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Actions of the 0-Hecke monoids of affine symmetric groups

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Abstract. There are left and right actions of the 0-Hecke monoid of the affine symmetric group $\tilde{S}_n$ on involutions whose cycles are labeled periodically by nonnegative integers. Using these actions we construct two bijections, which are length-preserving in an appropriate sense, from the set of involutions in $\tilde{S}_n$ to the set of $\mathbb{N}$-weighted matchings in the n-element cycle graph. As an application, we show that the bivariate generating function counting the involutions in $\tilde{S}_n$ by length and absolute length is a rescaled Lucas polynomial. The 0-Hecke monoid of $\tilde{S}_n$ also acts on involutions (without any cycle labelling) by Demazure conjugation. The atoms of an involution $z \in \tilde{S}_n$ are the minimal length permutations w which transform the identity to z under this action. We prove that the set of atoms for an involution in $\tilde{S}_n$ is naturally a bounded, graded poset, and give a formula for the set's minimum and maximum elements.

Keywords: Affine symmetric groups, 0-Hecke monoids, involutions, Bruhat order

1 Introduction

For each integer $n \geq 1$, let $\tilde{S}_n$ be the *affine symmetric group* of rank n, consisting of the bijections $w : \mathbb{Z} \to \mathbb{Z}$ with $w(i+n) = w(i) + n$ for all $i \in \mathbb{Z}$ and $w(1) + w(2) + \cdots + w(n) = \binom{n+1}{2}$. If $n = 1$ then $\tilde{S}_1 = \{1\}$. Assume $n \geq 2$, and define $s_i \in \tilde{S}_n$ for $i \in \mathbb{Z}$ as the permutation which exchanges $i + mn$ and $i + 1 + mn$ for each $m \in \mathbb{Z}$, while fixing every integer not congruent to i or $i + 1$ modulo n. The elements $s_1, s_2, \ldots, s_n$ generate $\tilde{S}_n$, and with respect to these generators $\tilde{S}_n$ is the Coxeter group of type $\tilde{A}_{n-1}$.

If W is any Coxeter group with simple generating set S and length function $\ell : W \to \mathbb{N}$, then there is a unique associative product $\circ : W \times W \to W$ such that $w \circ s = w$ if $\ell(ws) < \ell(w)$ and $w \circ s = ws$ if $\ell(ws) > \ell(w)$ for $w \in W$ and $s \in S$ [16, Theorem 7.1]. The product $\circ$ is often called the *Demazure product*, and the pair $(W, \circ)$ is usually referred to as the *0-Hecke monoid* or *Richardson–Springer monoid* of (W, S). This extended abstract discusses three actions of the 0-Hecke monoid of $\tilde{S}_n$. Each action will be on objects related to the group's involutions, that is, the elements $z \in \tilde{S}_n$ with $z^2 = 1$.

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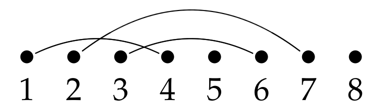
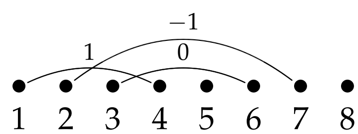
Let I_n be the set of involutions in the finite symmetric group S_n, which we identify with the parabolic subgroup of $\tilde{S}_n$ generated by $s_1, s_2, \dots, s_{n-1}$. Elements of I_n may be viewed as matchings on $\{1, 2, \dots, n\}$. For example,

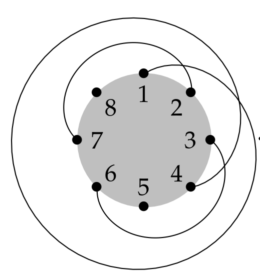
$(1,4)(2,7)(3,6) \in I_8$ corresponds to  . (1.1)

One adapts this model to the elements of $\tilde{I}_n = \{z \in \tilde{S}_n : z^2 = 1\}$ by representing $z \in \tilde{I}_n$ as the matching on $\mathbb{Z}$ in which i and j are connected by an edge whenever $z(i) = j \neq i = z(j)$. This gives a bijection between $\tilde{I}_n$ and matchings on $\mathbb{Z}$ which are "n-periodic" in the sense of having $\{i, j\}$ as an edge if and only if $\{i + n, j + n\}$ is also an edge. We can make this model more compact by converting n-periodic matchings on $\mathbb{Z}$ to $\mathbb{Z}$-weighted matchings on $\{1, 2, \dots, n\}$: to represent $z \in \tilde{I}_n$, include the edge $\{i, j\}$ labeled by $m \in \mathbb{Z}$ whenever $i < j$ and $z(i) = j + mn$ and $z(j) = i - mn$. For example,



(1.2)

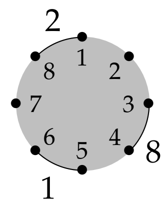
corresponds to $z = \prod_{m \in \mathbb{Z}} (1 + mn, 12 + mn)(7 + mn, 10 + mn)(3 + mn, 6 + mn) \in \tilde{I}_8$. Diagrams of this type are most useful when $\tilde{S}_n$ is viewed as a semidirect product $S_n \ltimes \mathbb{Z}^{n-1}$. When the structure of $\tilde{S}_n$ as a Coxeter group is significant, a better approach is to view n-periodic matchings as *winding diagrams*. To construct the *winding diagram* of $z \in \tilde{I}_n$, arrange $1, 2, \dots, n$ clockwise on a circle, and whenever $i < z(i) \equiv j \pmod{n}$, connect i to j by an arc winding $\frac{z(i)-i}{n}$ times in the clockwise direction around the circle's exterior. For the involution in (1.2), this produces the picture

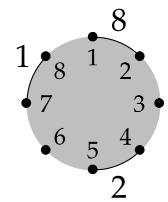
 .

(1.3)

Formally, a winding diagram is a collection of continuous paths between disjoint pairs of marked points on the boundary of the plane minus an open disc, up to homotopy. Each winding diagram corresponds to a unique involution in some affine symmetric group.

Write $\ell(w)$ for the usual Coxeter length of $w \in \tilde{S}_n$, and define the *absolute length* $\ell'(z)$ of $z \in \tilde{I}_n$ to be the number of arcs in its winding diagram. Our first main result, Theorem 2.12, identifies two bijections ω_R and ω_L from $\tilde{I}_n$ to the set $\mathcal{M}_n$ of $\mathbb{N}$-weighted matchings in $\mathcal{C}_n$, the cycle graph on n vertices. These bijections preserve length and absolute length, where the absolute length of an $\mathbb{N}$-weighted matching is its number of edges and the length is its number of edges plus twice the sum of their weights. The images of the element $z \in \tilde{I}_8$ in our running example (1.3) are

$\omega_R(z) =$ 

and $\omega_L(z) =$ 

and indeed it holds that $\ell'(z) = 3$ and $\ell(z) = 25$. The statement of Theorem 2.12 relies on the construction of a left and right action of the 0-Hecke monoid of $\tilde{S}_n$ on the set of *weighted involutions*. As an application, we show that $\sum_{z \in \tilde{I}_n} q^{\ell(z)} x^{\ell'(z)} = \frac{1}{1+q^n} \mathrm{Luc}_n(1 + q, qx)$ where $\mathrm{Luc}_n(x, s)$ is the nth bivariate *Lucas polynomial*; see Corollary 2.17. This is an analogue of a more complicated identity proved in [20].

The 0-Hecke monoid of $\tilde{S}_n$ also acts directly on $\tilde{I}_n$ by *Demazure conjugation*: the right action mapping $(z, w) \mapsto w^{-1} \circ z \circ w$ for $z \in \tilde{I}_n$ and $w \in \tilde{S}_n$. This monoid action is a degeneration of the Iwahori–Hecke algebra representation studied by Lusztig and Vogan in [18, 19]. The orbit of the identity under Demazure conjugation is all of $\tilde{I}_n$, and we define $\mathcal{A}(z)$ for $z \in \tilde{I}_n$ as the set of elements $w \in \tilde{S}_n$ of minimal length such that $z = w^{-1} \circ w$. Following [9, 10], we call these permutations the *atoms* of z. There are a few reasons why these elements merit further study, beyond their interesting combinatorial properties. The sets $\mathcal{A}(z)$ may be defined for involutions in any Coxeter group and, in the case of finite Weyl groups, are closely related to the sets $W(Y)$ which Brion [3] attaches to B-orbit closures Y in a spherical homogeneous space G/H (where G is a connected complex reductive group, B a Borel subgroup, and H a spherical subgroup). Results of Hultman [14, 15], extending work of Richardson and Springer [21], show the atoms to be intimately connected to the Bruhat order of a Coxeter group restricted to its involutions. Finally, the atoms of involutions in S_n play a central role in recent work of Can, Joyce, Wyser, and Yong on the geometry of the orbits of the orthogonal group on the type A flag variety; see [4, 5, 24].

Our second objective is to generalise a number of results about the atoms of involutions in finite symmetric groups to the affine case. In Section 3, extending results in [10, 13], we show that there is a natural partial order which makes $\mathcal{A}(z)$ for $z \in \tilde{I}_n$ into a bounded, graded poset. We conjecture that this poset is a lattice. Generalising results of Can, Joyce, and Wyser [4, 5], we describe in Section 4 a "local" criterion for membership in $\mathcal{A}(z)$ involving a notion of *(affine) standardisation*; see Theorem 4.4. Using this result, one can prove that involutions in $\tilde{S}_n$ have what we call the *Bruhat covering property*. For $i < j \not\equiv i \pmod{n}$, let $t_{ij} = t_{ji} \in \tilde{S}_n$ be the permutation which interchanges $i + mn$ and $j + mn$ for each $m \in \mathbb{Z}$ and which fixes all integers not in $\{i, j\} + n\mathbb{Z}$. The elements t_{ij} are precisely the *reflections* in $\tilde{S}_n$.

Theorem 1.1 (Bruhat covering property). *If $y \in \tilde{I}_n$ and $t \in \tilde{S}_n$ is a reflection, then there exists at most one $z \in \tilde{I}_n$ such that $\{wt : w \in \mathcal{A}(y) \text{ and } \ell(wt) = \ell(w) + 1\} \cap \mathcal{A}(z) \neq \varnothing$.*

The analogue of this result for involutions in S_n was shown in [12], and served as a key lemma in proofs of "transition formulas" for certain *involution Schubert polynomials*. We conjecture that the same property holds for arbitrary Coxeter systems, in the following sense. Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$ and Demazure product $\circ : W \times W \to W$. Suppose $w \mapsto w^*$ is an automorphism of W with $S^* = S$. The corresponding set of *twisted involutions* is $I_* = \{w \in W : w^{-1} = w^*\}$. For $y \in I_*$ let $\mathcal{A}_*(y)$ be the set of elements of minimal length with $(w^*)^{-1} \circ w = y$.

Conjecture 1.2. *If $y \in I_*$ is a twisted involution in an arbitrary Coxeter group and $t \in \{wsw^{-1} : w \in W,\ s \in S\}$, then there exists at most one twisted involution $z \in I_*$ such that $\{wt : w \in \mathcal{A}_*(y) \text{ and } \ell(wt) = \ell(w) + 1\} \cap \mathcal{A}_*(z) \neq \varnothing$.*

We anticipate that these results will be useful in developing a theory of *affine involution Stanley symmetric functions*, simultaneously generalising [17] and [9, 11, 8]. For the sake of brevity, we have omitted in this extended abstract most proofs, which will appear elsewhere.

2 Weighted involutions

Throughout, let $\mathbb{Z}$ be the set of all integers and let $\mathbb{N} = \{0, 1, 2, \dots\}$, and define $[n] = \{1, 2, \dots, n\}$ for $n \in \mathbb{N}$. For any map $w : \mathbb{Z} \to \mathbb{Z}$, let

$$\mathrm{Inv}(w) = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i < j \text{ and } w(i) > w(j)\}.$$

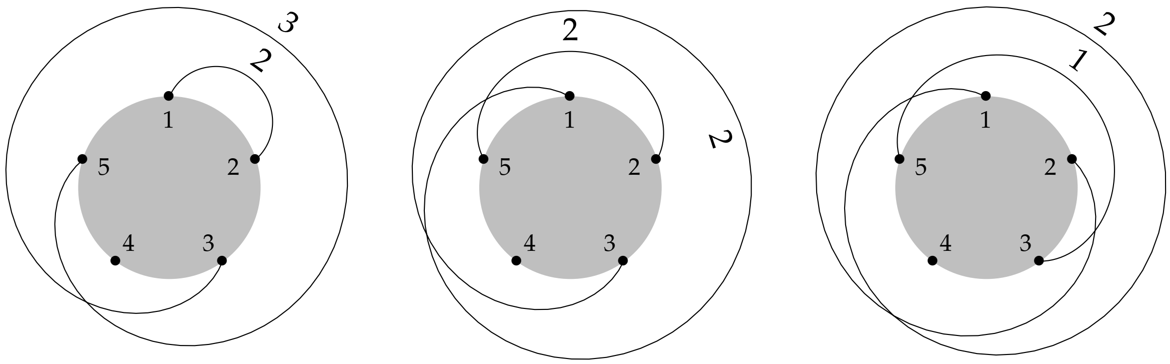
For $w \in \tilde{S}_n$, it then holds that $\ell(w)$ is the number of equivalence classes in $\mathrm{Inv}(w)$ under the relation on $\mathbb{Z} \times \mathbb{Z}$ generated by $(i, j) \sim (i + n, j + n)$ [2, Section 8.3]. For $z \in \tilde{I}_n$, let

$$\mathcal{C}(z) = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i < j = z(i)\}$$

The absolute length $\ell'(z)$ is then similarly the number of equivalence classes in $\mathcal{C}(z)$ under the relation on $\mathbb{Z} \times \mathbb{Z}$ generated by $(i, j) \sim (i + n, j + n)$.

Definition 2.1. A *weighted involution* in $\tilde{S}_n$ is a pair (w, ϕ) where $w \in \tilde{I}_n$ and ϕ is a map $\mathcal{C}(w) \to \mathbb{N}$ with $\phi(i, j) = \phi(i + n, j + n)$ for all $(i, j) \in \mathcal{C}(w)$. We refer to ϕ as the *weight map* of (w, ϕ). Define the *weight* of (w, ϕ) as the number $\mathrm{wt}(w, \phi) = \sum_\gamma \phi(\gamma)$ where the sum is over a set of cycles γ representing the distinct equivalence classes in $\mathcal{C}(w)$ under the relation $(i, j) \sim (i + n, j + n)$. Let $\mathcal{W}_n$ be the set of all weighted involutions in $\tilde{S}_n$.

Example 2.2. We draw a weighted involution $(w, \phi) \in \mathcal{W}_n$ as the winding diagram of w with its arcs labeled by the values of ϕ. For example, if $\theta_1, \theta_2, \theta_3 \in \mathcal{W}_5$ are



and $\theta_i = (w_i, \phi_i)$, then $w_1 = t_{1,2}t_{3,10}$, $w_2 = t_{0,2}t_{3,11}$, and $w_3 = t_{0,3}t_{2,11}$, while $\phi_1(1,2) = 2$ and $\phi_1(3,10) = 3$, $\phi_2(3,11) = \phi_2(5,7) = 2$, and $\phi_3(2,11) = 2$ and $\phi_3(5,8) = 1$.

We identify $\tilde{I}_n$ with the subset of weighted involutions of the form $(w, 0) \in \mathcal{W}_n$ with 0 denoting the unique weight map $\mathcal{C}(w) \to \{0\}$. We extend ℓ and ℓ' to $\mathcal{W}_n$ by setting

$$\ell(\theta) = \ell(w) + 2\,\mathrm{wt}(\theta) \qquad \text{and} \qquad \ell'(\theta) = \ell'(w) \qquad \text{for } \theta = (w, \phi) \in \mathcal{W}_n.$$

Given $(w, \phi) \in \mathcal{W}_n$, define the *right form* of ϕ to be the map $\phi_R : \mathbb{Z} \to \mathbb{N}$ with $\phi_R(i) = \phi(w(i), i)$ if $w(i) < i$ and with $\phi_R(i) = 0$ otherwise. Likewise, define the *left form* of ϕ to be the map $\phi_L : \mathbb{Z} \to \mathbb{N}$ with $\phi_L(i) = \phi(i, w(i))$ if $i < w(i)$ and with $\phi_L(i) = 0$ otherwise. Clearly ϕ_L and ϕ_R each determine ϕ, given w.

Definition 2.3. Let $\theta = (w, \phi) \in \mathcal{W}_n$ and $i \in \mathbb{Z}$. We define $\theta\pi_i, \pi_i\theta \in \mathcal{W}_n$ as follows:

(a) If $\phi_R(i) > \phi_R(i+1)$ then let $\theta\pi_i = (s_i w s_i, \psi) \in \mathcal{W}_n$ where ψ is the unique weight map with $\psi_R(j) = \phi_R(i) - 1$ if $j \equiv i+1 \pmod{n}$, $\psi_R(j) = \phi_R(i+1)$ if $j \equiv i \pmod{n}$, and $\psi_R(j) = \phi_R(j)$ otherwise. If $\phi_R(i) \leq \phi_R(i+1)$ then let $\theta\pi_i = \theta$.

(b) If $\phi_L(i+1) > \phi_L(i)$ then let $\pi_i\theta = (s_i w s_i, \chi) \in \mathcal{W}_n$ where χ is the unique weight map with $\chi_L(j) = \phi_L(i+1) - 1$if $j \equiv i \pmod{n}$, $\chi_L(j) = \phi_L(i)$ if $j \equiv i+1 \pmod{n}$, and $\chi_L(j) = \phi_L(j)$ otherwise. If $\phi_L(i+1) \leq \phi_L(i)$ then let $\pi_i\theta = \theta$.

Example 2.4. Define $\theta_1, \theta_2, \theta_3 \in \mathcal{W}_5$ as in Example 2.2. Then $\theta_1\pi_5 = \theta_2\pi_1 = \theta_2$ and $\theta_2\pi_2 = \theta_3$. Form $\theta_2' \in \mathcal{W}_5$ from θ_2 by replacing the label of the short arc in the picture in Example 2.2 by 1 and the label of the long arc by 3. Then $\pi_5\theta_1 = \theta_2'$ and $\pi_2\theta_2' = \theta_3$.

It may hold that $(\pi_i\theta)\pi_j \neq \pi_i(\theta\pi_j)$; for example, if $\theta = (w, \phi)$ where $w = s_1 \in \tilde{S}_2$ and $\phi(1,2) = 1$, then $\pi_0\theta = (\pi_0\theta)\pi_2 \neq \pi_0(\theta\pi_2) = \theta\pi_2$. We always have $\pi_i = \pi_{i+n}$.

Proposition 2.5. *The left (respectively, right) operators π_i satisfy (a) $\pi_i^2 = \pi_i$, (b) $\pi_i\pi_j = \pi_j\pi_i$ if $i \not\equiv j \pm 1 \pmod{n}$, and (c) $\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}$ for all $i, j \in \mathbb{Z}$.*

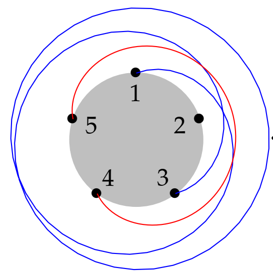
For each $g \in \tilde{S}_n$, we may therefore define a right (respectively, left) operator π_g on $\mathcal{W}_n$ by setting $\pi_g = \pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ where $g = s_{i_1}s_{i_2}\cdots s_{i_k}$ is any reduced expression.

Corollary 2.6. *The map $g \mapsto \pi_g$ defines a right and left action of $(\tilde{S}_n, \circ)$ on $\mathcal{W}_n$.*

Theorem-Definition 2.7. *Let $\theta = (w, \phi) \in \mathcal{W}_n$. There are unique permutations $g, h \in \tilde{S}_n$ with $\ell(g) = \ell(h) = \mathrm{wt}(\theta)$ and $\mathrm{wt}(\pi_g\theta) = \mathrm{wt}(\theta\pi_h) = 0$. Define $g_L(\theta) = g^{-1}$ and $g_R(\theta) = h$, and set $\omega_L(\theta) = \pi_g\theta$ and $\omega_R(\theta) = \theta\pi_h$.*

Example 2.8. If $\theta_1, \theta_2, \theta_3 \in \mathcal{W}_5$ are as in Example 2.2, then we have

$$\omega_R(\theta_1) = \omega_R(\theta_2) = \omega_R(\theta_3) = t_{1,13}t_{5,9} =$$



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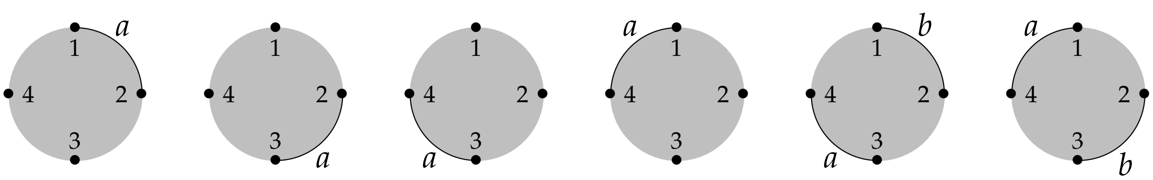
There is a simple relationship between the left and right versions of these constructions. Define $\tau : \mathbb{Z} \to \mathbb{Z}$ by $\tau(i) = n + 1 - i$ and let $w^* = \tau w \tau$ for $w \in \tilde{S}_n$. For $\phi : \mathcal{C}(w) \to \mathbb{N}$ let ϕ^* be the map $\mathcal{C}(w^*) \to \mathbb{N}$ given by $(\tau(j), \tau(i)) \mapsto \phi(i, j)$. Extend $*$ to $\mathcal{W}_n$ by setting $\theta^* = (w^*, \phi^*)$ for $\theta = (w, \phi) \in \mathcal{W}_n$. Clearly $(\theta^*)^* = \theta$.

Lemma 2.9. *If $\theta \in \mathcal{W}_n$ then $g_L(\theta^*) = g_R(\theta)^*$ and $\omega_L(\theta^*) = \omega_R(\theta)^*$.*

An involution $w \in \tilde{I}_n$ has $\ell(w) = \ell'(w)$ if and only if w is a product of commuting simple generators, i.e., $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ where $i_j \not\equiv i_k \pm 1 \pmod{n}$ for all $j, k \in [l]$.

Definition 2.10. Define $\mathcal{M}_n$ as the set of $\theta = (w, \phi) \in \mathcal{W}_n$ with $\ell'(w) = \ell(w)$.

The elements of $\mathcal{M}_n$ are in bijection with $\mathbb{N}$-weighted matchings in $\mathcal{C}_n$, the cycle graph on n vertices, which explains our notation. The set $\mathcal{M}_4$ consists of



where $a, b \in \mathbb{N}$ are arbitrary. The following is well-known; see [22, A034807].

Proposition 2.11. *There are $\frac{n}{n-k}\binom{n-k}{k}$ distinct k-element matchings in $\mathcal{C}_n$.*

Recall the definition of $t_{ij} \in \tilde{S}_n$ for $i < j \not\equiv i \pmod{n}$. Let $\prec$ be the partial order on $\mathcal{M}_n$ with $(w,\phi) \preceq (w',\phi')$ if and only if $w = w'$ and $\phi(a,b) \leq \phi'(a,b)$ for $(a,b) \in \mathcal{C}(w)$. Next, define $\prec_R$ as the transitive closure of the relation on $\tilde{I}_n$ with $z \prec_R t_{ij}zt_{ij}$ whenever $z(i) < i$ and $j = \min\{e \in \mathbb{Z} : i < e \text{ and } z(i) < z(e)\}$. Finally, define $\prec_L$ similarly as the transitive closure of the relation on $\tilde{I}_n$ with $z \prec_L t_{ij}zt_{ij}$ whenever $j < z(j)$ and $i = \max\{e \in \mathbb{Z} : e < j \text{ and } z(e) < z(j)\}$. The posets $(\tilde{I}_n, \prec_R)$ and $(\tilde{I}_n, \prec_L)$ are isomorphic via the map $z \mapsto z^*$. One can show that these posets are graded with rank function $z \mapsto \frac{1}{2}\ell(z)$, and that both are subposets of the Bruhat order restricted to $\tilde{I}_n$.

Theorem 2.12. *The maps $\omega_R : (\mathcal{M}_n, \prec) \to (\tilde{I}_n, \prec_R)$ and $\omega_L : (\mathcal{M}_n, \prec) \to (\tilde{I}_n, \prec_L)$ are isomorphisms of partially ordered sets which preserve ℓ and ℓ'.*

Consider the following variations of $g_L(\theta)$ and $g_R(\theta)$ from Theorem-Definition 2.7.

Definition 2.13. For $z \in \tilde{I}_n$ let θ_R and θ_L be the unique elements of $\mathcal{M}_n$ such that $\omega_R(\theta_R) = \omega_L(\theta_L) = z$, and define $\alpha_R(z) = g_R(\theta_R) \in \tilde{S}_n$ and $\alpha_L(z) = g_L(\theta_L) \in \tilde{S}_n$.

One can derive a more explicit formula for these elements.

Proposition-Definition 2.14. *If $a_1, a_2, \dots, a_n \in \mathbb{Z}$ represent the distinct congruence classes modulo n then there is a unique $m \in \mathbb{Z}$ and a unique $w \in \tilde{S}_n$ such that $w(m+i) = a_i$ for $i \in [n]$. Moreover, it holds that $m = \frac{1}{n}\sum_{i=1}^{n}(a_i - i)$. Define $[a_1, a_2, \dots, a_n] = w \in \tilde{S}_n$.*

If $a_1, a_2, \dots, a_N \in \mathbb{Z}$ represent all congruence classes modulo n, and $i_1 < i_2 < \cdots < i_n$ are the indices of the first representative of each class, then we define $[a_1, a_2, \dots, a_N] = [a_{i_1}, a_{i_2}, \dots, a_{i_n}] \in \tilde{S}_n$. For example, if $n = 3$ then $[1,0,1,3,8,4,2] = [1,0,8]$.

Theorem 2.15. *Let $z \in \tilde{I}_n$ and $m \in \mathbb{Z}$. Suppose $a_1 < a_2 < \cdots < a_l$ and $d_1 < d_2 < \cdots < d_l$ are the elements of $m + [n]$ with $a_i \leq z(a_i)$ and $z(d_i) \leq d_i$. Define $b_i = z(a_i)$ and $c_i = z(d_i)$. Then $\alpha_R(z) = [a_1, b_1, a_2, b_2, \dots, a_l, b_l]^{-1}$ and $\alpha_L(z) = [c_1, d_1, c_2, d_2, \dots, c_l, d_l]^{-1}$.*

Example 2.16. One has $\alpha_R(1) = \alpha_L(1) = [1,1,2,2,\dots,n,n]^{-1} = 1$. If $z = t_{1,8}t_{2,7} \in \tilde{I}_4$ then $\alpha_R(z) = [1,8,2,7]^{-1} = [3,5,2,0]$ and $\alpha_L(z) = [-2,3,-3,4]^{-1} = [5,3,0,2]$.

Slightly abusing notation, we write $\tilde{I}_n(q,x) = \sum_{w \in \tilde{I}_n} q^{\ell(w)} x^{\ell'(w)} \in \mathbb{N}[[q,x]]$.

Corollary 2.17. *If $n \geq 1$ then $\tilde{I}_n(q,x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k}\binom{n-k}{k}\left(\frac{qx}{1-q^2}\right)^k$.*

Proof. Theorem 2.12 implies that $\tilde{I}_n(q,x) = \sum_{\theta \in \mathcal{M}_n} q^{2\,\mathrm{wt}(\theta)}(qx)^{\ell'(\theta)}$. By Proposition 2.11, the coefficient of x^k in the latter power series is $\frac{n}{n-k}\binom{n-k}{k} q^k (1 + q^2 + q^4 + q^6 + \dots)^k$. □

Corollary 2.18. *If $n \geq 3$ then $\tilde{I}_n(q,x) = \tilde{I}_{n-1}(q,x) + \frac{qx}{1-q^2}\tilde{I}_{n-2}(q,x)$.*

Define $\hat{\ell}(w) = \frac{1}{2}(\ell(w) + \ell'(w))$ for $w \in \tilde{I}_n$. Corollary 2.17 shows that $\hat{\ell}(w) \in \mathbb{N}$. Let $\hat{N}_n(m)$ be the number of involutions $w \in \tilde{I}_n$ with $\hat{\ell}(w) = m$.

Corollary 2.19. *For each $n \geq 2$ and $m \geq 1$, it holds that $\hat{N}_n(m) = \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{n}{n-j}\binom{n-j}{j}\binom{m-1}{j-1}$.*

Remark. The numbers $\{\hat{N}_n(n)\}_{n=1,2,3,\dots} = (0,2,3,10,25,71,196,554,1569,\dots)$ are sequence [22, A246437], which gives the "type B analog for Motzkin sums." The numbers $\{\hat{N}_n(2n)\}_{n=1,2,3,\dots} = (0,2,3,18,50,215,735,2898,\dots)$ are sequence [22, A211867].

3 Demazure conjugation

Recall the definition of the Demazure product $\circ : \tilde{S}_n \times \tilde{S}_n \to \tilde{S}_n$ from the introduction. The operation $(z,w) \mapsto w^{-1} \circ z \circ w$ for $z \in \tilde{I}_n$ and $w \in \tilde{S}_n$ defines another right action of the monoid $(\tilde{S}_n, \circ)$, which we call *Demazure conjugation*. If $z \in \tilde{I}_n$ and $i \in \mathbb{Z}$ then

$$s_i \circ z \circ s_i = \begin{cases} s_i z s_i & \text{if } z(i) < z(i+1) \text{ and } (i,i+1) \notin \mathcal{C}(z) \\ z s_i & \text{if } z(i) < z(i+1) \text{ and } (i,i+1) \in \mathcal{C}(z) \\ z & \text{otherwise.} \end{cases} \tag{3.1}$$

Every $z \in \tilde{I}_n$ can be expressed as $z = w^{-1} \circ w$ for some $w \in \tilde{S}_n$, and we define $\mathcal{A}(z)$ as the set of elements $w \in \tilde{S}_n$ of shortest possible length such that $z = w^{-1} \circ w$. For example, if $z = t_{0,5} = [-4,2,3,9] \in \tilde{I}_4$ then $\mathcal{A}(z) = \{s_1s_2s_3s_4, s_2s_1s_3s_4, s_3s_2s_1s_4\}$.

The set $\mathcal{A}(z)$ is nonempty for all $z \in \tilde{I}_n$, and we refer to its elements as the *atoms* of z. Recall from Corollary 2.19 that $\hat{\ell}(z) = \frac{1}{2}(\ell(z) + \ell'(z))$. By (3.1), we have:

Proposition 3.1. *If $z \in \tilde{I}_n$ then $\hat{\ell}(z)$ is the common value of $\ell(w)$ for $w \in \mathcal{A}(z)$.*

Results in [10], building on work of Can, Joyce, and Wyser [4, 5], show that the sets $\mathcal{A}(z)$ for involutions $z \in I_n \subset \tilde{I}_n$ in the finite symmetric group are naturally bounded, graded posets. This phenomenon extends to all involutions in $\tilde{S}_n$. Recall the elements $\alpha_R(z)$ and $\alpha_L(z)$ from Definitions 2.13.

Definition 3.2. Given $z \in \tilde{I}_n$, let $\alpha_{\min}(z) = \alpha_R(z)z$ and $\alpha_{\max}(z) = \alpha_L(z)z$.

Corollary 3.3. *Let $z \in \tilde{I}_n$ and $m \in \mathbb{Z}$ and define a_i, b_i, c_i, d_i as in Theorem 2.15. Then $\alpha_{\min}(z) = [b_1, a_1, b_2, a_2, \dots, b_l, a_l]^{-1}$ and $\alpha_{\max}(z) = [d_1, c_1, d_2, c_2, \dots, d_l, c_l]^{-1}$.*

Example 3.4. If $z = t_{1,8}t_{2,7} \in \tilde{I}_4$ then $\alpha_{\min}(z) = [8,1,7,2]^{-1}$and $\alpha_{\max}(z) = [3,-2,4,-3]^{-1}$.

Let $\lessdot_{\mathcal{A}}$ be the relation on $\tilde{S}_n$ with $u \lessdot_{\mathcal{A}} v$ if and only if $u < s_{i+1}u = s_i v > v$ for some $i \in \mathbb{Z}$. Let $<_{\mathcal{A}}$ be the transitive closure of $\lessdot_{\mathcal{A}}$.

Theorem 3.5. *Let $z \in \tilde{I}_n$. Restricted to $\mathcal{A}(z)$, the relation $<_{\mathcal{A}}$ is a bounded, graded partial order, and it holds that $\mathcal{A}(z) = \{w \in \tilde{S}_n : \alpha_{\min}(z) \leq_{\mathcal{A}} w\} = \{w \in \tilde{S}_n : w \leq_{\mathcal{A}} \alpha_{\max}(z)\}$.*

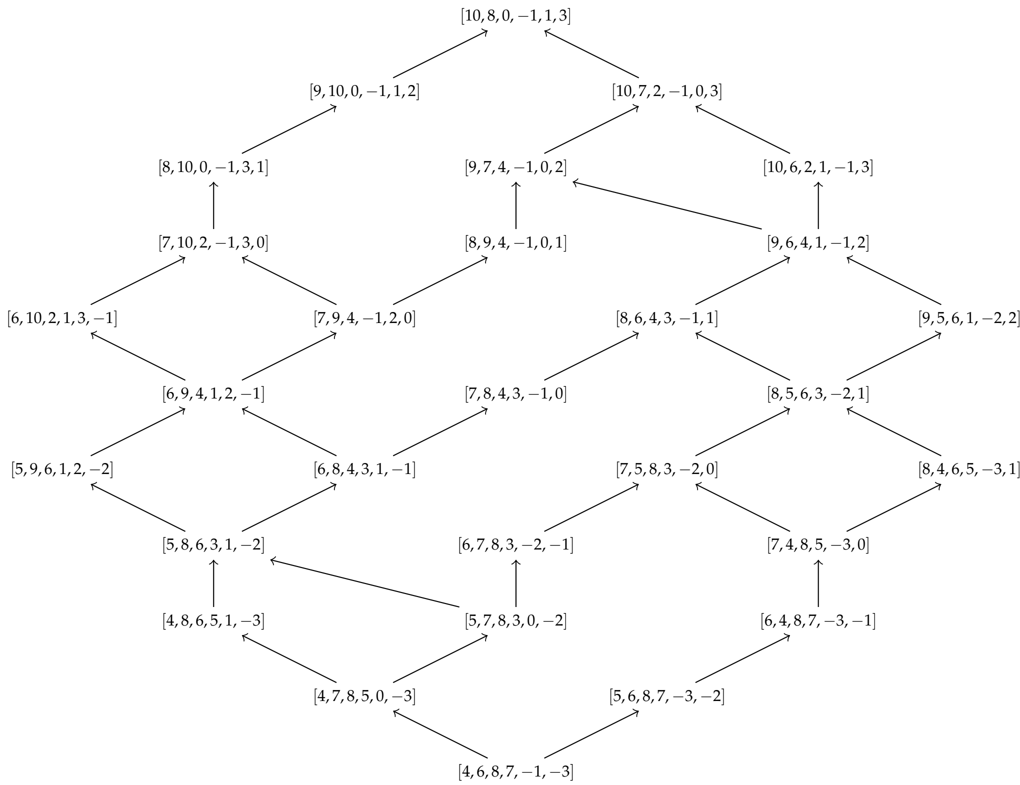


Figure 1: Hasse diagram of $(\mathcal{A}(z), <_{\mathcal{A}})$ for $z = t_{1,12}t_{2,11}t_{3,4} \in \tilde{I}_6$

The situation described by the preceding theorem has some formal similarities to Stembridge's results in [23, Section 4] about the top and bottom classes of a permutation.

Figure 1 shows an example of $(\mathcal{A}(z), <_{\mathcal{A}})$. The lattice structure evident in this picture appears to be typical; we have used a computer to check the following conjecture for $z \in \tilde{I}_n$ in the 333,307 cases when $0 < \hat{\ell}(z)n \leq 100$.

Conjecture 3.6. *The graded poset $(\mathcal{A}(z), <_{\mathcal{A}})$ is a lattice for all n and $z \in \tilde{I}_n$.*

An element $w \in \tilde{S}_n$ is *321-avoiding* if no integers $a < b < c$ have $w(a) > w(b) > w(c)$, and *fully commutative* if we cannot write $w = us_is_{i+1}s_iv$ for $u, v \in \tilde{S}_n$ and $i \in \mathbb{Z}$ with $\ell(w) = \ell(u) + \ell(v) + 3$. The following extends [10, Corollary 6.11] to affine type A.

Corollary 3.7. *Let $z \in \tilde{I}_n$. The following are equivalent: (a) $|\mathcal{A}(z)| = 1$, (b) $\alpha_{\min}(z) = \alpha_{\max}(z)$, (c) $\alpha_R(z) = \alpha_L(z)$, (d) z is 321-avoiding, and (e) z is fully commutative.*

The equivalence of (d) and (e) is well-known; see the results of Green [7, Theorem 2.7], Lam [17, Proposition 35], or Fan and Stembridge [6]. Biagioli, Jouhet, and Nadeau [1, Proposition 3.3] have derived a length generating function for the involutions in $\tilde{S}_n$ with these equivalent properties.

4 Local characterisations of atoms

Fix a subset $E \subset [n]$ of size m. Let $\phi_E : [m] \to E$ and $\psi_E : E \to [m]$ be order-preserving bijections. The *standardisation* of $w \in S_n$ is the permutation $[w]_E = \psi_{w(E)} \circ w \circ \phi_E \in S_m$. If $w^2 = 1$ and $w(E) = E$, then $([w]_E)^2 = 1$.

The Demazure product $\circ$ on $\tilde{S}_n$ restricts to an associative product $S_n \times S_n \to S_n$ and each involution $y \in I_n = \tilde{I}_n \cap S_n$ has $\mathcal{A}(y) \subset S_n$. Can, Joyce, and Wyser's description of $\mathcal{A}(y)$ for $y \in I_n$ in [5] implies that $w \in S_n$ belongs to $\mathcal{A}(y)$ if and only if $[w]_E \in \mathcal{A}([y]_E)$ for all subsets $E \subset [n]$ which are invariant under y and contain at most two y-orbits; see Corollary 3.19 in [12]. This "local" criterion for membership in $\mathcal{A}(y)$ was an important tool in the proofs of the main results in [12].

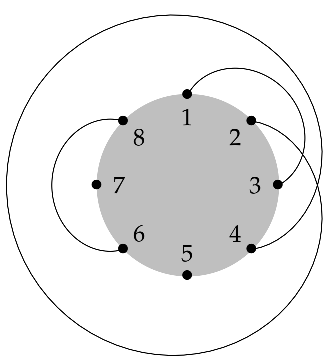
This result can be extended to the affine case, provided we give the right definition of the standardisation of an affine permutation. Fix $E \subset \mathbb{Z}$ with $|(E + n\mathbb{Z}) \cap [n]| = m$, and define $\tilde{\phi}_{E,n}$ as the unique order-preserving $\mathbb{Z} \to E + n\mathbb{Z}$ with $\tilde{\phi}_{E,n}([m]) \subset [n]$.

Lemma 4.1. *Let $w \in \tilde{S}_n$. There is a unique order-preserving bijection $\tilde{\psi}_{E,w} : w(E) + n\mathbb{Z} \to \mathbb{Z}$ with $\tilde{\psi}_{E,w} \circ w \circ \tilde{\phi}_{E,n} \in \tilde{S}_m$. If $w \in \tilde{I}_n$ and $w(E) = E$ then $\tilde{\phi}_{E,n}$ and $\tilde{\psi}_{E,w}$ are inverses.*

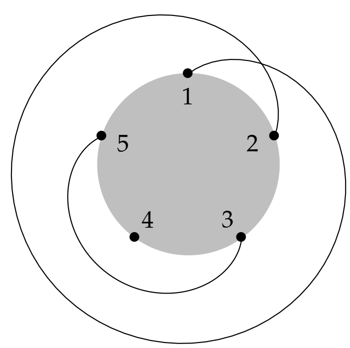
Given $w \in \tilde{S}_n$ and $E \subset \mathbb{Z}$ with $|(E + n\mathbb{Z}) \cap [n]| = m$, define $[w]_{E,n} = \tilde{\psi}_{E,w} \circ w \circ \tilde{\phi}_{E,n} \in \tilde{S}_m$. We refer to $[w]_{E,n}$ as the *(affine) standardisation* of w. One has $[w]_{E,n} = [w]_{E+mn,n}$ for all $m \in \mathbb{Z}$. When n is clear from context, we write $[w]_E$ instead of $[w]_{E,n}$. If $E \subset [n]$ and $w \in S_n \subset \tilde{S}_n$, then $\tilde{\phi}_{E,n}|_{[m]} = \phi_E$ and $\tilde{\psi}_{E,w}|_E = \psi_{w(E)}$.

Corollary 4.2. *If $E \subset \mathbb{Z}$, $y \in \tilde{I}_n$, and $y(E) = E$, then $[y]_E \in \tilde{I}_n$.*

Example 4.3. Standardisation has a simple interpretation in terms of winding diagrams. If $E = y(E) \subset \mathbb{Z}$ then the winding diagram of $[y]_E$ is formed from that of $y \in \tilde{I}_n$ by erasing the vertices in $[n] \setminus (E + n\mathbb{Z})$ and their incident edges, and then relabelling the remaining numbers as consecutive integers. If $n = 8$ and $E = \{2, 4, 6, 7, 8\}$, then



and



represent $y = t_{1,3}t_{2,12}t_{6,8} \in \tilde{I}_8$ and $[y]_E = t_{1,7}t_{3,5} \in \tilde{I}_5$, respectively.

Theorem 4.4. *Let $y \in \tilde{I}_n$, $w \in \tilde{S}_n$, and $X = [n] \cup y([n])$. Then $w \in \mathcal{A}(y)$ if and only if $[w]_E \in \mathcal{A}([y]_E)$ for each subset $E = y(E) \subset X$ containing at most two y-orbits.*

This result is the starting point for the proof of Theorem 1.1 in the introduction.

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