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# Crystals and Schur $P$-positive expansions

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**Abstract.** We give a new characterization of Littlewood–Richardson–Stembridge tableaux for Schur $P$-functions by using the theory of $\mathfrak{q}(n)$-crystals. We also give alternate proofs of the Schur $P$-expansion of a skew Schur function due to Ardila and Serrano, and the Schur expansion of a Schur $P$-function due to Stembridge using the associated crystal structures.

**Keywords:** Schur P-function, crystals, Littlewood–Richardson rule

## 1 Introduction

Let $\mathscr{P}^+$ be the set of strict partitions and let $P_\lambda$ be the Schur $P$-function corresponding to $\lambda \in \mathscr{P}^+$. The set of Schur $P$-functions is an important class of symmetric functions, which is closely related with representation theory and algebraic geometry. For example, the Schur $P$-polynomial $P_\lambda(x_1, \ldots, x_n)$ in $n$ variables is the character of a finite-dimensional irreducible representation $V_n(\lambda)$ of the queer Lie superalgebra $\mathfrak{q}(n)$ with highest weight $\lambda$ up to a power of 2 when the length $\ell(\lambda)$ of $\lambda$ is no more than $n$ [10].

The set of Schur $P$-functions forms a basis of a subring of the ring of symmetric functions, and the structure constants with respect to this basis are nonnegative integers, that is, given $\mu, \nu \in \mathscr{P}^+$,

$$P_\mu P_\nu = \sum_{\lambda \in \mathscr{P}^+} f^\lambda_{\mu\nu} P_\lambda,$$

for some nonnegative integers $f^\lambda_{\mu\nu}$. The first and the most well-known result on a combinatorial description of $f^\lambda_{\mu\nu}$ was obtained by Stembridge [12] using shifted Young tableaux, which is a combinatorial model for Schur $P$- or $Q$-functions [9]. It is shown that $f^\lambda_{\mu\nu}$ is equal to the number of semistandard tableaux with entries in a $\mathbb{Z}_2$-graded set $\mathcal{N} = \{1' < 1 < 2' < 2 < \cdots\}$ of shifted skew shape $\lambda/\mu$ and weight $\nu$ such that (i) for each integer $k \geq 1$ the southwesternmost entry with value $k$ is unprimed or of even degree and (ii) the reading words satisfy the *lattice property*. Here we say that the value $|x|$ is $k$ when $x$ is either $k$ or $k'$ in a tableau. Let us call these tableaux the *Littlewood–Richardson–Stembridge (LRS) tableaux* (see Definitions 3.5 and 3.6).

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Recently, two more descriptions of $f^{\lambda}_{\mu\nu}$ were obtained in terms of semistandard decomposition tableaux, which is another combinatorial model for Schur $P$-functions introduced by Serrano [11]. It is shown by Cho that $f^{\lambda}_{\mu\nu}$ is given by the number of semistandard decomposition tableaux of shifted shape $\mu$ and weight $w_0(\lambda-\nu)$ whose reading words satisfy the $\lambda$-*good property* (see [2, Corollary 5.14]). Here we assume that $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$, and $w_0$ denotes the longest element in the symmetric group $\mathfrak{S}_n$. Another description is given by Grantcharov, Jung, Kang, Kashiwara, and Kim [7] based on their crystal base theory for the quantized enveloping algebra of $\mathfrak{q}(n)$ [6]. They realize the crystal $\mathbf{B}_n(\lambda)$ associated to $V_n(\lambda)$ as the set of semistandard decomposition tableaux of shape $\lambda$ with entries in $\{1 < \cdots < n\}$, and describe $f^{\lambda}_{\mu\nu}$ by characterizing the lowest weight vectors of weight $w_0\lambda$ in the tensor product $\mathbf{B}_n(\mu) \otimes \mathbf{B}_n(\nu)$. We also remark that bijections between the above mentioned combinatorial models for $f^{\lambda}_{\mu\nu}$ are studied in [4] using insertion schemes for semistandard decomposition tableaux.

The main result in this paper is to give another description of $f^{\lambda}_{\mu\nu}$ using the theory of $\mathfrak{q}(n)$-crystals, and show that it is indeed equivalent to that of Stembridge. More precisely, we show that $f^{\lambda}_{\mu\nu}$ is equal to the number of semistandard tableaux with entries in $\mathcal{N}$ of shifted skew shape $\lambda/\mu$ and weight $\nu$ such that (i) for each integer $k \geq 1$ the southwesternmost entry with value $k$ is unprimed or of even degree and (ii) the reading words satisfy the *"lattice property"* (see Definitions 3.1, 3.2 and Theorem 3.3). It is obtained by semistandardizing the standard tableaux which parametrize the lowest weight vectors counting $f^{\lambda}_{\mu\nu}$ in [7], where the "lattice property" naturally arises from the configuration of entries in semistandard decomposition tableaux. We show that these tableaux for $f^{\lambda}_{\mu\nu}$ are equal to LRS tableaux (Theorem 3.7), and hence obtain a new characterization of LRS tableaux. For more details, see Section 3.

We next consider the Schur $P$-positive expansion of a skew Schur function

$$s_{\lambda/\delta_r} = \sum_{\nu \in \mathscr{P}^+} a_{\lambda/\delta_r\,\nu} P_\nu$$

for a skew diagram $\lambda/\delta_r$ contained in a rectangle $((r+1)^{r+1})$, where $\delta_r = (r, r-1, \ldots, 1)$ (cf. [1, 5]). We give a combinatorial description of $a_{\lambda/\delta_r\,\nu}$ (Theorem 4.4) by considering a $\mathfrak{q}(n)$-crystal structure on the set of usual semistandard tableaux of shape $\lambda/\delta_r$ and characterizing the lowest weight vectors corresponding to each $\nu \in \mathscr{P}^+$. We refer to Section 4.

Finally, we consider the Schur expansion of a Schur $P$-function

$$P_\lambda = \sum_{\mu} g_{\lambda\mu} s_\mu,$$

for $\lambda \in \mathscr{P}^+$. We give a simple and alternate proof of Stembridge's description for $g_{\lambda\mu}$ [12] (Theorem 5.2) by characterizing the type $A$ lowest weight vectors of weight $w_0\mu$ in the $\mathfrak{q}(n)$-crystal $\mathbf{B}_n(\lambda)$ when $\ell(\lambda), \ell(\mu) \leq n$ (see Section 5).

A full version of this paper including detailed proofs has appeared in [3].

# 2 Crystals for queer Lie superalgebras

## 2.1 Notation and terminology

Let $\mathbb{Z}_+$ be the set of nonnegative integers. We fix a positive integer $n \geq 2$ throughout this paper. Let

$$\begin{aligned}\mathscr{P} &= \{\, \lambda = (\lambda_i)_{i\geq 1} \,|\, \lambda_i \in \mathbb{Z}_+,\ \lambda_i \geq \lambda_{i+1}\ (i \geq 1),\ |\lambda| := \textstyle\sum \lambda_i < \infty \,\},\\ \mathscr{P}^+ &= \{\, \lambda = (\lambda_i)_{i\geq 1} \,|\, \lambda \in \mathscr{P},\ \lambda_i = \lambda_{i+1} \Rightarrow \lambda_i = 0\ (i \geq 1) \,\}.\end{aligned}$$

Let $\mathscr{P}_n = \{\, \lambda \,|\, \ell(\lambda) \leq n \,\} \subseteq \mathscr{P}$, where $\ell(\lambda)$ is the length of $\lambda$, and $\mathscr{P}_n^+ = \mathscr{P}^+ \cap \mathscr{P}_n$.

The (unshifted) diagram of $\lambda \in \mathscr{P}$ and the shifted diagram of $\lambda \in \mathscr{P}^+$ are defined to be

$$\begin{aligned}D_\lambda &= \{\, (i,j) \in \mathbb{N}^2 \,:\, 1 \leq j \leq \lambda_i,\ 1 \leq i \leq \ell(\lambda) \,\},\\ D_\lambda^+ &= \{\, (i,j) \in \mathbb{N}^2 \,:\, i \leq j \leq \lambda_i + i - 1,\ 1 \leq i \leq \ell(\lambda) \,\},\end{aligned}$$

respectively. We identify $D_\lambda$ and $D_\lambda^+$ with diagrams where a box is placed at the $i$-th row from the top and the $j$-th column from the left for each $(i,j) \in D_\lambda$ and $D_\lambda^+$, respectively.

Let $\mathcal{A}$ be a linearly ordered set, and $\mathcal{W}_\mathcal{A}$ be the set of words of finite length with letters in $\mathcal{A}$. For $w \in \mathcal{W}_\mathcal{A}$ and $a \in \mathcal{A}$, let $c_a(w)$ be the number of occurrences of $a$ in $w$.

For $\lambda, \mu \in \mathscr{P}$ with $D_\mu \subseteq D_\lambda$, a *tableau of shape* $\lambda/\mu$ means a filling on the skew diagram $D_\lambda \setminus D_\mu$ with entries in $\mathcal{A}$. For $\lambda, \mu \in \mathscr{P}^+$ with $D_\mu^+ \subseteq D_\lambda^+$, a *tableau of shifted shape* $\lambda/\mu$ is defined in a similar way. For a tableau $T$ of (shifted) shape $\lambda/\mu$, let $w(T)$ be the word given by reading the entries of $T$ row by row from top to bottom, and from right to left in each row. We denote by $T_{i,j}$ the $j$-th entry (from the left) of the $i$-th row of $T$ from the top. For $1 \leq i \leq \ell(\lambda)$, let $T^{(i)} = T_{i,\lambda_i} \cdots T_{i,1}$ be the subword of $w(T)$ corresponding to the $i$-th row of $T$. Then we have $w(T) = T^{(1)} \cdots T^{(\ell(\lambda))}$. We denote by $w_{\mathrm{rev}}(T)$ the reverse word of $w(T)$. For $a \in \mathcal{A}$, let $c_a(T) = c_a(w(T))$ be the number of occurrences of $a$ in $T$.

Suppose that $\mathcal{A}$ is a linearly ordered set with a $\mathbb{Z}_2$-grading $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$. For $\lambda, \mu \in \mathscr{P}$ with $D_\mu \subseteq D_\lambda$, let $SST_\mathcal{A}(\lambda/\mu)$ be the set of tableaux of shape $\lambda/\mu$ with entries in $\mathcal{A}$ which is semistandard, that is, (a) the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), (b) the entries in $\mathcal{A}_0$ (resp. $\mathcal{A}_1$) are strictly increasing in each column (resp. row). Similarly, for $\lambda, \mu \in \mathscr{P}^+$ with $D_\mu^+ \subseteq D_\lambda^+$, we define $SST_\mathcal{A}^+(\lambda/\mu)$ to be the set of semistandard tableaux of shifted shape $\lambda/\mu$ with entries in $\mathcal{A}$.

Let $\mathcal{N} = \{\, 1' < 1 < 2' < 2 < \cdots \}$ be a linearly ordered set with a $\mathbb{Z}_2$-grading $\mathcal{N}_0 = \mathbb{N}$ and $\mathcal{N}_1 = \mathbb{N}' = \{1', 2', \cdots\}$. Put $[n] = \{\, 1, \ldots, n \,\}$ and $[n]' = \{\, 1', \ldots, n' \,\}$, where the $\mathbb{Z}_2$-grading and linear ordering are induced from $\mathcal{N}$. For $a \in \mathcal{N}$, we write $|a| = k$ when $a$ is either $k$ or $k'$.

## 2.2 Semistandard decomposition tableaux and Schur $P$-functions

Let us recall the notion of semistandard decomposition tableaux [7, 11], which is our main combinatorial object. A word $u = u_1 \cdots u_s$ in $\mathcal{W}_{\mathbb{N}}$ is called a *hook word* if it satisfies $u_1 \geq u_2 \geq \cdots \geq u_k < u_{k+1} < \cdots < u_s$ for some $1 \leq k \leq s$. In this case, let $u{\downarrow} = u_1 \cdots u_k$ be the weakly decreasing subword of maximal length and $u{\uparrow} = u_{k+1} \cdots u_s$ the remaining strictly increasing subword in $u$.

**Definition 2.1.** *For $\lambda \in \mathscr{P}^+$, let $T$ be a tableau of shifted shape $\lambda$ with entries in $\mathbb{N}$. Then $T$ is called a semistandard decomposition tableau of shape $\lambda$ if (a) $T^{(i)}$ is a hook word of length $\lambda_i$ for $1 \leq i \leq \ell(\lambda)$, (b) $T^{(i)}$ is a hook subword of maximal length in $T^{(i+1)}T^{(i)}$, the concatenation of $T^{(i+1)}$ and $T^{(i)}$, for $1 \leq i < \ell(\lambda)$.*

For $\lambda \in \mathscr{P}^+$, let $SSDT(\lambda)$ be the set of semistandard decomposition tableaux of shape $\lambda$. Let $x = \{x_1, x_2, \ldots\}$ be a set of formal commuting variables, and let $P_\lambda = P_\lambda(x)$ be the Schur $P$-function in $x$ corresponding to $\lambda \in \mathscr{P}^+$. It is shown in [11] that $P_\lambda$ is given by the weight generating function of $SSDT(\lambda)$: $P_\lambda = \sum x^T$, where the sum runs over all $T \in SSDT(\lambda)$, and $x^T = \prod_{i\geq 1} x_i^{c_i(T)}$.

**Remark 2.2.** *Recall that the Schur $P$-function $P_\lambda$ can be realized as the character of tableaux $T \in SST^+_{\mathcal{N}}(\lambda)$ with no primed entry or entry of odd degree on the main diagonal (cf. [9]). The notion of semistandard decomposition tableaux was introduced in [11] to give a plactic monoid model for Schur $P$-functions. In this paper, we follow its modified version (Definition 2.1) introduced in [7], by which it is easier to describe $\mathfrak{q}(n)$-crystals [7, Remark 2.6]. We also refer the reader to [4] for more details on relation between the combinatorics of these two models.*

For $\lambda \in \mathscr{P}^+$, let $SSDT_n(\lambda)$ be the set of tableaux $T \in SSDT(\lambda)$ with entries in $[n]$. By [7, Proposition 2.3], we see that $SSDT_n(\lambda) \neq \emptyset$ if and only if $\lambda \in \mathscr{P}^+_n$. We denote by $P_\lambda(x_1, \ldots, x_n)$ the Schur $P$-polynomial in $x_1, \ldots, x_n$ given by specializing $P_\lambda$ at $x_{n+1} = x_{n+2} = \cdots = 0$. Then we have $P_\lambda(x_1, \ldots, x_n) = \sum_{T\in SSDT_n(\lambda)} x^T$.

For $\lambda \in \mathscr{P}^+_n$, let $H^\lambda_n$ be the element in $SSDT_n(\lambda)$ where the subtableau with entry $\ell(\lambda) - i + 1$ is a connected border strip of size $\lambda_{\ell(\lambda)-i+1}$ starting at $(i, i) \in D^+_\lambda$ for each $i = 1, \ldots, \ell(\lambda)$, and let $L^\lambda_n$ be the one where the subtableau with entry $n - i + 1$ is a connected horizontal strip of size $\lambda_i$ starting at $(i, i) \in D^+_\lambda$ for each $i = 1, \ldots, \ell(\lambda)$. Indeed, $H^\lambda_n$ and $L^\lambda_n$ are the unique tableaux in $SSDT_n(\lambda)$ such that $(c_1(H^\lambda_n), \ldots, c_n(H^\lambda_n)) = \lambda$ and $(c_1(L^\lambda_n), \ldots, c_n(L^\lambda_n)) = w_0\lambda$. Here we assume that $\mathscr{P}^+_n \subset \mathbb{Z}^n_+$ and the symmetric group $\mathfrak{S}_n$ acts on $\mathbb{Z}^n_+$ by permutation, where $w_0$ is the longest element in $\mathfrak{S}_n$.

## 2.3 Crystals

Let us briefly review the crystals for the general linear Lie algebra $\mathfrak{gl}(n)$ in [8]. Let $P^\vee = \bigoplus_{i=1}^n \mathbb{Z}e_i$ be the dual weight lattice and $P = \mathrm{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z}) = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ the weight lattice with $\langle \epsilon_i, e_j \rangle = \delta_{ij}$ for $1 \le i,j \le n$. Define a symmetric bilinear form $(\,\cdot\,|\,\cdot\,)$ on $P$ by $(\epsilon_i|\epsilon_j) = \delta_{ij}$ for $1 \le i,j \le n$. Let $\{\, \alpha_i = \epsilon_i - \epsilon_{i+1}\ (i = 1, \ldots, n-1) \,\}$ be the set of simple roots, and $\{\, h_i = e_i - e_{i+1}\ (i = 1, \ldots, n-1) \,\}$ the set of simple coroots of $\mathfrak{gl}(n)$. Let $P^+ = \{\, \lambda \,|\, \lambda \in P,\ \langle \lambda, h_i \rangle \ge 0\ (i = 1, \ldots, n-1) \,\}$ be the set of dominant integral weights.

A $\mathfrak{gl}(n)$-*crystal* is a set $B$ together with the maps $\mathrm{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\widetilde{e}_i, \widetilde{f}_i : B \to B \cup \{\mathbf{0}\}$ for $i = 1, \ldots, n-1$ satisfying the following conditions: for $b \in B$,

(a) $\varphi_i(b) = \langle \mathrm{wt}(b), h_i \rangle + \varepsilon_i(b)$,
(b) $\varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1$, $\mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\widetilde{e}_i b \in B$,
(c) $\varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1$, $\mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\widetilde{f}_i b \in B$,
(d) $\widetilde{f}_i b = b'$ if and only if $b = \widetilde{e}_i b'$ for $b' \in B$,
(e) $\widetilde{e}_i b = \widetilde{f}_i b = \mathbf{0}$ when $\varphi_i(b) = -\infty$.

Here $\mathbf{0}$ is a formal symbol and $-\infty$ is the smallest element in $\mathbb{Z} \cup \{-\infty\}$ such that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$. For $\mu \in P$, let $B_\mu = \{\, b \in B \,|\, \mathrm{wt}(b) = \mu \,\}$. When $B_\mu$ is finite for all $\mu$, we define the character of $B$ by $\mathrm{ch} B = \sum_{\mu \in P} |B_\mu| e^\mu$, where $e^\mu$ is a basis element of the group algebra $\mathbb{Q}[P]$.

Let $B_1$ and $B_2$ be $\mathfrak{gl}(n)$-crystals. A tensor product $B_1 \otimes B_2$ is a $\mathfrak{gl}(n)$-crystal, which is defined to be $B_1 \times B_2$ as a set with elements denoted by $b_1 \otimes b_2$, where

(a) $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$,
(b) $\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), h_i \rangle\}$,
(c) $\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1) + \langle \mathrm{wt}(b_2), h_i \rangle, \varphi_i(b_2)\}$,
(d) $\widetilde{e}_i(b_1 \otimes b_2) = \widetilde{e}_i b_1 \otimes b_2$, if $\varphi_i(b_1) \ge \varepsilon_i(b_2)$, and $b_1 \otimes \widetilde{e}_i b_2$ otherwise,
(e) $\widetilde{f}_i(b_1 \otimes b_2) = \widetilde{f}_i b_1 \otimes b_2$, if $\varphi_i(b_1) > \varepsilon_i(b_2)$, and $b_1 \otimes \widetilde{f}_i b_2$ otherwise,

for $i = 1, \ldots, n-1$. Here we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$.

For $\lambda \in \mathscr{P}_n$, let $B_n(\lambda)$ be the crystal associated to an irreducible $\mathfrak{gl}(n)$-module with highest weight $\lambda$, where we regard $\lambda$ as $\sum_{i=1}^n \lambda_i \epsilon_i \in P^+$. We may regard $[n]$ as $B_n(\epsilon_1)$, where $\mathrm{wt}(k) = \epsilon_k$ for $k \in [n]$, and hence $\mathcal{W}_{[n]}$ as a $\mathfrak{gl}(n)$-crystal where we identify $w = w_1 \ldots w_r$ with $w_1 \otimes \cdots \otimes w_r \in B_n(\epsilon_1)^{\otimes r}$. The crystal structure on $\mathcal{W}_{[n]}$ is easily described by the so-called the signature rule (cf. [8, Section 2.1]). For $\lambda \in \mathscr{P}_n$, the set $SST_{[n]}(\lambda)$ becomes a $\mathfrak{gl}(n)$-crystal under the identification of $T$ with $w(T) \in \mathcal{W}_{[n]}$, and it is isomorphic to $B_n(\lambda)$ [8]. In general, one can define a $\mathfrak{gl}(n)$-crystal structure on $SST_{[n]}(\lambda/\mu)$ for a skew diagram $\lambda/\mu$. By abuse of notation, we set $B_n(\lambda/\mu) := SST_{[n]}(\lambda/\mu)$.

Next, let us review the notion of crystals associated to polynomial representations of the queer Lie superalgebra $\mathfrak{q}(n)$ developed in [7, 6].

**Definition 2.3.** *A* $\mathfrak{q}(n)$-crystal *is a set $B$ together with the maps* $\mathrm{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ *and* $\widetilde{e}_i, \widetilde{f}_i : B \to B \cup \{\mathbf{0}\}$ *for* $i \in I := \{\, 1, \ldots, n-1, \overline{1} \,\}$ *satisfying the following conditions:*

*(a)* $B$ *is a* $\mathfrak{gl}(n)$*-crystal with respect to* $\mathrm{wt}$, $\varepsilon_i$, $\varphi_i$, $\widetilde{e}_i$, $\widetilde{f}_i$ *for* $i = 1, \ldots, n-1$,
*(b)* $\mathrm{wt}(b) \in \bigoplus_{i \in [n]} \mathbb{Z}_+ \epsilon_i$ *for* $b \in B$,

(c) $\mathrm{wt}(\tilde{e}_{\overline{1}}b) = \mathrm{wt}(b) + \alpha_1$, $\mathrm{wt}(\tilde{f}_{\overline{1}}b) = \mathrm{wt}(b) - \alpha_1$ *for* $b \in B$,
(d) $\tilde{f}_{\overline{1}}b = b'$ *if and only if* $b = \tilde{e}_{\overline{1}}b'$ *for all* $b, b' \in B$,
(e) *for* $3 \leq i \leq n-1$, *we have*
  (i) *the operators* $\tilde{e}_{\overline{1}}$ *and* $\tilde{f}_{\overline{1}}$ *commute with* $\tilde{e}_i$, $\tilde{f}_i$,
  (ii) *if* $\tilde{e}_{\overline{1}}b \in B$, *then* $\varepsilon_i(\tilde{e}_{\overline{1}}b) = \varepsilon_i(b)$ *and* $\varphi_i(\tilde{e}_{\overline{1}}b) = \varphi_i(b)$.

Let $\mathbf{B}_n$ be a $\mathfrak{q}(n)$-crystal which is the $\mathfrak{gl}(n)$-crystal $B_n(\epsilon_1)$ together with $\tilde{f}_{\overline{1}}\,\boxed{1} = \boxed{2}$ (in dashed arrow): $\boxed{1} \overset{1}{\underset{\overline{1}}{\rightrightarrows}} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$. Here we write $b \xrightarrow{i} b'$ if $\tilde{f}_i b = b'$ for $b, b' \in B$ and $i \in I \setminus \{\overline{1}\}$ as usual, and $b \overset{}{\underset{\overline{1}}{\dashrightarrow}} b'$ if $\tilde{f}_{\overline{1}}b = b'$.

For $\mathfrak{q}(n)$-crystals $B_1$ and $B_2$, the tensor product $B_1 \otimes B_2$ is the $\mathfrak{gl}(n)$-crystal $B_1 \otimes B_2$ where the actions of $\tilde{e}_{\overline{1}}$ and $\tilde{f}_{\overline{1}}$ are given by

$$\begin{aligned} &\tilde{e}_{\overline{1}}(b_1 \otimes b_2) = \tilde{e}_{\overline{1}}b_1 \otimes b_2 \text{ if } \langle \epsilon_1, \mathrm{wt}(b_2)\rangle = \langle \epsilon_2, \mathrm{wt}(b_2)\rangle = 0, \text{ and } b_1 \otimes \tilde{e}_{\overline{1}}b_2 \text{ otherwise,} \\ &\tilde{f}_{\overline{1}}(b_1 \otimes b_2) = \tilde{f}_{\overline{1}}b_1 \otimes b_2 \text{ if } \langle \epsilon_1, \mathrm{wt}(b_2)\rangle = \langle \epsilon_2, \mathrm{wt}(b_2)\rangle = 0, \text{ and } b_1 \otimes \tilde{f}_{\overline{1}}b_2 \text{ otherwise.} \end{aligned} \tag{2.1}$$

Then it is easy to see that $B_1 \otimes B_2$ is a $\mathfrak{q}(n)$-crystal, and $\mathcal{W}_{[n]}$ is also a $\mathfrak{q}(n)$-crystal.

Let $B$ be a $\mathfrak{q}(n)$-crystal. Suppose that $B$ is a regular $\mathfrak{gl}(n)$-crystal, that is, each connected component in $B$ is isomorphic to $B_n(\lambda)$ for some $\lambda \in \mathscr{P}_n$. Let $W = \mathfrak{S}_n$ be the Weyl group of $\mathfrak{gl}(n)$ which is generated by the simple reflection $r_i$ corresponding to $\alpha_i$ for $i = 1, \ldots, n-1$. We have a group action of $W$ on $B$ denoted by $S$ such that $S_{r_i}(b) = \tilde{f}_i^{\langle \mathrm{wt}(b), h_i\rangle}b$ if $\langle \mathrm{wt}(b), h_i\rangle \geq 0$, and $\tilde{e}_i^{-\langle \mathrm{wt}(b), h_i\rangle}b$ otherwise, for $b \in B$ and $i = 1, \ldots, n-1$. For $2 \leq i \leq n-1$, let $w_i \in W$ be such that $w_i(\alpha_i) = \alpha_1$, and let $\tilde{e}_{\overline{i}} = S_{w_i^{-1}}\tilde{e}_{\overline{1}}S_{w_i}$ and $\tilde{f}_{\overline{i}} = S_{w_i^{-1}}\tilde{f}_{\overline{1}}S_{w_i}$. For $b \in B$, we say that $b$ is a $\mathfrak{q}(n)$-*highest weight vector* if $\tilde{e}_i b = \tilde{e}_{\overline{i}} b = \mathbf{0}$ for $1 \leq i \leq n-1$, and $b$ is a $\mathfrak{q}(n)$-*lowest weight vector* if $S_{w_0}b$ is a $\mathfrak{q}(n)$-highest weight vector.

For $\lambda \in \mathscr{P}^+$, let $\mathbf{B}_n(\lambda) = SSDT_n(\lambda)$, and consider an injective map

$$\mathbf{B}_n(\lambda) \hookrightarrow \mathcal{W}_{[n]}, \qquad T \longmapsto w_{\mathrm{rev}}(T). \tag{2.2}$$

**Theorem 2.4** ([7, Theorem 2.5]). *Let* $\lambda \in \mathscr{P}_n^+$ *be given.*

(a) *The image of* $\mathbf{B}_n(\lambda)$ *in* (2.2) *together with* $\{\mathbf{0}\}$ *is invariant under the action of* $\tilde{e}_i$ *and* $\tilde{f}_i$ *for* $i \in I$, *and hence* $\mathbf{B}_n(\lambda)$ *is a* $\mathfrak{q}(n)$*-crystal.*

(b) *The* $\mathfrak{q}(n)$*-crystal* $\mathbf{B}_n(\lambda)$ *is connected where* $H_n^\lambda$ *is a unique* $\mathfrak{q}(n)$*-highest weight vector and* $L_n^\lambda$ *is a unique* $\mathfrak{q}(n)$*-lowest weight vector.*

Let $B_1$ and $B_2$ be $\mathfrak{q}(n)$-crystals. For $b_1 \in B_1$ and $b_2 \in B_2$, let us say that $b_1$ and $b_2$ are equivalent and write $b_1 \equiv b_2$ if there exists an isomorphism of $\mathfrak{q}(n)$-crystals $\psi : C(b_1) \longrightarrow C(b_2)$ such that $\psi(b_1) = b_2$ where $C(b_i)$ denotes the connected component of $b_i \in B_i$ $(i = 1, 2)$ as a $\mathfrak{q}(n)$-crystal.

By [6, Theorem 4.6], each connected component in $\mathbf{B}_n^{\otimes N}$ ($N \geq 1$) is isomorphic to $\mathbf{B}_n(\lambda)$ for some $\lambda \in \mathscr{P}_n^+$ with $|\lambda| = N$. Indeed, for $b = b_1 \otimes \cdots \otimes b_N \in \mathbf{B}_n^{\otimes N}$, there exists a unique $\lambda \in \mathscr{P}_n^+$ and $T \in \mathbf{B}_n(\lambda)$ such that $b \equiv T$. In particular, $b$ is a $\mathfrak{q}(n)$-lowest (resp. $\mathfrak{q}(n)$-highest) weight vector if and only if $b \equiv L_n^\lambda$ (resp. $H_n^\lambda$). The following lemma plays a crucial role in characterization of $\mathfrak{q}(n)$-lowest weight vectors in $\mathbf{B}_n^{\otimes N}$ and hence describing the decompositions of $\mathbf{B}_n^{\otimes N}$ and $\mathbf{B}_n(\mu) \otimes \mathbf{B}_n(\nu)$ ($\mu, \nu \in \mathscr{P}_n^+$) into connected components in [7].

**Lemma 2.5** ([7, Lemma 1.15, Corollary 1.16])**.** *For $b = b_1 \otimes \cdots \otimes b_N \in \mathbf{B}_n^{\otimes N}$, the following are equivalent:*

(a) *$b$ is a $\mathfrak{q}(n)$-lowest weight vector,*

(b) *$b' = b_2 \otimes \cdots \otimes b_N$ is a $\mathfrak{q}(n)$-lowest weight vector and $\epsilon_{b_1} + \mathrm{wt}(b') \in w_0\mathscr{P}_n^+$,*

(c) *$\mathrm{wt}(b_M \otimes \cdots \otimes b_N) \in w_0\mathscr{P}_n^+$ for all $1 \leq M \leq N$.*

Hence, we have the following immediately by Lemma 2.5.

**Corollary 2.6.** *For $\lambda^{(1)}, \ldots, \lambda^{(s)} \in \mathscr{P}_n^+$ and $T_1 \otimes \cdots \otimes T_s \in \mathbf{B}_n(\lambda^{(1)}) \otimes \cdots \otimes \mathbf{B}_n(\lambda^{(s)})$, the following are equivalent:*

(a) *$T_1 \otimes \cdots \otimes T_s$ is a $\mathfrak{q}(n)$-lowest weight vector,*

(b) *$T_r \otimes \cdots \otimes T_s \in \mathbf{B}_n(\lambda^{(s)}) \otimes \cdots \otimes \mathbf{B}_n(\lambda^{(r)})$ is a $\mathfrak{q}(n)$-lowest weight vector for $1 \leq r \leq s$.*

**Remark 2.7.** *Let $m \geq n$ be a positive integer, and put $t = m - n$. For $N \geq 1$, let $\psi_t : \mathbf{B}_n^{\otimes N} \longrightarrow \mathbf{B}_m^{\otimes N}$ be the map given by $\psi_t(u_1 \otimes \cdots \otimes u_N) = (u_1 + t) \otimes \cdots \otimes (u_N + t)$. Then for $\lambda \in \mathscr{P}_n^+$ and $u \in \mathbf{B}_n^{\otimes N}$ we have $u \equiv L_n^\lambda$ if and only if $\psi_t(u) \equiv L_m^\lambda$. This implies that the multiplicity of $\mathbf{B}_n(\lambda)$ in $\mathbf{B}_n^{\otimes N}$ is equal to that of $\mathbf{B}_m(\lambda)$ in $\mathbf{B}_m^{\otimes N}$ for $\lambda \in \mathscr{P}_n^+$.*

# 3 Littlewood–Richardson rule for Schur $P$-functions

## 3.1 Shifted Littlewood–Richardson rule

Let $w = w_1 \cdots w_N$ be a word in $\mathcal{W}_N$. Let $m_k = c_k(w) + c_{k'}(w)$ for $k \geq 1$. We define $w^* = w_1^* \cdots w_N^*$ to be the word obtained from $w$ as follows: for each $k \geq 1$,

(1) consider the letters $w_i$'s with $|w_i| = k$. Label them with $1, 2, \ldots, m_k$ (as subscripts), first enumerating the $w_p$'s with $w_p = k$ from left to right, and then the $w_q$'s with $w_q = k'$ from right to left.

(2) After the step (1), remove all $\,'\,$ in each labeled letter $k'_j$, that is, replace any $k'_j$ by $k_j$ for $c_k(w) < j \leq m_k$.

**Definition 3.1.** *Let $w = w_1 \cdots w_N \in \mathcal{W}_N$ be given. We say that $w$ satisfies the* "lattice property" *if the word $w^* = w_1^* \cdots w_N^*$ associated to $w$ satisfies the following: for $k \geq 1$*

(L1) *if $w_i^* = k_1$, then no $k+1_j$ for $j \geq 1$ occurs in $w_1^* \cdots w_{i-1}^*$,*

(L2) *if $(w_s^*, w_t^*) = (k+1_i, k_{i+1})$ for $s < t$ and $i \geq 1$, then no $k+1_j$ $(i < j)$ occurs in $w_s^* \cdots w_t^*$,* (L3) *if* $(w_s^*, w_t^*) = (k_{j+1}, k+1_j)$ *for some* $s < t$ *and* $j \geq 1$, *then no* $k_i$ $(i \leq j)$ *occurs in* $w_s^* \cdots w_t^*$.

**Definition 3.2.** *For* $\lambda, \mu, \nu \in \mathscr{P}^+$, *let* $\mathtt{F}_{\mu\nu}^{\lambda}$ *be the set of tableaux* $Q$ *such that*

(a) $Q \in SST_{\mathcal{N}}^{+}(\lambda/\mu)$ *with* $c_k(Q) + c_{k'}(Q) = \nu_k$ *for* $k \geq 1$,
(b) *for* $k \geq 1$, *if* $x$ *is the rightmost letter in* $w(Q)$ *with* $|x| = k$, *then* $x = k$,
(c) $w(Q)$ *satisfies the "lattice property" in Definition 3.1.*

Then we have the following characterization of $f_{\mu\nu}^{\lambda}$.

**Theorem 3.3** ([3, Theorem 3.5])**.** *For* $\lambda, \mu, \nu \in \mathscr{P}^+$, *we have* $f_{\mu\nu}^{\lambda} = \left|\mathtt{F}_{\mu\nu}^{\lambda}\right|$.

Choose $n$ such that $\lambda, \mu, \nu \in \mathscr{P}_n^+$. Put $\mathtt{L}_{\mu\nu}^{\lambda} = \{\, T \,|\, T \in \mathbf{B}_n(\nu),\ T \otimes L_n^{\mu} \equiv L_n^{\lambda} \,\}$. By Corollary 2.6, we have $\mathbf{B}_n(\nu) \otimes \mathbf{B}_n(\mu) \cong \bigsqcup_{\lambda \in \mathscr{P}_n^+} \mathbf{B}_n(\lambda)^{\oplus |\mathtt{L}_{\mu\nu}^{\lambda}|}$. Hence we have $|\mathtt{L}_{\mu\nu}^{\lambda}| = f_{\mu\nu}^{\lambda} = f_{\nu\mu}^{\lambda}$. The key in the proof of this theorem is to construct a bijection

$$\mathtt{L}_{\mu\nu}^{\lambda} \longrightarrow \mathtt{F}_{\mu\nu}^{\lambda}, \quad T \longmapsto Q_T. \tag{3.1}$$

such that $w(Q_T)$ satisfies the "lattice property". We briefly explain this construction now. Let $T \in \mathtt{L}_{\mu\nu}^{\lambda}$ be given. Assume that $w_{\rm rev}(T) = u_1 \cdots u_N$ where $N = |\nu|$. By Lemma 2.5, there exists $\mu^{(m)} \in \mathscr{P}_n^+$ for $1 \leq m \leq N$ such that (i) $(u_{N-m+1} \cdots u_N) \otimes L_n^{\mu} \equiv L_n^{\mu^{(m)}}$ and $\mu^{(N)} = \lambda$, and (ii) $\mu^{(m)}$ is obtained by adding a box in the $(n - u_m + 1)$-st row of $\mu^{(m-1)}$. Here we assume that $\mu^{(0)} = \mu$. Recall that $w_{\rm rev}(T) = T^{(\ell(\nu))} \cdots T^{(1)}$, where $T^{(k)} = T_{k,1} \cdots T_{k,\lambda_k}$ is a hook word for $1 \leq k \leq \ell(\nu)$. Define $Q_T$ to be a tableau of shifted shape $\lambda/\mu$ with entries in $\mathcal{N}$, where $\mu^{(m)}/\mu^{(m-1)}$ is filled with

$$\begin{cases} k', & \text{if } u_m \text{ belongs to } T^{(k)}\!\uparrow, \\ k, & \text{if } u_m \text{ belongs to } T^{(k)}\!\downarrow, \end{cases} \tag{3.2}$$

for some $1 \leq k \leq \ell(\nu)$. In other words, the boxes in $Q_T$ corresponding to $T^{(k)}\!\uparrow$ are filled with $k'$ from right to left as a vertical strip and then those corresponding to $T^{(k)}\!\downarrow$ are filled with $k$ from left to right as a horizontal strip. (cf. the proof of [3, Theorem 3.5]).

**Remark 3.4.** *For* $T \in \mathtt{L}_{\mu\nu}^{\lambda}$, *let* $\widehat{Q}_T$ *be the tableau of shifted shape* $\lambda/\mu$, *which is defined in the same way as* $Q_T$ *in the proof of Theorem 3.3 except that we fill* $\mu^{(m)}/\mu^{(m-1)}$ *with* $m$ *in* (3.2) *for* $1 \leq m \leq N$. *Then the set* $\{\, \widehat{Q}_T \,|\, T \in \mathtt{L}_{\mu\nu}^{\lambda} \,\}$ *is equal to the one given in [7, Theorem 4.13] to describe* $f_{\mu\nu}^{\lambda}$. *For example,*

$$T_1 = \begin{array}{ccc} 3 & 3 & 4 \\ & 2 & \end{array} \quad T_2 = \begin{array}{ccc} 4 & 2 & 3 \\ & 3 & \end{array} \quad \widehat{Q}_{T_1} = \begin{array}{cccc} \cdot & \cdot & \cdot & 1 \\ & \cdot & 2 & 3 \\ & & 4 & \end{array} \quad \widehat{Q}_{T_2} = \begin{array}{cccc} \cdot & \cdot & \cdot & 3 \\ & \cdot & 1 & 4 \\ & & 2 & \end{array} \quad Q_{T_1} = \begin{array}{cccc} \cdot & \cdot & \cdot & 1' \\ & \cdot & 1 & 1 \\ & & 2 & \end{array} \quad Q_{T_2} = \begin{array}{cccc} \cdot & \cdot & \cdot & 1 \\ & \cdot & 1' & 2 \\ & & 1 & \end{array}$$

### 3.2 Stembridge's description of $f^{\lambda}_{\mu\nu}$

**Definition 3.5.** *Let $w = w_1 \cdots w_N$ be a word in $\mathcal{W}_{\mathcal{N}}$ and $w_{\rm rev}$ be the reverse word of $w$. Let $\widehat{w}$ be the word obtained from $w$ by replacing $k$ by $(k+1)'$ and $k'$ by $k$ for each $k \geq 1$. Suppose that $w\widehat{w}_{\rm rev} = a_1 \cdots a_{2N}$, and let $m_k(i) = c_k(a_1 \cdots a_i)$ for $k \geq 1$ and $0 \leq i \leq 2N$. Then we say that $w$ satisfies the lattice property if*

$$m_{k+1}(i) = m_k(i) \text{ implies } |a_{i+1}| \neq k+1 \text{ for } k \geq 1 \text{ and } i \geq 0. \tag{3.3}$$

*Here we assume that $m_k(0) = 0$.*

**Definition 3.6.** *For $\lambda, \mu, \nu \in \mathscr{P}^+$, let $\mathtt{LRS}^{\lambda}_{\mu\nu}$ be the set of tableaux $Q$ such that*

(a) *$Q \in SST^+_{\mathcal{N}}(\lambda/\mu)$ with $c_k(Q) + c_{k'}(Q) = \nu_k$ for $k \geq 1$,*
(b) *for $k \geq 1$, if $x$ is the rightmost letter in $w(Q)$ with $|x| = k$, then $x = k$,*
(c) *$w(Q)$ satisfies the lattice property in Definition 3.5.*

We call $\mathtt{LRS}^{\lambda}_{\mu\nu}$ the set of *Littlewood–Richardson–Stembridge tableaux*. Due to Stembridge it provided that for $\lambda, \mu, \nu \in \mathscr{P}^+$ the shifted LR coefficient $f^{\lambda}_{\mu\nu}$ is equal to the number of tableaux in $\mathtt{LRS}^{\lambda}_{\mu\nu}$ (cf. ([12, Theorem 8.3]). So we have

**Theorem 3.7** ([3, Theorem 3.11])**.** *For $\lambda, \mu, \nu \in \mathscr{P}^+$, we have $\mathtt{F}^{\lambda}_{\mu\nu} = \mathtt{LRS}^{\lambda}_{\mu\nu}$.*

**Corollary 3.8.** *Let $w \in \mathcal{W}_{\mathcal{N}}$ be such that $(c_k(Q) + c_{k'}(Q))_{k\geq 1} \in \mathscr{P}^+$, and for $k \geq 1$, if $x$ is the rightmost letter in $w$ with $|x| = k$, then $x = k$. Then $w$ satisfies the "lattice property" in Definition 3.1 if and only if $w$ satisfies the lattice property in Definition 3.5.*

**Remark 3.9.** *A bijection from $\mathtt{LRS}^{\lambda}_{\mu\nu}$ to $\mathtt{L}^{\lambda}_{\mu\nu}$ is also given in [4, Theorem 4.7], which coincides with the inverse of the map $T \mapsto Q_T$ in (3.1) (see also the remarks in [4, p.82]). The proof of [4, Theorem 4.7] use insertion schemes for two versions of semistandard decomposition tableaux and another combinatorial model for $f^{\lambda}_{\mu\nu}$ by Cho [2] as an intermediate object between $\mathtt{LRS}^{\lambda}_{\mu\nu}$ and $\mathtt{L}^{\lambda}_{\mu\nu}$.*

*On the other hand, we prove more directly that the map $T \mapsto Q_T$ in (3.1) is a bijection from $\mathtt{L}^{\lambda}_{\mu\nu}$ to $\mathtt{LRS}^{\lambda}_{\mu\nu}$ by using a new characterization of the lattice property in Theorem 3.7.*

## 4 Schur $P$-expansions of skew Schur functions

For $r \geq 0$, let us denote by $\delta_r$ the partition $(r, r-1, \ldots, 1)$ if $r \geq 1$, and $(0)$ if $r = 0$. We fix a nonnegative integer $r$. Let $\lambda \in \mathscr{P}$ be such that $D_{\delta_r} \subseteq D_\lambda \subseteq D_{((r+1)^{r+1})}$. Here $((r+1)^{r+1})$ means the rectangular partition $(r+1, \ldots, r+1)$ with length $r+1$. It is shown in [1, 5] that the skew Schur function $s_{\lambda/\delta_k}$ has a nonnegative integral expansion in terms of Schur $P$-functions

$$s_{\lambda/\delta_r} = \sum_{\nu \in \mathscr{P}^+} a_{\lambda/\delta_r\, \nu} P_\nu,$$

together with a combinatorial description of $a_{\lambda/\delta_r\,\nu}$. Moreover it is shown that these skew Schur functions are the only ones (up to rotation of shape by 180°), which have Schur $P$-positivity. In this section, we give a new simple description of $a_{\lambda/\delta_r\,\nu}$ using $\mathfrak{q}(n)$-crystals.

**Proposition 4.1** ([3, Proposition 4.1]). *Let $\lambda \in \mathscr{P}_n$ be such that $D_{\delta_r} \subseteq D_\lambda \subseteq D_{(r+1)^{r+1}}$. Then the $\mathfrak{gl}(n)$-crystal $B_n(\lambda/\delta_r)$, regarded as a subset of $\mathcal{W}_{[n]}$ together with $\mathbf{0}$ is invariant under $\widetilde{e}_{\overline{1}}$ and $\widetilde{f}_{\overline{1}}$. Hence $B_n(\lambda/\delta_r)$ is a $\mathfrak{q}(n)$-crystal.*

Since $B_n(\lambda/\delta_r)$ is a $\mathfrak{q}(n)$-crystal, the skew Schur polynomial $s_{\lambda/\delta_r}(x_1,\ldots,x_n)$ is a nonnegative integral linear combination of $P_\nu(x_1,\ldots,x_n)$. By applying Remark 2.7, we have

**Corollary 4.2.** *Under the above hypothesis, the skew Schur function $s_{\lambda/\delta_r}$ is Schur $P$-positive.*

**Definition 4.3.** *Let $\lambda \in \mathscr{P}$ be such that $D_{\delta_r} \subseteq D_\lambda \subseteq D_{((r+1)^{r+1})}$ and $\nu \in \mathscr{P}^+$. Let $\mathtt{A}_{\lambda/\delta_r\,\nu}$ be the set of tableaux $Q$ such that*

*(a) $Q \in SST^+_{[r+1]}(\nu)$ with $c_k(Q) = \lambda_{r-k+2} - k + 1$ for $1 \leq k \leq r+1$,*

*(b) $m_k(i) \leq m_{k+1}(i) + 1$ for $1 \leq k \leq r$ and $1 \leq i \leq N$, where $w_{\mathrm{rev}}(Q) = w_1 \cdots w_N$ and $m_k(i) = c_k(w_1 \cdots w_i)$.*

Then we have the following combinatorial description of $a_{\lambda/\delta_r\,\nu}$.

**Theorem 4.4** ([3, Theorem 4.4]). *For $\lambda \in \mathscr{P}$ with $D_{\delta_r} \subseteq D_\lambda \subseteq D_{((r+1)^{r+1})}$ and $\nu \in \mathscr{P}^+$, we have $a_{\lambda/\delta_r\,\nu} = |\mathtt{A}_{\lambda/\delta_r\,\nu}|$.*

**Example 4.5.** *Let $\lambda = (5,5,4,3,1)$ with $D_\lambda \subseteq D_{(5^5)}$ and $n = 7$. For $\nu = (4,3,1)$, we have $\mathtt{L}_{\lambda/\delta_4\,\nu} = \{ T_1, T_2 \}$ and $\mathtt{A}_{\lambda/\delta_4\,\nu} = \{ Q_{T_1}, Q_{T_2} \}$ as follows.*

$$T_1 = \begin{array}{ccccc} & & & & 6 \\ & & & 5 & 7 \\ & & 6 & 6 & \\ & 7 & 7 & & \\ 7 & & & & \end{array} \qquad T_2 = \begin{array}{ccccc} & & & & 5 \\ & & & 6 & 6 \\ & & 6 & 7 & \\ & 7 & 7 & & \\ 7 & & & & \end{array} \qquad Q_{T_1} = \begin{array}{cccc} 1 & 2 & 2 & 4 \\ & 3 & 3 & 5 \\ & & 4 & \end{array} \qquad Q_{T_2} = \begin{array}{cccc} 1 & 2 & 2 & 3 \\ & 3 & 4 & 4 \\ & & 5 & \end{array}$$

*Moreover, we have $s_{(5,5,4,3,1)/\delta_4} = 2P_{(4,3,1)} + P_{(5,2,1)} + P_{(5,3)}$.*

**Remark 4.6.** *Ardila–Serrano [1] already gave a result on the Schur $P$-expansion of the staircase skew Schur function $s_{\delta_{r+1}/\mu}$ (see [1, Theorem 4.10]). It is well known that $s_{\lambda/\delta_r} = s_{(\lambda/\delta_r)^\pi}$, where $\pi$ means the 180° rotation operation on skew diagrams. It implies that the coefficient of $P_\nu$ of $s_{\lambda/\delta_r}$ is equal to that of $s_{\delta_{r+1}/\mu}$. Via the map in [3, Corollary 4.8] we provided a bijection between them.*

## 5 Schur expansion of Schur $P$-function

For $\lambda \in \mathscr{P}^+$ and $\mu \in \mathscr{P}$, let $g_{\lambda\mu}$ be the coefficient of $s_\mu$ in the Schur expansion of $P_\lambda$, that is,

$$P_\lambda = \sum_\mu g_{\lambda\mu} s_\mu.$$

The purpose of this section is to give an alternate proof of the following combinatorial description of $g_{\lambda\mu}$ due to Stembridge.

**Definition 5.1.** *For $\lambda \in \mathscr{P}^+$ and $\mu \in \mathscr{P}$, let $\mathtt{G}_{\lambda\mu}$ be the set of tableaux $Q$ such that*
(a) *$Q \in SST_{\mathcal{N}}(\mu)$ with $c_k(Q) + c_{k'}(Q) = \lambda_k$ for $k \geq 1$,*
(b) *for $k \geq 1$, if $x$ is the rightmost letter in $w(Q)$ with $|x| = k$, then $x = k$,*
(c) *$w(Q)$ satisfies the lattice property.*

**Theorem 5.2** ([12, Theorem 9.3])**.** *For $\lambda \in \mathscr{P}^+$ and $\mu \in \mathscr{P}$, we have $g_{\lambda\mu} = |\mathtt{G}_{\lambda\mu}|$.*

Choose $n$ such that $\lambda \in \mathscr{P}_n^+$ and $\mu \in \mathscr{P}_n$. Let $\mathbf{L}_{\lambda\mu} = \{\, T \mid T \in \mathbf{B}_n(\lambda),\ \widetilde{f}_i T = \mathbf{0}\ (1 \leq i \leq n-1),\ \mathrm{wt}(T) = w_0\mu \,\}$. Then we have as a $\mathfrak{gl}(n)$-crystal $\mathbf{B}_n(\lambda) \cong \bigsqcup_\mu B_n(\mu)^{\oplus |\mathbf{L}_{\lambda\mu}|}$, and hence $g_{\lambda\mu} = |\mathbf{L}_{\lambda\mu}|$ by linear independence of Schur polynomials. The proof is similar to that of Theorem 3.3. That is, the key is to construct a bijection between $\mathbf{L}_{\lambda\mu}$ and $\mathtt{G}_{\lambda\mu}$. For $T \in \mathbf{L}_{\lambda\mu}$, let $w_{\mathrm{rev}}(T) = u_1 \cdots u_N$ ($N = |\lambda|$). Since $T$ is a $\mathfrak{gl}(n)$-lowest weight vector, we have by the tensor product of crystals that $u_{N-m+1} \otimes \cdots \otimes u_N \in \mathbf{B}_n^{\otimes m}$ is a $\mathfrak{gl}(n)$-lowest weight element for $1 \leq m \leq N$. This implies that there exists $\mu^{(m)} \in \mathscr{P}_n$ for $1 \leq m \leq N$ such that $u_{N-m+1} \cdots u_N$ is equivalent as an element of $\mathfrak{gl}(n)$-crystal to a $\mathfrak{gl}(n)$-lowest weight element in $B_n(\mu^{(m)})$, where $\mu^{(N)} = \mu$ and $\mu^{(m)}$ is obtained by adding a box in the $(n - u_m + 1)$-st row of $\mu^{(m-1)}$ with $\mu^{(0)} = \emptyset$. Define $Q_T$ to be a tableau of shape $\mu$ whose entry at $\mu^{(m)}/\mu^{(m-1)}$ is

$$\begin{cases} k', & \text{if } u_m \text{ belongs to } T^{(k)}\uparrow, \\ k, & \text{if } u_m \text{ belongs to } T^{(k)}\downarrow, \end{cases}$$

for some $1 \leq k \leq \ell(\lambda)$. Then we obtain the desired mapping $T \mapsto Q_T$.

**Example 5.3.** *Let $\lambda = (3,1)$ and $n = 3$. The mapping $T \mapsto Q_T$ gives*

$$T_1 = \begin{array}{|c|c|c|}\hline 3&3&3\\\hline \end{array}\!\!\begin{array}{l}\\ \end{array}\ \text{with } \begin{array}{c|c|}\hline & 2\\\hline\end{array} \quad$$

$$T_1 = \begin{array}{ccc} \boxed{3} & \boxed{3} & \boxed{3} \\ & \boxed{2} & \end{array} \quad T_2 = \begin{array}{ccc} \boxed{3} & \boxed{2} & \boxed{3} \\ & \boxed{2} & \end{array} \quad T_3 = \begin{array}{ccc} \boxed{3} & \boxed{2} & \boxed{3} \\ & \boxed{1} & \end{array} \qquad Q_{T_1} = \begin{array}{ccc} \boxed{1} & \boxed{1} & \boxed{1} \\ \boxed{2} & & \end{array} \quad Q_{T_2} = \begin{array}{cc} \boxed{1'} & \boxed{1} \\ \boxed{1} & \boxed{2} \end{array} \quad Q_{T_3} = \begin{array}{cc} \boxed{1'} & \boxed{1} \\ \boxed{1} & \\ \boxed{2} & \end{array}.$$

*Thus $P_{(3,1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}$.*

**Remark 5.4.** *Let $\lambda \in \mathscr{P}^+$ be such that $D_\lambda^+ \subseteq D_{\delta_{r+1}}^+$ for some $r \geq 0$. Let $\lambda^{\mathsf{c}+}$ be the complement of $\lambda$ in $D_{\delta_{r+1}}^+$. It is shown in [5] that $s_{\delta_{r+1}/\lambda} = \sum g_{\nu\lambda} P_{\nu^{\mathsf{c}+}}$, where the sum runs over all $\nu \in \mathscr{P}^+$ with $|\nu| = |\lambda|$. So we have $g_{\nu\lambda} = a_{\lambda^{\mathsf{c}}/\delta_r\,(\nu^{\mathsf{c}+})'}$, where $\lambda^{\mathsf{c}}$ is the complement of $\lambda$ in $D_{((r+1)^{r+1})}$. One may expect that there is a natural bijection between $\mathtt{G}_{\nu\lambda}$ and $\mathtt{A}_{\lambda^{\mathsf{c}}/\delta_r\,(\nu^{\mathsf{c}+})'}$ that we have not yet make explicit.*

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