J \subseteq [n-1]$, the group $\mathfrak{S}_n$ acts transitively on the simplices whose vertices have color set J. By the Euler–Poincaré relation for Δ_J

$$\mathrm{ch}(\tilde{H}_{\#J-1}(\Delta_J)) = \sum_{I \subseteq J} (-1)^{\#(J \setminus I)} h_{\alpha(I,n)} = s_{\alpha(J,n)},$$

where ch is the *Frobenius characteristic map*.

We wish to similarly re-interpret, for $\varnothing \neq J \subseteq [n]$, the affine ribbon skew Schur function $\tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)}$ in terms of the type A_{n-1} case of what Dilks, Petersen and Stembridge [8] call the *Steinberg torus*. This is a regular cell complex which we shall denote $\widetilde{\Delta}$. It is a *Boolean cell complex*: all lower intervals in the partial ordering of cells are Boolean algebras, so that the cells are essentially simplices, but their intersections are not necessarily common faces. It triangulates an $(n-1)$-dimensional torus, with a simply transitive action of $\mathfrak{S}_n$ on the maximal cells. It also has a balanced coloring of its vertices by $[n]$, so each maximal simplex has exactly one vertex of each color. Furthermore, for $\varnothing \neq J \subseteq [n]$, the group $\mathfrak{S}_n$ again acts transitively on the set of all cells whose vertices have color set J, but this time with $\mathfrak{S}_n$-stabilizers conjugate to the Young subgroup $\mathfrak{S}_{\alpha^{\mathrm{cyc}}(J,n)}$ associated to the *cyclic* composition $\alpha^{\mathrm{cyc}}(J,n)$. Thus the permutation representation of $\mathfrak{S}_n$ on the cells of color set J has image, under the Frobenius characteristic map ch, equal to the symmetric function $h_{\alpha^{\mathrm{cyc}}(J,n)}$. This describes the action of $\mathfrak{S}_n$ on the simplices of $\widetilde{\Delta}$.

On the homological side, since $\widetilde{\Delta}$ triangulates a torus, it is not Cohen–Macaulay. In fact, its (non-reduced) cohomology ring $H^*(\widetilde{\Delta})$ with rational coefficients is isomorphic to an exterior algebra $\wedge V$, where $V = H^1(\widetilde{\Delta})$ carries the irreducible reflection representation of $\mathfrak{S}_n$. Since $\wedge^i V$ has Frobenius image $\mathrm{ch}(\wedge^i V) = s_{(n-i,1^i)}$, this describes the $\mathfrak{S}_n$-action on homology. An analysis via the Euler–Poincaré relation, as before, shows that

$$\sum_{i \geq 0} (-1)^i \mathrm{ch}(C^i(\widetilde{\Delta})) = \sum_{i \geq 0} (-1)^i \mathrm{ch}(H^i(\widetilde{\Delta}))$$

which becomes exactly (4.1) by the above discussion. More generally, for each $\varnothing \neq J \subseteq [n]$, the *type-selected subcomplex* $\widetilde{\Delta}_J$ consisting of the cells that only use vertices whose colors lie in J has Euler–Poincaré relation

$$\sum_{i \geq 0} (-1)^i \mathrm{ch}(C^i(\widetilde{\Delta}_J)) = \sum_{i \geq 0} (-1)^i \mathrm{ch}(H^i(\widetilde{\Delta}_J))$$

giving the re-interpretation

$$\tilde{s}_{\alpha^{\mathrm{cyc}}(J,n)} = \sum_{\varnothing \neq I \subseteq J} (-1)^{\#(J \setminus I)} h_{\alpha^{\mathrm{cyc}}(I,n)} = \sum_{i \geq 0} (-1)^{\#J-1-i} \mathrm{ch}(H^i(\widetilde{\Delta}_J)).$$

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