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Topology of Posets with Special Partial Matchings

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Abstract. Special partial matchings (SPMs) are a generalisation of Brenti's special matchings. Let a *pircon* be a poset in which every non-trivial principal order ideal is finite and admits an SPM. Thus pircons generalise Marietti's zircons. We prove that every open interval in a pircon is a PL ball or a PL sphere. It is then demonstrated that Bruhat orders on certain twisted identities and quasiparabolic W-sets constitute pircons. Together, these results extend a result of Can, Cherniavsky, and Twelbeck, prove a conjecture of Hultman, and confirm a claim of Rains and Vazirani.

Résumé. Les couplages partiels distingués (SPMs) forment une généralisation des couplages distingués définis par Brenti. Un pircon est un EPO (ensemble partiellement ordonné) dans lequel tout idéal ordonné principal et non-trivial est fini et admet un SPM. Ainsi les pircons généralisent les zircons définis par Marietti. On montre que tout intervalle ouvert d'un pircon est une PL-boule ou une PL-sphère (où PL signifie linéaire par morceaux). Ensuite on montre que l'ordre de Bruhat sur certaines identités tordues et W-ensembles quasiparaboliques forment des pircons. Ces résultats généralisent un résultat de Can, Cherniavsky et Twelbeck, montre une conjecture de Hultman et confirme une réclamation de Rains et Vazirani.

Keywords: Special partial matching, PL topology, pircons, twisted identities, quasiparabolic sets

1 Introduction

A special matching on a poset is a complete matching of the Hasse diagram satisfying certain extra conditions. The concept was introduced by Brenti [6]. For eulerian posets, an equivalent notion was also independently introduced by du Cloux [9]. Their main motivation was to provide an abstract framework in which to study the Bruhat order on a Coxeter group. Namely, every non-trivial lower interval in the Bruhat order admits a special matching. Thus, Bruhat orders provide examples of *zircons*, posets in which every non-trivial principal order ideal is finite and has a special matching. Beginning with Marietti [18], zircons have been the focal point of a lot of attention; see, e.g., [8, 11, 17]. Notably, (the order complex of) any open interval in a zircon is a PL sphere; this is

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essentially a result of du Cloux [9, Corollary 3.6], which is based on results from Dyer's thesis [10]. Reading [20] provided a different proof.[1]

In [2], two of the present authors generalised the special matching concept to special partial matchings (SPMs), which are not necessarily complete matchings satisfying similar conditions. Generalising zircons, let us say that a *pircon* is a poset in which every non-trivial principal order ideal is finite and admits an SPM. These notions, too, are originally motivated by Coxeter group theory: the dual of the Bruhat order on the fixed point free involutions in the symmetric group is a pircon [2]. This is generalised considerably in Section 4, where it is demonstrated that the Bruhat order on the twisted identities $\iota(\theta)$ is a pircon whenever the involution θ has the so-called NOF property. Moreover, Bruhat orders on Rains and Vazirani's [19] quasiparabolic W-sets (under a boundedness assumption) form pircons. In particular, this applies to all parabolic quotients of Coxeter groups.

We investigate the topology of posets with SPMs. Our first main result roughly states that an SPM provides a way to "lift" the PL ball or sphere property from a subinterval; this is Theorem 3.4. It follows that every open interval in a pircon is a PL ball or a PL sphere, which is our second main result. In particular, this proves a conjecture from [12] on Bruhat orders on twisted identities, and confirms a claim from [19] about quasiparabolic W-sets.

The overall proof strategy is inspired by that of Reading's aforementioned proof in [20]. Roughly, if P is a poset with minimum $\hat{0}$, maximum $\hat{1}$, and an SPM M, we prove that P can be obtained from the interval $[\hat{0}, M(\hat{1})]$ using modifications (including a version of Reading's "zippings") that preserve PL balls or spheres. Investigating the effect of these modifications on the poset topology forms the technical backbone of the paper.

The remainder of the extended abstract is structured in the following way. In the next section, we recall basic definitions and review useful results from the literature. We introduce pircons and study ways to locally modify posets, including Reading's zippings from [20]. These modifications turn out to preserve PL balls and spheres. Section 3 gives our main results. We explain how a poset which admits an SPM can be obtained from one which in some sense is easier to understand, using the modifications studied in the previous section. In Section 4, we explain how examples of pircons are provided by Bruhat orders, first on twisted identities and second on quasiparabolic W-sets in Coxeter groups. The implications of our second main result in these contexts are discussed. Finally, in the last section, we raise some open questions.

The details that are omitted in this extended abstract will appear in the complete work [1].

[1] Although Reading worked in the context of Bruhat orders, his proof is valid in the more general zircon setting.

2 Background

Posets Let P be a poset (partially ordered set). If P contains an element denoted $\hat{0}$ or $\hat{1}$, it is assumed to be a minimum or a maximum, respectively, i.e., $x \geq \hat{0}$ and $x \leq \hat{1}$ for all $x \in P$. The *proper part* of P is then $\overline{P} = P - \{\hat{0}, \hat{1}\}$.

Standard interval notation is employed for posets. Thus, if $x, y \in P$, then

$$[x, y] = \{z \in P \mid x \leq z \leq y\},$$

with the induced order from P, and similarly for open and half-open intervals.

An *order ideal* $J \subseteq P$ is an induced subposet closed under going down, i.e., $x \leq y \in J \Rightarrow x \in J$. An order ideal is *principal* if it has a maximum. For principal order ideals, the notation $P_{\leq y} = \{x \in P \mid x \leq y\}$ is convenient.

Suppose every principal order ideal in P is finite. If, for any $y \in P$, all maximal chains (totally ordered subsets) in $P_{\leq y}$ have the same number of elements, P is called *graded*. In this case, there is a unique *rank function*, i.e., a function $\mathrm{rk} : P \to \{0, 1, \ldots\}$ such that $\mathrm{rk}(x) = 0$ if x is minimal, and $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$ if y covers x.

Suppose $\pi : P \to P'$ is an order-preserving map of posets. Then π is called an *order projection* if for every ordered pair $x' \leq_{P'} y'$ in P' there exist $x \leq_P y$ in P such that $\pi(x) = x'$ and $\pi(y) = y'$. In particular, any order projection is surjective. We construct the quotient $\mathcal{F}_\pi$ as follows. The elements of $\mathcal{F}_\pi$ are the fibres $\pi^{-1}(x') = \{x \in P \mid \pi(x) = x'\}$ for $x' \in P'$. A relation on $\mathcal{F}_\pi$ is given by $F_1 \leq_{\mathcal{F}_\pi} F_2$ if $x \leq_P y$ for some $x \in F_1$ and $y \in F_2$. This is a partial order if π is an order projection. We then call $\mathcal{F}_\pi$ the *fibre poset*. It is isomorphic to P' (see [20, Proposition 1.1]).

Special Partial Matchings On a poset P we define a matching concept which generalises the special matchings defined by Brenti in [6].

Definition 2.1 ([2])**.** Suppose P is a finite poset with $\hat{1}$, and let $\lhd$ denote its cover relation. A *special partial matching*, or *SPM*, on P is a function $M : P \to P$ such that

- $M^2 = \mathrm{id}$,
- $M(\hat{1}) \lhd \hat{1}$,
- for all $x \in P$, $M(x) \lhd x$, $M(x) = x$, or $x \lhd M(x)$, and
- if $x \lhd y$ and $M(x) \neq y$, then $M(x) < M(y)$.

An SPM without fixed points is precisely a special matching.

A poset with an SPM is isomorphic to a certain fibre poset. We describe below the fundamental construction that gives this isomorphism. This extends Reading's corresponding construction for Bruhat intervals [20, Section 5].

Let P be a finite poset with $\hat{0}$ and $\hat{1}$, and let $\mathbf{2}$ denote the totally ordered two-element poset $\{\alpha, \beta\}$ with $\alpha < \beta$. Assume M is an SPM on P, and define $\pi : [\hat{0}, M(\hat{1})] \times \mathbf{2} \to P$ by

$$(p, \gamma) \mapsto \begin{cases} M(p) & \text{if } \gamma = \beta \text{ and } p \lhd M(p), \\ p & \text{otherwise.} \end{cases}$$

It is readily checked that the fibres of π are as follows:

$$\pi^{-1}(p) = \begin{cases} \{(M(p), \beta)\} & \text{if } p \not\leq M(\hat{1}), \\ \{(p, \alpha)\} & \text{if } p < M(p), \\ \{(p, \alpha), (p, \beta)\} & \text{if } p = M(p), \\ \{(p, \alpha), (M(p), \beta), (p, \beta)\} & \text{if } M(p) < p \leq M(\hat{1}). \end{cases} \tag{2.1}$$

Lemma 2.2. *The map π is an order projection. In particular, P is isomorphic to the fibre poset $\mathcal{F}_\pi$.*

Pircons A zircon is a poset in which for every non-minimal element x the order ideal $P_{\leq x}$ is finite and admits a special matching. The original definition given by Marietti ([18]) is actually slightly different but it is equivalent to the definition given here (see [11, Proposition 2.3]). Zircons can be extended to pircons as follows:

Definition 2.3. A poset P is a *pircon* if, for every non-minimal element $x \in P$, the order ideal $P_{\leq x}$ is finite and admits an SPM.

Zippings and Removals In [20], Reading introduced the concept of a zipper in a poset P. We restrict his definition somewhat, and we introduce a special class of zippers that we call *clean zippers*.

Definition 2.4. Let P be a finite poset with $\hat{0}$ and $\hat{1}$, and distinct elements $x, y, z \in P$. Call (x, y, z) a *zipper* if

(i) z covers only x and y,

(ii) $z = x \vee y$, where $\vee$ denotes join (supremum), and

(iii) $[\hat{0}, x) = [\hat{0}, y)$.

The zipper is *proper* if $z \neq \hat{1}$.

A zipper (x, y, z) is called *clean* if it is proper, and for some coatom c there is a poset isomorphism $\varphi : [x, \hat{1}] \to [x, c] \times \mathbf{2}$ such that $\varphi(z) = (x, \beta)$.

Reading proved in [20] that one can construct a new poset P' by identifying the elements in a zipper.

Definition 2.5. Given P with a partial order $\leq$ and a proper zipper (x, y, z), let $P' = (P - \{x, y, z\}) \uplus \{x'\}$, and define a partial order $\leq'$ on P' by

- $a \leq' b$ if $a \leq b$,
- $x' \leq' a$ if $x \leq a$ or $y \leq a$,
- $a \leq' x'$ if $a \leq x$ (or, equivalently, $a \leq y$), and
- $x' \leq' x'$,

for all a and b in $P - \{x, y, z\}$. We say that the poset P' is obtained from P by *zipping*.

In addition to zippings, we need another way to modify posets.

Definition 2.6. Let P be a finite poset with $\hat{0}$ and $\hat{1}$. An element $z \neq \hat{1}$ is *removable* if z covers exactly one element x, and for some coatom c there is a poset isomorphism $\varphi : [x, \hat{1}] \to [x, c] \times \mathbf{2}$ such that $\varphi(z) = (x, \beta)$.

If $z \in P$ is removable, we shall refer to $P - \{z\}$ as obtained by a *removal*. Alternatively, in analogy with zippings, we may consider $P - \{z\}$ as being obtained by identifying x and z.

Simplicial Complexes and PL Topology *Throughout the present abstract, all simplicial complexes are finite.* By convention, the empty set is considered to be a simplex of every non-void simplicial complex. Given an abstract simplicial complex Δ, we shall denote its geometric realisation (defined up to linear homeomorphism) by $\|\Delta\|$, a polyhedron in some real euclidean space. The simplices of Δ are sometimes called its *faces*, and maximal faces are referred to as *facets*.

Suppose Δ and Δ' are (abstract) simplicial complexes. A continuous map $f : \|\Delta\| \to \|\Delta'\|$ is *piecewise linear*, or *PL*, if its graph is a euclidean polyhedron. This is equivalent to asserting that there are simplicial subdivisions $\tilde{\Delta}$ and $\tilde{\Delta}'$ of Δ and Δ', respectively, with respect to which f is a simplicial map of the corresponding triangulations of $\|\Delta\|$ and $\|\Delta'\|$.

Say that Δ and Δ' are *PL homeomorphic* if there exists a PL homeomorphism $f : \|\Delta\| \to \|\Delta'\|$ (it follows that f^{-1} is also PL).

A *PL d-ball* is a simplicial complex which is PL homeomorphic to the simplicial complex Δ^d whose only facet is the d-dimensional simplex. A *PL $(d-1)$-sphere* is a simplicial complex which is PL homeomorphic to the simplicial complex obtained by removing the facet from Δ^d.

Given a finite poset P, its *order complex* $\Delta(P)$ is the abstract simplicial complex whose faces are the chains in P. In order to prevent proliferation of brackets when taking order complexes of poset intervals, we shall write $\Delta(x, y)$ instead of $\Delta((x, y))$.

3 Topology of pircons

The aim of this section is to show that open intervals in pircons are PL balls or spheres. Lemma 2.2 shows that every principal order ideal in a pircon is isomorphic to a fibre poset $\mathcal{F}_\pi$. The next result is proven by passing to $\mathcal{F}_\pi$ by identifying the elements of one non-trivial fibre at a time in a suitable order. Such an identification is either a clean zipping or a removal, depending on the cardinality of the fibre. Recall that the fibres are given in Equation 2.1. This is analogous to Reading's [20, Theorem 5.5].

Theorem 3.1. *Let P be a finite poset with $\hat{0}$ and $\hat{1}$. If M is an SPM on P, then P can be obtained from $[\hat{0}, M(\hat{1})] \times \mathbf{2}$ by a sequence of clean zippings and removals.*

Reading proved that proper zipping preserves PL spheres ([20, Theorem 4.7]). Furthermore, clean zippings and removals preserve PL balls, as the next two theorems state. They form the foundation of our work.

Theorem 3.2. *If P' is obtained from P by zipping a clean zipper (x, y, z) and $\Delta(\overline{P})$ is a PL d-ball, then so is $\Delta(\overline{P'})$.*

The main proof idea is to obtain $\Delta(\overline{P'})$ by removing a PL ball from $\Delta(\overline{P})$ (which corresponds to the simplicial complex whose facets are the maximal chains passing through x or y) and adding another PL ball (which corresponds to the simplicial complex whose facets are the maximal chains passing through x'). It turns out that this modification preserves PL balls under certain conditions on their boundaries, which are satisfied in the context of Theorem 3.2.

Using a similar argument and the fact that one gets a PL ball by deleting a vertex from a PL sphere, we also prove the following result.

Theorem 3.3. *Suppose $z \in P$ is removable. If $\Delta(\overline{P})$ is a PL d-ball or a PL d-sphere, then $\Delta(\overline{P} - \{z\})$ is a PL d-ball.*

Combining the aforementioned results, we obtain strong topological statements about posets with special partial matchings. These are our main results.

Theorem 3.4. *Let P be a finite poset with $\hat{0}$ and $\hat{1}$, and suppose M is an SPM on P. If $\Delta(\hat{0}, M(\hat{1}))$ is a PL d-ball, then $\Delta(\overline{P})$ is a PL $(d+1)$-ball. If $\Delta(\hat{0}, M(\hat{1}))$ is a PL d-sphere, then $\Delta(\overline{P})$ is a PL $(d+1)$-ball or a PL $(d+1)$-sphere; the latter holds if and only if M is actually a special matching.*

Applying Theorem 3.4 on the principal order ideals of a pircon and using an inductive argument on the cardinality of a longest chain in the ideal, we deduce our second main theorem, which in particular characterizes zircons among pircons.

Theorem 3.5. *Suppose P is a pircon and $x < y$ in P. Then $\Delta(x, y)$ is a PL ball or a PL sphere. Moreover, there exist $x < y$ in P such that $\Delta(x, y)$ is a ball if and only if P is not a zircon.*

4 Applications in Coxeter Groups

In this section, we demonstrate how Theorem 3.5 can be applied to certain posets appearing in Coxeter group theory. Acquaintance with the basics of this theory, as explained for example in [4] or [13], is assumed.

4.1 Twisted identities

As a first application, we prove [12, Conjecture 6.3], see Corollary 4.3 below. The reader may consult [12] for context. Here we only describe the necessary ingredients for the statement.

Let (W, S) be a Coxeter system with an involutive automorphism θ. Define two subsets of W as follows. The set of *twisted involutions* is

$$\mathfrak{I}(\theta) = \{w \in W \mid \theta(w) = w^{-1}\},$$

and the set of *twisted identities* is

$$\iota(\theta) = \{\theta(w)w^{-1} \mid w \in W\}.$$

It is clear that $\iota(\theta) \subseteq \mathfrak{I}(\theta)$.

Say that θ has the *no odd flip*, or *NOF*, *property* if $s\theta(s)$ has even or infinite order for every $s \in S$ with $s \neq \theta(s)$.[2] For any $X \subseteq W$, let $\mathrm{Br}(X)$ denote the subposet of the Bruhat order on W which is induced by X. The identity element $e \in W$ is the minimum in $\mathrm{Br}(W)$, hence in $\mathrm{Br}(\iota(\theta))$.

When W is of type A_{2n+1} and θ is the unique non-trivial involution, [2, Theorem 4.3] shows that $\mathrm{Br}(\iota(\theta))$ is a pircon. This is generalised substantially in the next result.

Theorem 4.1. *If θ has the NOF property, then $\mathrm{Br}(\iota(\theta))$ is a pircon.*

Remark 4.2. *In general, Theorem 4.1 is false without the NOF assumption. For example, suppose W is of type A_4 with generating set $S = \{s_1, s_2, s_3, s_4\}$ such that s_1s_2, s_2s_3, and s_3s_4 have order 3, and all other generator pairs commute. Let θ be the unique non-trivial involution of (W, S), mapping s_i to s_{5-i}. Define $w = s_2s_1s_3s_2s_4s_3$. One readily checks that $\mathrm{Br}(\mathfrak{I}(\theta))_{\leq w}$ is isomorphic to the rank 3 boolean lattice, and that $\mathrm{Br}(\iota(\theta))_{\leq w}$ is obtained from $\mathrm{Br}(\mathfrak{I}(\theta))_{\leq w}$ by removing the rank 2 element $s_2s_3s_2$. The resulting poset does not admit an SPM, hence $\mathrm{Br}(\iota(\theta))$ cannot be a pircon.*

In light of Theorem 3.5, Theorem 4.1 immediately implies the following result, which is the previously mentioned conjecture.

[2] This means that θ does not flip any edges with odd labels in the Coxeter graph.

Corollary 4.3 ([12, Conjecture 6.3])**.** *Suppose θ has the NOF property and let I be an open interval in* $\mathrm{Br}(\iota(\theta))$*. Then $\Delta(I)$ is a PL ball or a PL sphere.*

Remark 4.4.

1. Can, Cherniavsky, and Twelbeck [7] established Corollary 4.3 for W of type A_{2n+1} using shellability methods.

2. It follows from [12, Theorem 4.12] that $\Delta(I)$ is a sphere precisely when I is full, *meaning that it coincides with an interval in* $\mathrm{Br}(\mathfrak{I}(\theta))$*, i.e., $I = \{x \in \iota(\theta) \mid u < x < w\} = \{x \in \mathfrak{I}(\theta) \mid u < x < w\}$ for some $u, w \in \iota(\theta)$.*

3. Remark 4.2 shows that $\mathrm{Br}(\iota(\theta))$ *is not a pircon if W is of type A_{2m}, $m \geq 2$, with $\theta \neq \mathrm{id}$. It is, however, an open question whether the open intervals are PL balls or spheres. This is not true for arbitrary W and θ. For example, as shown in [12, Example 4.7], if W is of type $\widetilde{A}_2$ with $\theta \neq \mathrm{id}$, there are intervals in* $\mathrm{Br}(\iota(\theta))$ *which are not even graded.*

4.2 Quasiparabolic W-sets

Our second application concerns quasiparabolic W-sets as introduced by Rains and Vazirani [19] as a context to which many nice properties of parabolic quotients extend. Let us recall some crucial definitions and results from [19]. The reader should consult the original source for much more background and motivation.

Again (W, S) denotes a Coxeter system. Say that X is a *scaled W-set* if X is a (left) W-set equipped with a function $\mathrm{ht} : X \to \mathbf{Z}$ such that $|\mathrm{ht}(sx) - \mathrm{ht}(x)| \leq 1$ for all $x \in X$ and all $s \in S$. An element $x \in X$ is called *W-minimal* if $\mathrm{ht}(sx) \geq \mathrm{ht}(x)$ for all $s \in S$. Say that X is *bounded from below* if the function ht is bounded from below.

Let $T = \{wsw^{-1} \mid w \in W,\ s \in S\}$ denote the set of reflections.

Definition 4.5 ([19, Definition 2.3])**.** A scaled W-set is called *quasiparabolic* if it satisfies the following two properties.

1. For all $t \in T$ and $x \in X$, if $\mathrm{ht}(tx) = \mathrm{ht}(x)$, then $tx = x$.

2. For all $t \in T$, $x \in X$, and $s \in S$, if $\mathrm{ht}(tx) > \mathrm{ht}(x)$ and $\mathrm{ht}(stx) < \mathrm{ht}(sx)$, then $tx = sx$.

Lemma 4.6 ([19, Corollary 2.10])**.** *Each orbit of a quasiparabolic W-set contains at most one W-minimal element.*

Suppose now that X is quasiparabolic with a W-minimal element x_0. Assume, without loss of generality, that $\mathrm{ht}(x_0) = 0$. If $y \in X$ with $\mathrm{ht}(y) = k$ and $y = s_1 \cdots s_k x_0$ for some $s_1, \cdots, s_k \in S$ then we call $s_1 \cdots s_k x_0$ a *reduced expression* for y. All elements in the orbit of x_0 have reduced expressions [19]. Rather than taking the original definition of Rains and Vazirani, we use the following result as our definition of the *Bruhat order* $\leq$ on X.

Definition 4.7 ([19, Theorem 5.15])**.** Let $y = s_1 \cdots s_k x_0$ be a reduced expression. Then

$$x \le y \iff x = s_{i_1} \cdots s_{i_j} x_0 \text{ for some } 1 \le i_1 < \cdots < i_j \le k.$$

In particular, elements in different W-orbits are incomparable. Although not obvious from Definition 4.7, the Bruhat order is indeed a partial order on X, which we denote by $\mathrm{Br}(X)$; it is graded with rank function ht [19, Section 5]. In particular, W-minimal elements are minimal in the Bruhat order.

Quasiparabolic W-sets have a lifting property:

Lemma 4.8 ([19, Lemma 5.7])**.** *Suppose $x, y \in X$ and $s \in S$. If $x \le y$ and $sx \not\le sy$, then $sx \le y$ and $x \le sy$.*

For every non-minimal z in a quasiparabolic W-set bounded from below, there exists a unique minimal element $x_0 < z$. Let $s_1 \cdots s_k x_0$ be a reduced expression of z. Using the lifting property one can prove that the function $M : X \to X$ given by $M(x) = s_1 x$ gives an SPM on $\mathrm{Br}(X)_{\le z} = [x_0, z]$. This proves the following theorem:

Theorem 4.9. *If X is a quasiparabolic W-set which is bounded from below, then $\mathrm{Br}(X)$ is a pircon. In particular, the order complex of every open interval in $\mathrm{Br}(X)$ is a PL ball or a PL sphere.*

The topological conclusion of Theorem 4.9 is implied by [19, Theorem 6.4], which claims CL-shellability of the intervals. Unfortunately, the proof of that result has turned out to be flawed; see the discussion in [7].

A familiar example of a quasiparabolic W-set is the parabolic quotient W^J, $J \subseteq S$, which consists of the minimal length representatives of the left cosets of the parabolic subgroup W_J in W. In this setting, the topological conclusion of Theorem 4.9 was established by Björner and Wachs using shellability.

Other examples include several instances of $\iota(\theta)$ (with W acting by twisted conjugation, i.e., the action of w on x is given by $wx\theta(w^{-1})$), including the odd rank type A case. In fact, it seems possible that $\iota(\theta)$ is always a quasiparabolic W-set with this action whenever θ has the NOF property; if so, Theorem 4.1 would be a special case of Theorem 4.9. We neither know of a proof nor of a counterexample.

5 Open questions

We conclude the paper with a couple of questions that suggest themselves naturally.

Clearly, all zircons and pircons have rank functions. Indeed, the rank of an element x equals the dimension of the ball or sphere $\Delta(\hat{0}, x)$ plus two, where $\hat{0}$ is the unique minimal element below x.

Let Z be a zircon with rank function rk. For a non-minimal element $z \in Z$, let M_z denote a fixed special matching of $Z_{\leq z}$. Given an induced subposet $P \subseteq Z$ and $p \in P$, let us define

$$M'_p(x) = \begin{cases} M_p(x) & \text{if } M_p(x) \in P, \\ x & \text{otherwise.} \end{cases}$$

Suppose M'_p is an SPM on $P_{\leq p}$ for every non-minimal element $p \in P$. If, moreover, the restriction of rk to P is a rank function of P, call P an *induced pircon* of Z.

Whenever $\mathrm{Br}(\iota(\theta))$ is graded, it has the induced rank function of $\mathrm{Br}(\mathfrak{I}(\theta))$ [12]. In fact, it follows from the proof of Theorem 4.1 that every pircon of the form $\mathrm{Br}(\iota(\theta))$ is an induced pircon of the corresponding zircon $\mathrm{Br}(\mathfrak{I}(\theta))$. Similarly, $\mathrm{Br}(W^J)$ is an induced pircon of $\mathrm{Br}(W)$ for any $J \subseteq S$.

Question 5.1. Is every pircon an induced pircon of some zircon?

A common way to establish topological consequences such as those of Theorem 3.5 is to prove shellability. Since Björner [3], there are several variations of lexicographic shellability; see, e.g., Wachs' survey [21]. Under this umbrella are gathered several similarly flavoured combinatorial methods that can be used to establish shellability of order complexes by means of certain labellings of the posets.

Concerning zircons, the following question is known to have an affirmative answer for $\mathrm{Br}(W)$ in arbitrary type [5], as well as for $\mathrm{Br}(\mathfrak{I}(\theta))$ in types A, B, and D [16, 15, 14]. For other pircons, it has been established for $\mathrm{Br}(W^J)$ [5] and for $\mathrm{Br}(\iota(\theta))$ in type A of odd rank [7].

Question 5.2. Is every interval in every pircon lexicographically shellable?

In case both the previous questions turn out to have affirmative answers, one may speculate that even more could be true. The aforementioned result from [7] can be interpreted in the following way. For W of type A_n, Incitti [16] established lexicographic shellability of $\mathrm{Br}(\mathfrak{I}(\theta))$ by producing an EL-labelling of this poset. When n is odd and $\theta \neq \mathrm{id}$, Can, Cherniavsky, and Twelbeck proved that the restriction of this labelling to the induced pircon $\mathrm{Br}(\iota(\theta))$ is an EL-labelling, too.

Question 5.3. Is it true that every induced pircon has an EL-labelling which is induced from an EL-labelling of the corresponding zircon?

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