Séminaire Lotharingien de Combinatoire **80B** (2018)
Article #66, 12 pp.

Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover)

On the number of λ-unimodal involutions

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Abstract. For a given composition λ of the positive integer n, a λ-unimodal permutation is a permutation comprised of contiguous unimodal segments whose lengths are determined by λ. In this extended abstract, the authors present a generating function for the number of λ-unimodal involutions and address its relationship to the Gelfand character of the symmetric group.

Keywords: involutions, enumeration, λ-unimodal permutations, descents

1 Introduction

A permutation is *unimodal* provided its one-line notation is increasing, then decreasing. Given any composition λ of the positive integer n, we say that a permutation is *λ-unimodal* if it is comprised of contiguous unimodal segments whose lengths are determined by the composition λ. These λ-unimodal permutations are the topic of research by numerous authors, although they are not often studied as purely combinatorial objects, and first appeared in the study of characters of the symmetric group; see for example [1, 2, 4, 6, 9, 11]. In [3], the first author of this extended abstract investigated λ-unimodal cycles and their application to a specific character of the symmetric group. Here, we investigate λ-unimodal involutions, i.e., those λ-unimodal permutations that are their own (algebraic) inverse.

In [1, 2], it is shown that these involutions have a direct relationship to the so-called Gelfand character, χ^G. This character is associated to the representation of $\mathcal{S}_n$ obtained by taking the multiplicity-free direct sum of the irreducible representations of $\mathcal{S}_n$. For example, see [1]. Specifically, if $\mathcal{I}^\lambda$ denotes the set of λ-unimodal involutions and $\text{des}_\lambda(\pi)$ denotes the number of λ-descents of a permutation π (defined in Section 2), then

$$\chi^G_\lambda = \sum_{\pi \in \mathcal{I}^\lambda} (-1)^{\text{des}_\lambda(\pi)}. \tag{1.1}$$

In this extended abstract, we enumerate λ-unimodal involutions via a recursive generating function. This can be further refined to a generating function for λ-unimodal involutions with a given number of λ-descents, which in turn gives a generating function

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for the Gelfand character (see Theorem 4.1). This gives us a new way of computing the Gelfand character (other than the Murnaghan–Nakayama rule; see [7, 8, 5, 11] for more). Because of space restrictions, most of the details of the proof regarding the computation of these involutions by λ-descents are omitted. However, this gives us an approach to address an open question in [10], in which Roichman comments on the desirability of combinatorial proofs to the given character formulas, such as Equation (1.1).

2 Background and Notation

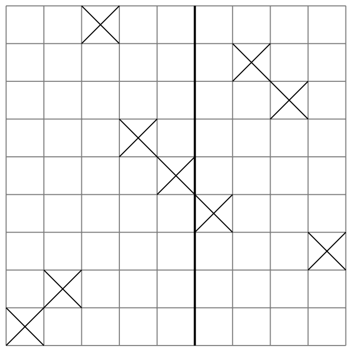
Let $\mathcal{S}_n$ be the set of permutations on $[n] = \{1, 2, \ldots, n\}$, and write $\pi \in \mathcal{S}_n$ in its one-line notation as $\pi = \pi_1\pi_2 \ldots \pi_n = \pi(1)\pi(2) \ldots \pi(n)$. A permutation $\pi \in \mathcal{S}_n$ is *unimodal* if there exists $i \in [n]$ such that

$$\pi_1 < \pi_2 < \cdots < \pi_{i-1} < \pi_i > \pi_{i+1} > \cdots > \pi_{n-1} > \pi_n;$$

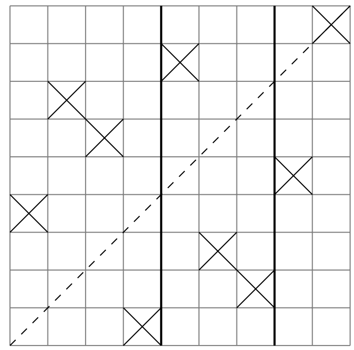
that is, π is increasing then decreasing. A *composition* of the integer n, denoted $\lambda \vDash n$, is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\sum \lambda_i = n$. Given a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n, we say that $\pi \in \mathcal{S}_n$ is λ*-unimodal* provided π is composed of k contiguous segments, where the i-th segment is unimodal of length λ_i. For example, the permutation $\pi = 129654873 \in \mathcal{S}_9$ is $(5, 4)$-unimodal because the first five entries 12965 and the last four entries 4873 both form unimodal segments of π; the pictorial representation of this permutation can be seen in Figure 1(a). The permutation $\pi \in \mathcal{S}_n$ has a *descent* at position i if $\pi_i > \pi_{i+1}$. The *descent set* of π, denoted $\mathrm{Des}(\pi)$, is the set of descents and the *descent number* of π, denoted $\mathrm{des}(\pi)$, is the number of descents of π. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vDash n$, we say that i is a λ*-descent* of π if i is a descent that occurs within a segment of length λ_i for some $i \in [n]$. In other words, we define the set of λ-descents of π, denoted $\mathrm{Des}_\lambda(\pi)$, to be the set

$$\mathrm{Des}_\lambda(\pi) = \mathrm{Des}(\pi) \setminus \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \lambda_2 + \cdots + \lambda_{k-1}\}.$$

We let $\mathrm{des}_\lambda(\pi)$ denote the number of λ-descents of π. For example, for the permutation $\pi = 129654873 \in \mathcal{S}_9$, we have $\mathrm{des}(\pi) = 5$ and $\mathrm{des}_\lambda(\pi) = 4$, where $\lambda = (5, 4)$. Finally, $\pi \in \mathcal{S}_n$ is an *involution* if it is comprised of only transpositions and fixed points. Equivalently, every involution is it own inverse and in its pictorial representation, every involution is symmetric about the diagonal. The $(4, 3, 2)$-unimodal involution $\pi = 476183259 \in \mathcal{S}_9$ is depicted in Figure 1(b). Finally, let S^λ denote the set of λ-unimodal permutations and let $\mathcal{I}^\lambda$ denote the set of λ-unimodal involutions. For example, $129654873 \in S^{(5,4)}$ and $129654873 \in \mathcal{I}^{(5,4)}$; also $476183259 \in \mathcal{I}^{(4,3,2)}$.



(a) $(5,4)$-unimodal permutation $\pi = 129654873 \in \mathcal{S}_9$



(b) $(4,3,2)$-unimodal involution $\pi = 476183259 \in \mathcal{S}_9$

Figure 1: Pictorial representations of λ-unimodal permutations.

We conclude this section with two enumerations of the set of λ-unimodal permutations that do not appear anywhere in the literature, in part because these permutations are not yet well-studied. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vDash n$, then the number of λ-unimodal permutations in $\mathcal{S}_n$ is

$$\binom{n}{\lambda_1, \lambda_2, \ldots, \lambda_k} \prod_{i=1}^{k} 2^{\lambda_i - 1} = \binom{n}{\lambda_1, \lambda_2, \ldots, \lambda_k} 2^{n-k},$$

where $\binom{n}{\lambda_1, \lambda_2, \ldots, \lambda_k}$ denotes the multinomial coefficient. The number of λ-unimodal permutations in $\mathcal{S}_n$ with m λ-descents is

$$\binom{n}{\lambda_1, \lambda_2, \ldots, \lambda_k} \sum_{\mathbf{d} \in \Omega_\lambda^m} \prod_{i=1}^{k} \binom{\lambda_i - 1}{d_i} = \binom{n}{\lambda_1, \lambda_2, \ldots, \lambda_k} \binom{n-k}{m},$$

where $\Omega_\lambda^m = \{(d_1, d_2, \ldots, d_k) : \sum d_i = m \text{ and } 0 \leq d_i \leq \lambda_i - 1\}$. The proofs of these equations follow quickly from the fact that there are exactly 2^{n-1} unimodal permutations in $\mathcal{S}_n$ and exactly $\binom{n-1}{m}$ unimodal permutations in $\mathcal{S}_n$ with m descents.

3 Main Theorem

In this section, let Λ_k be the set of integer compositions into k positive integer parts and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Lambda_k$. Let $\boldsymbol{x}$ denote the set of indeterminates $\{x_1, x_2, \ldots, x_k\}$. For a generating function F on variables $\boldsymbol{x} \setminus \{x_{i_1}, x_{i_2}, \ldots, x_{i_j}\}$, we write $F(\boldsymbol{x}; \hat{x}_{i_1}, \hat{x}_{i_2}, \ldots, \hat{x}_{i_j})$. For example, if $k = 3$, then $\boldsymbol{x} = \{x_1, x_2, x_3\}$ and $F(\boldsymbol{x}; \hat{x}_2)$ is a function on variables x_1 and x_3 only. Let x^λ denote the monomial $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$.

We first define three generating functions as follows. Recall that $\mathcal{I}^\lambda$ is the set of λ-unimodal involutions. Let $\mathcal{I}_j^\lambda$ be the set of λ-unimodal involutions, where either $\pi_1 \leq$

$\lambda_1 + \cdots + \lambda_j$ and the first segment is decreasing or where $\pi_1 > \lambda_1 + \cdots + \lambda_j$. Finally, let $\mathcal{D}_i^\lambda$ be the set of λ-unimodal involutions, where $\pi_1 \leq \lambda_1 + \cdots + \lambda_i$ and the first segment is decreasing. We define the generating functions $L^k(\boldsymbol{x}), L_j^k(\boldsymbol{x})$, and $D_i^k(\boldsymbol{x})$ as follows:

$$L^k(\boldsymbol{x}) = \sum_{\lambda \in \Lambda_k} |\mathcal{I}^\lambda| x^\lambda, \quad L_j^k(\boldsymbol{x}) = \sum_{\lambda \in \Lambda_k} |\mathcal{I}_j^\lambda| x^\lambda, \quad \text{and} \quad D_i^k(\boldsymbol{x}) = \sum_{\lambda \in \Lambda_k} |\mathcal{D}_i^\lambda| x^\lambda.$$

We find recursive formulas for these generating functions below.

Theorem 3.1. *We have $L^0(\boldsymbol{x}) = 1$, $L^1(\boldsymbol{x}) = \dfrac{x_1}{(1-x_1)^2}$ and for $k \geq 2$,*

$$L^k(\boldsymbol{x}) = \frac{1}{1-x_1}\Big[x_1^2 L_1^k(\boldsymbol{x}) + (x_1 + x_1^2) L^{k-1}(\boldsymbol{x}; \hat{x}_1)$$
$$+ \sum_{i=2}^{k} x_1 x_i [2L^{k-1}(\boldsymbol{x}; \hat{x}_1) + L_{i-1}^{k-1}(\boldsymbol{x}; \hat{x}_i) + L^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i) + L_i^k(\boldsymbol{x}) + L_{i-1}^k(\boldsymbol{x})]\Big];$$

if $k \geq 1$, then $L_k^k(\boldsymbol{x}) = D_k^k(\boldsymbol{x})$ and for $1 \leq j < k$,

$$L_j^k(\boldsymbol{x}) = \frac{1}{1 - x_1 x_{j+1}}\Big[D_j^k(\boldsymbol{x}) + \sum_{i=j+2}^{k} x_1 x_i L_{i-1}^k(\boldsymbol{x})$$
$$+ \sum_{i=j+1}^{k} x_1 x_i [2L^{k-1}(\boldsymbol{x}; \hat{x}_1) + L_{i-1}^{k-1}(\boldsymbol{x}; \hat{x}_i) + L^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i) + L_i^k(\boldsymbol{x})]\Big];$$

and if $k \geq 1$, then $D_1^k(\boldsymbol{x}) = \dfrac{x_1}{1-x_1} L^{k-1}(\boldsymbol{x}; \hat{x}_1)$ and for $1 \leq i < k$, we have

$$D_i^k(\boldsymbol{x}) = \frac{1}{1 - x_1 x_i}\Big[D_{i-1}^k(\boldsymbol{x}) + x_1 x_i [D_{i-1}^k(\boldsymbol{x}) + D_{i-1}^{k-1}(\boldsymbol{x}; \hat{x}_i) + L^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i) + 2L^{k-1}(\boldsymbol{x}; \hat{x}_1)]\Big].$$

In the extended abstract, we provide proof sketches due to the space limitations. To prove Theorem 3.1, we start with a few lemmas to establish the base cases and recurrences. For $\pi \in S_n$ with $\pi = \pi_1 \pi_2 \ldots \pi_n$ and $\sigma_m \in \mathcal{S}_m$ with $\sigma = \sigma_1 \sigma_2 \ldots \sigma_m$, we let $\pi \oplus \sigma \in \mathcal{S}_{n+m}$ denote the permutation

$$\pi \oplus \sigma = \pi_1 \pi_2 \ldots \pi_n (\sigma_1 + n)(\sigma_2 + n) \ldots (\sigma_m + n).$$

For example, if $\pi = 312 \in \mathcal{S}_3$ and $\sigma = 635421 \in \mathcal{S}_6$, then $\pi \oplus \sigma = 312968754 \in \mathcal{S}_9$.

Lemma 3.2. *If $n \geq 1$, then there are n unimodal involutions in $\mathcal{S}_n$. Consequently, the generating function $L^1(x) = \sum_{\pi \in \mathcal{I}^{(n)}} x^n$ is given by*

$$L^1(x) = \frac{x}{(1-x)^2}.$$

Proof Sketch. Clearly there is only one unimodal permutation of length 1 and it is an involution. We proceed by induction. For any $\pi \in \mathcal{I}^{(n)}$ with $n \geq 2$, either $\pi_1 = 1$ or $\pi_n = 1$. Notice that if $\pi_1 = 1$, then $\pi \in \mathcal{I}^{(n)}$ if and only if $\pi = 1 \oplus \sigma$ with $\sigma \in \mathcal{I}^{(n-1)}$. If $\pi_n = 1$ then necessarily $\pi_1 = n$, and thus π is the decreasing permutation which is indeed an involution. Therefore, $|\mathcal{I}^{(n)}| = |\mathcal{I}^{(n-1)}| + 1$, which in turn implies that $|\mathcal{I}^{(n)}| = n$, and thus the result follows. □

Clearly, the number of unimodal involutions in $\mathcal{S}_n$ that are strictly decreasing is 1 for each n and thus $D_1^1(\boldsymbol{x}) = \dfrac{x_1}{1-x_1}$. By the definition of $L_k^k(\boldsymbol{x})$, it is clear that we must have $L_k^k(\boldsymbol{x}) = D_k^k(\boldsymbol{x})$ for all $k \geq 1$.

Lemma 3.3. *For any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vDash n$, the number of λ-unimodal involutions for which the first λ_1 elements are decreasing and $\pi_1 \leq \lambda_1$ is equal to the number of λ'-unimodal involutions, where $\lambda' = (\lambda_2, \ldots, \lambda_k) \vDash n - \lambda_1$. Consequently, the generating function for these permutations is given by*

$$D_1^k(\boldsymbol{x}) = \frac{x_1}{1-x_1} L^{k-1}(\boldsymbol{x}; \hat{x}_1).$$

Proof Sketch. If the first λ_1 elements of a λ-unimodal involution π form a decreasing sequence and $\pi_1 \leq \lambda_1$, then $\pi_i = \lambda_1 - i + 1$ for all $i \in [\lambda_1]$. For any $\sigma \in \mathcal{I}^{\lambda'}$, we can obtain a λ-unimodal involution with the necessary property by taking $\pi = \delta_{\lambda_1} \oplus \sigma$, where δ_{λ_1} is the decreasing permutation of length λ_1. The result now follows. □

With the base cases of Theorem 3.1 established, we can now prove the recurrences given in Theorem 3.1. For convenience, we establish the following notation. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vDash n$, then let $\bar{\lambda} = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_k)$; when $\lambda_1 = 1$, we implicitly assume that $\bar{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_k)$. Also, let $\bar{\lambda}^1 = (\lambda_1 - 2, \lambda_2, \ldots, \lambda_k)$, and for $i \in \{2, 3, \ldots, k\}$, we let $\bar{\lambda}^i = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_k)$, where again if $\lambda_1 = 1$ or if $\lambda_i = 1$, we omit these terms from $\bar{\lambda}^i$ altogether. For example, if $\lambda = (4,3,1,2)$, then $\bar{\lambda} = (3,3,1,2)$, $\bar{\lambda}^1 = (2,3,1,2)$, $\bar{\lambda}^2 = (3,2,1,2)$, and $\bar{\lambda}^3 = (3,3,2)$. Finally, let

$$s_\lambda^i = \lambda_1 + \lambda_2 + \cdots + \lambda_i.$$

For example, if $\lambda = (4,2,6,1)$, then $s_\lambda^1 = 4$, $s_\lambda^2 = 6$, $s_\lambda^3 = 12$, and $s_\lambda^4 = 13$.

Lemma 3.4. *For $k \geq 2$, the generating function $L^k(\boldsymbol{x})$ satisfies the recurrence*

$$\begin{aligned} L^k(\boldsymbol{x}) = \frac{1}{1-x_1} \Bigg[& x_1^2 L_1^k(\boldsymbol{x}) + (x_1 + x_1^2) L^{k-1}(\boldsymbol{x}; \hat{x}_1) \\ & + \sum_{i=2}^{k} x_1 x_i [2L^{k-1}(\boldsymbol{x}; \hat{x}_1) + L_{i-1}^{k-1}(\boldsymbol{x}; \hat{x}_i) + L^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i) + L_i^k(\boldsymbol{x}) + L_{i-1}^k(\boldsymbol{x})] \Bigg]. \end{aligned}$$

Proof Sketch. We will establish the following equivalent formula:

$$\begin{aligned} L^k(\boldsymbol{x}) = {} & x_1 L^k(\boldsymbol{x}) + x_1^2 L_1^k(\boldsymbol{x}) + (x_1 + x_1^2) L^{k-1}(\boldsymbol{x}; \hat{x}_1) \\ & + \sum_{i=2}^{k} x_1 x_i [2L^{k-1}(\boldsymbol{x}; \hat{x}_1) + L_{i-1}^{k-1}(\boldsymbol{x}; \hat{x}_i) + L^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i) + L_i^k(\boldsymbol{x}) + L_{i-1}^k(\boldsymbol{x})]. \end{aligned}$$

Suppose that $\pi \in \mathcal{I}^\lambda$, and consider where 1 is in the permutation π. Notice that since π is λ-unimodal, 1 must occur at the beginning or end of a unimodal segment. That is, if $\pi_j = 1$, then $j \in \{\lambda_i : i \in [n]\} \cup \{\lambda_i + 1 : i \in [n-1]\} \cup \{1\}$. Two cases follow.

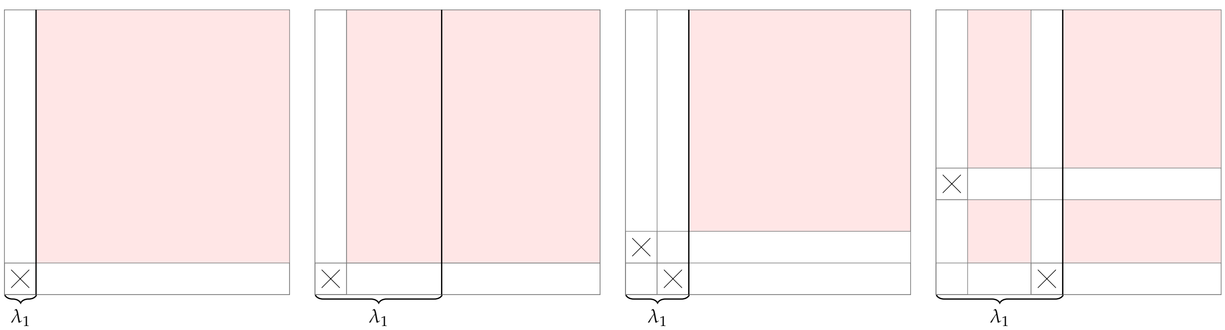


Figure 2: This figure illustrates the case in proof of Lemma 3.4 when 1 lies in the first segment (i.e. when $\pi_1 = 1$ or $\pi_{\lambda_1} = 1$). In the left-most picture, the $\times$ in the bottom left corner together with the shaded portion contribute $x_1 L^{k-1}(\boldsymbol{x}; \hat{x}_1)$. In the second picture, we get $x_1 L^k(\boldsymbol{x})$; in the third picture, we get $x_1^2 L^{k-1}(\boldsymbol{x}; \hat{x}_1)$; in the fourth picture we get $x_1^2 L_1^k(\boldsymbol{x})$.

First, assume that 1 lies in the first segment (of length λ_1) and consider the following four possible subcases, pictured in Figure 2. If $\lambda_1 = 1$, then we have $\pi_1 = 1$ and $\pi = 1 \oplus \sigma$, where σ is a $\bar{\lambda}$-unimodal involution. This contributes $x_1 L^{k-1}(\boldsymbol{x}; \hat{x}_1)$ to the sum. If $\lambda_1 > 1$ and $\pi_1 = 1$, then $\pi = 1 \oplus \sigma$, where σ is a $\bar{\lambda}$-unimodal involution, which contributes $x_1 L^k(\boldsymbol{x})$ to the sum. If $\lambda_1 = 2$ and $\pi_2 = 1$, then we necessarily have that $\pi_1 = 2$ since π is an involution. Hence we must have $\pi = 21 \oplus \sigma$, where σ is a $\bar{\lambda}^1$-unimodal involution. This contributes $x_1^2 L^{k-1}(\boldsymbol{x}; \hat{x}_1)$ to the sum. Finally, if $\lambda_1 > 2$ and $\pi_{\lambda_1} = 1$, then we must have $\pi_1 = \lambda_1$ and in turn $\pi = \lambda_1 \alpha 1 \beta$, where the permutation $\sigma = \alpha\beta$ is order-isomorphic to a $\bar{\lambda}^1$-unimodal involution with the added condition that $\sigma_1 > \bar{\lambda}_1^1$ or $\sigma_1 \leq \bar{\lambda}_1^1$ and $\sigma_1 \ldots \sigma_{\bar{\lambda}_1^1}$ is decreasing. This contributes $x_1^2 L_1^k(\boldsymbol{x})$ to the sum.

Now assume that 1 lies in the i-th segment with $i > 1$, and consider the six subcases pictured in Figure 3. If $\lambda_1 = \lambda_i = 1$ and $\pi(s_\lambda^i) = 1$, then we must have $\pi = s_\lambda^i \alpha 1 \beta$, where the permutation $\sigma = \alpha\beta$ is order-isomorphic to a $\bar{\lambda}^i$-unimodal involution. This contributes $x_1 x_i L^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i)$ to the sum for each $i > 1$. If $\lambda_1 > 1$ and $\lambda_i = 1$ (and thus $\pi(s_\lambda^i) = 1$), then we must have that $\pi = s_\lambda^i \alpha 1 \beta$, where the permutation $\sigma = \alpha\beta$ is

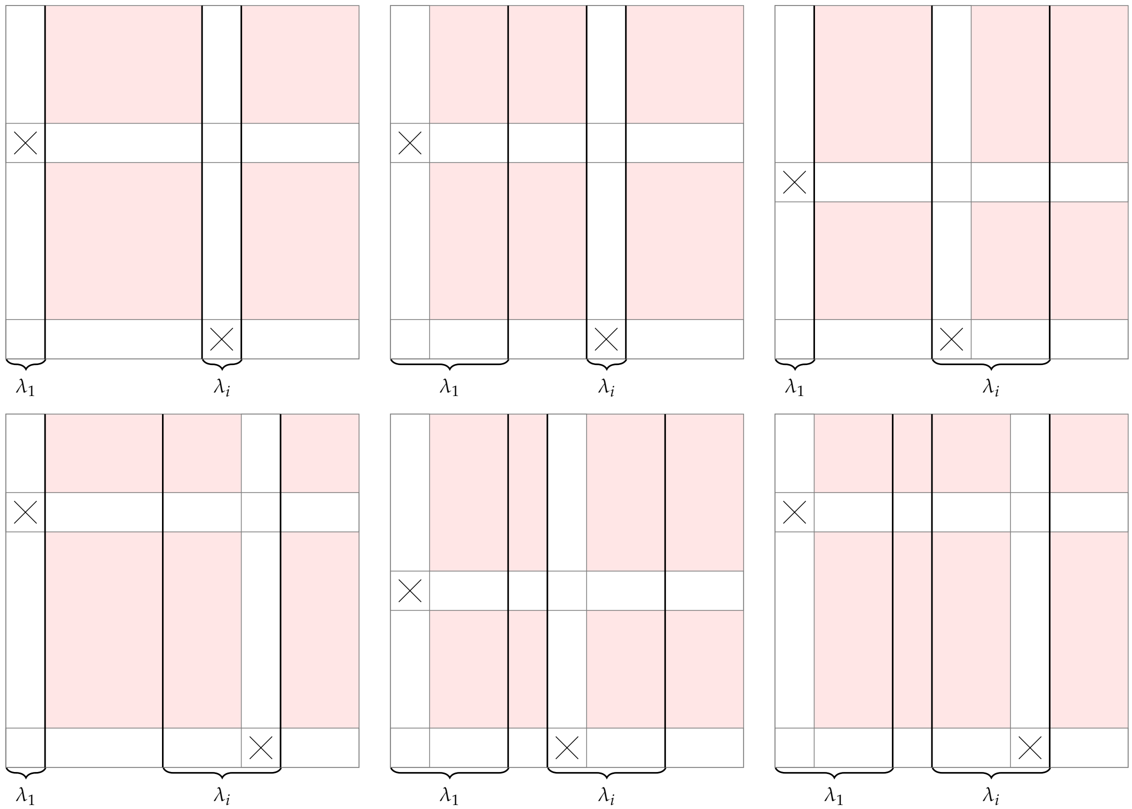


Figure 3: This figure illustrates the case in the proof of Lemma 3.4 when 1 lies in the i-th segment with $i > 1$. From left-to-right along each row starting at the top left picture, these figures illustrate the following terms of the recurrence: $x_1x_iL^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i)$, $x_1x_iL^{k-1}_{i-1}(\boldsymbol{x}; \hat{x}_i)$, $x_1x_iL^{k-1}(\boldsymbol{x}; \hat{x}_1)$, $x_1x_iL^{k-1}(\boldsymbol{x}; \hat{x}_1)$, $x_1x_iL^{k}_{i-1}(\boldsymbol{x})$, and $x_1x_iL^{k}_{i}(\boldsymbol{x})$.

order-isomorphic to a $\bar{\lambda}^i$-unimodal involution with the added condition that $\sigma_1 > s^{i-1}_{\bar{\lambda}^i}$ or $\sigma_1 \leq s^{i-1}_{\bar{\lambda}^i}$ and $\sigma_1 \dots \sigma_{\bar{\lambda}^i_1}$ is decreasing. Thus, this contributes $x_1x_iL^{k-1}_{i-1}(\boldsymbol{x}; \hat{x}_i)$ to the sum for each $i > 1$. If $\lambda_1 = 1$ and $\lambda_i > 1$, then either $\pi(s^i_\lambda) = 1$ or $\pi(s^{i-1}_\lambda + 1) = 1$. In the former case, we have that $\pi = s^i_\lambda \alpha 1 \beta$, where the permutation $\sigma = \alpha\beta$ is order-isomorphic to a $\bar{\lambda}^i$-unimodal involution and in the latter case, we must have that $\pi = (s^{i-1}_\lambda + 1)\alpha 1 \beta$, where the permutation $\sigma = \alpha\beta$ is order-isomorphic to a $\bar{\lambda}^i$-unimodal involution. Together, these contribute $2x_1x_iL^{k-1}(\boldsymbol{x}; \hat{x}_1)$ to the sum for each $i > 1$. Finally, we have the subcase when $\lambda_1 > 1$ and $\lambda_i > 1$. In this instance, we must have either $\pi(s^{i-1}_\lambda + 1) = 1$ or $\pi(s^i_\lambda) = 1$. If $\pi(s^{i-1}_\lambda + 1) = 1$, then $\pi = (s^{i-1}_\lambda + 1)\alpha 1 \beta$, where the permutation $\sigma = \alpha\beta$ is order-isomorphic to a $\bar{\lambda}^i$-unimodal involution with the added condition that $\sigma_1 > s^{i-1}_{\bar{\lambda}^i}$ or $\sigma_1 \leq s^{i-1}_{\bar{\lambda}^i}$ and $\sigma_1 \dots \sigma_{\bar{\lambda}^i_1}$ is decreasing. If $\pi(s^i_\lambda) = 1$, then $\pi = s^i_\lambda \alpha 1 \beta$, where the permutation $\sigma = \alpha\beta$ is order-isomorphic to a $\bar{\lambda}^i$-unimodal involution with the added condition that $\sigma_1 > s^i_{\bar{\lambda}^i}$ or $\sigma_1 \leq s^i_{\bar{\lambda}^i}$ and $\sigma_1 \ldots \sigma_{\bar{\lambda}^i_1}$ is decreasing. Together, these contribute $x_1 x_i [L^k_{i-1}(\boldsymbol{x}) + L^k_i(\boldsymbol{x})]$ to the sum for each $i > 1$. $\square$

Lemma 3.5. *For $1 \leq j \leq k$, the generating function $L^k_j(\boldsymbol{x})$ satisfies the recurrence*

$$L^k_j(\boldsymbol{x}) = \frac{1}{1 - x_1 x_{j+1}} \Bigg[D^k_j(\boldsymbol{x}) + \sum_{i=j+2}^{k} x_1 x_i L^k_{i-1}(\boldsymbol{x}) + \sum_{i=j+1}^{k} x_1 x_i [L^k_i(\boldsymbol{x}) + 2L^{k-1}(\boldsymbol{x}; \hat{x}_1) + L^{k-1}_{i-1}(\boldsymbol{x}; \hat{x}_i) + L^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i)] \Bigg].$$

Proof Sketch. We will establish the equivalent formula:

$$L^k_j(\boldsymbol{x}) = D^k_j(\boldsymbol{x}) + \sum_{i=j+1}^{k} x_1 x_i [L^k_{i-1}(\boldsymbol{x}) + L^k_i(\boldsymbol{x}) + 2L^{k-1}(\boldsymbol{x}; \hat{x}_1) + L^{k-1}_{i-1}(\boldsymbol{x}; \hat{x}_i) + L^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i)].$$

The proof is similar to the one for Lemma 3.4, so we omit some of the details; the associated figures are also very similar to those in Figure 3. Let $\pi \in \mathcal{I}^\lambda$, where $\pi_1 \leq s^j_\lambda$ and $\pi_1 \ldots \pi_{\lambda_1}$ is decreasing or $\pi_1 > s^j_\lambda$. In the first case, we get exactly those permutations enumerated using the generating function $D^k_j(\boldsymbol{x})$. Otherwise, we must have that $\pi_1 > s^j_\lambda$, which implies that if $\pi_k = 1$, then $k > s^j_\lambda$. Thus for any $i \geq j+1$, we can add i either to the beginning or end of the i-th unimodal segment. Again, there are six cases. The case when $\lambda_1 = \lambda_i = 1$ contributes $x_1 x_i L^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i)$ to the sum for each $i \geq j+1$. In the case where $\lambda_1 = 1$ and $\lambda_i > 1$, we can either have $\pi(s^i_\lambda) = 1$ or $\pi(s^{i-1}_\lambda + 1) = 1$. This contributes $2x_1 x_i L^{k-1}(\boldsymbol{x}; \hat{x}_1)$ to the sum for each $i \geq j+1$. The case when $\lambda_1 > 1$ and $\lambda_i = 1$ contributes $x_1 x_i L^{k-1}_{i-1}(\boldsymbol{x}; \hat{x}_i)$ for each $i \geq j+1$. In the case when $\lambda_1 > 1$ and $\lambda_i > 1$, we can either have $\pi(s^i_\lambda) = 1$ or $\pi(s^{i-1}_\lambda + 1) = 1$. This contributes $x_1 x_i [L^k_{i-1}(\boldsymbol{x}) + L^k_i(\boldsymbol{x})]$ to the sum for each $i \geq j+1$. $\square$

Lemma 3.6. *For $1 < i \leq k$, the generating function $D^k_i(\boldsymbol{x})$ satisfies the recurrence*

$$D^k_i(\boldsymbol{x}) = \frac{1}{1 - x_1 x_i} \left[D^k_{i-1}(\boldsymbol{x}) + x_1 x_i [D^k_{i-1}(\boldsymbol{x}) + D^{k-1}_{i-1}(\boldsymbol{x}; \hat{x}_i) + L^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i) + 2L^{k-1}(\boldsymbol{x}; \hat{x}_1)] \right].$$

Proof Sketch. We will establish the equivalent formula:

$$D^k_i(\boldsymbol{x}) = D^k_{i-1}(\boldsymbol{x}) + x_1 x_i [D^k_i(\boldsymbol{x}) + D^k_{i-1}(\boldsymbol{x}) + D^{k-1}_{i-1}(\boldsymbol{x}; \hat{x}_i) + L^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i) + 2L^{k-1}(\boldsymbol{x}; \hat{x}_1)].$$

Again, because of similarities to the proof of Lemma 3.4, we omit most of the details. The associated figures can be found in Figure 4.

Let $\pi \in \mathcal{I}^\lambda$, where $\pi_1 \leq s^i_\lambda$ and $\pi_1 \ldots \pi_{\lambda_1}$ is decreasing. In the case when $\pi_1 \leq s^{i-1}_\lambda$, we have exactly those permutations enumerated by D^k_{i-1}. If $s^{i-1}_\lambda < \pi_1 \leq s^i_\lambda$, then

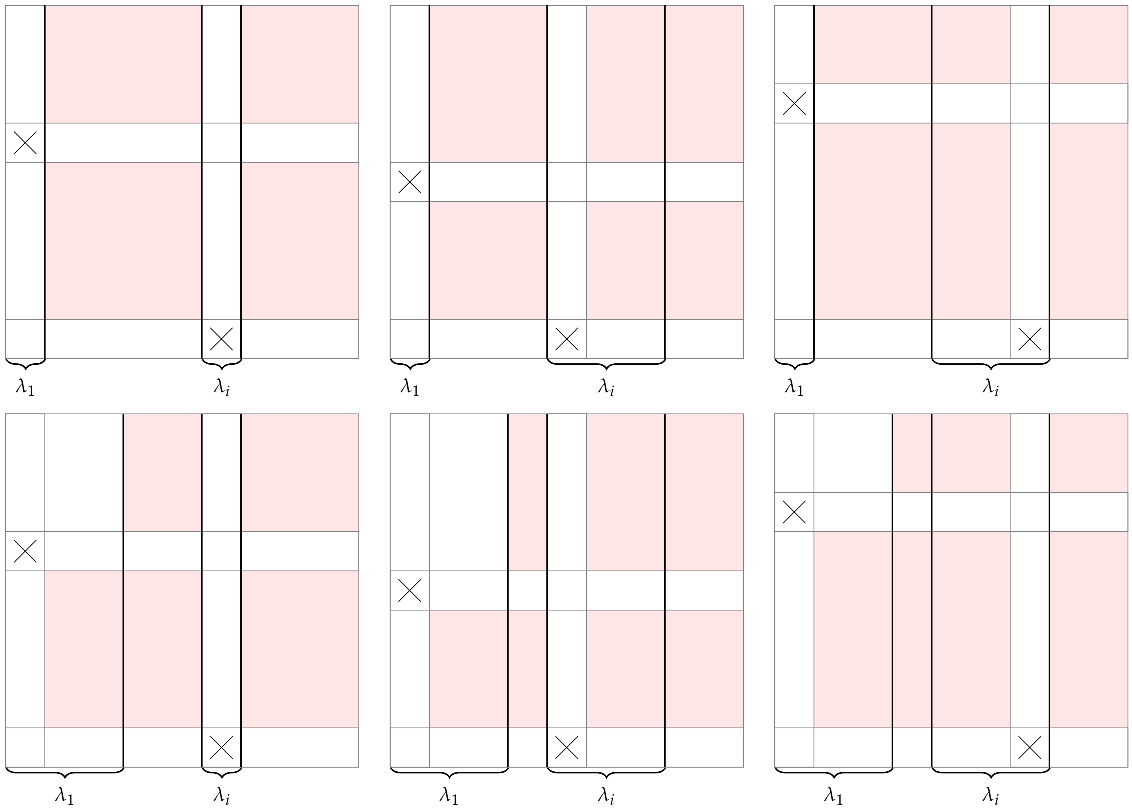


Figure 4: This figure illustrates Lemma 3.6 when $s_\lambda^{i-1} < \pi_1 \leq s_\lambda^i$. From left-to-right along each row starting at the top left picture, these figures illustrate the following terms of the recurrence: $x_1x_iL^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i)$, $x_1x_iL^{k-1}(\boldsymbol{x}; \hat{x}_1)$, $x_1x_iL^{k-1}(\boldsymbol{x}; \hat{x}_1)$, $x_1x_iD_{i-1}^{k-1}(\boldsymbol{x}; \hat{x}_i)$, $x_1x_iD_{i-1}^{k}(\boldsymbol{x})$, and $x_1x_iD_i^k(\boldsymbol{x})$.

we must have that $\pi_1 = s_\lambda^i$ or $\pi_1 = s_\lambda^{i-1} + 1$. The case when $\lambda_1 = \lambda_i = 1$ contributes $x_1x_iL^{k-2}(\boldsymbol{x}; \hat{x}_1, \hat{x}_i)$ to the sum. The case when $\lambda_1 = 1$ and $\lambda_i > 1$ contributes $2x_1x_iL^{k-1}(\boldsymbol{x}; \hat{x}_1)$ to the sum. The case when $\lambda_1 > 1$ and $\lambda_i = 1$ contributes $x_1x_iD_{i-1}^{k-1}(\boldsymbol{x}; \hat{x}_i)$ to the sum. Finally, the case when $\lambda_1 > 1$ and $\lambda_i > 1$ contributes $x_1x_i[D_i^k(\boldsymbol{x}) + D_{i-1}^k(\boldsymbol{x})]$ to the sum. □

4 The Gelfand character

Let $G^k(\boldsymbol{x})$ be defined to be as follows:

$$G^k(\boldsymbol{x}) = \sum_\lambda \chi_\lambda^G x^\lambda,$$

where χ^G is the Gelfand character mentioned in the introduction, λ is a partition of length k, and χ_λ^G is the value the character takes on the conjugacy class given by λ. Then

$G^k(\boldsymbol{x})$ can be computed recursively from itself and two other series, G_j^k and H_i^k, both defined in the next theorem.

Theorem 4.1. *We have $G^0(\boldsymbol{x}) = 1$, $G^1(\boldsymbol{x}) = \dfrac{x_1}{1-x_1^2}$ and for $k \geq 2$,*

$$G^k(\boldsymbol{x}) = \frac{1}{1-x_1}\Bigg[(x_1 - x_1^2)G^{k-1}(\boldsymbol{x};\hat{x}_1) - x_1^2(G_1^k(\boldsymbol{x}) - 2H_1^k(\boldsymbol{x})) + \sum_{i=2}^{k} x_1x_i[G^{k-2}(\boldsymbol{x};\hat{x}_1,\hat{x}_i) + G_{i-1}^{k-1}(\boldsymbol{x};\hat{x}_i) - 2H_{i-1}^{k-1}(\boldsymbol{x};\hat{x}_i) - G_i^k(\boldsymbol{x}) + 2H_i^k(\boldsymbol{x}) + G_{i-1}^k(\boldsymbol{x}) - 2H_{i-1}^k(\boldsymbol{x})]\Bigg];$$

if $k \geq 1$, then $G_k^k(\boldsymbol{x},t) = H_k^k(\boldsymbol{x},t)$ and for $1 \leq j < k$,

$$G_j^k(\boldsymbol{x}) = \frac{1}{1-x_1x_{j+1}}\Bigg[H_j^k(\boldsymbol{x}) + \sum_{i=j+2}^{k} x_1x_iG_{i-1}^k(\boldsymbol{x}) + \sum_{i=j+1}^{k} x_1x_i[G^{k-2}(\boldsymbol{x};\hat{x}_1,\hat{x}_i) + G_{i-1}^{k-1}(\boldsymbol{x};\hat{x}_i) - 2H_{i-1}^{k-1}(\boldsymbol{x};\hat{x}_i) - G_i^k(\boldsymbol{x}) + 2H_i^k(\boldsymbol{x}) - 2H_{i-1}^k(\boldsymbol{x})]\Bigg];$$

and if $k \geq 1$, then $H_1^k(\boldsymbol{x}) = \dfrac{x_1}{1+x_1}G^{k-1}(\boldsymbol{x};\hat{x}_1)$ and for $1 \leq i < k$, we have

$$H_i^k(\boldsymbol{x}) = \frac{1}{1-x_1x_i}\Bigg[H_{i-1}^k(\boldsymbol{x}) + x_1x_i[G^{k-2}(\boldsymbol{x};\hat{x}_1,\hat{x}_i) - H_{i-1}^k(\boldsymbol{x}) - H_{i-1}^{k-1}(\boldsymbol{x};\hat{x}_i)]\Bigg].$$

Proof Sketch. The results of Section 3 can quickly be extended to include λ-descents. Let $\mathcal{I}^\lambda(d)$ be the set of permutations $\pi \in \mathcal{I}^\lambda$ such that $\mathrm{des}_\lambda(\pi) = d$. Similarly, let $\mathcal{I}_j^\lambda(d)$ be the set of permutations $\pi \in \mathcal{I}_j^\lambda$ such that $\mathrm{des}_\lambda(\pi) = d$, and let $\mathcal{D}_i^\lambda(d)$ be the set of permutations $\pi \in \mathcal{D}_i^\lambda$ such that $\mathrm{des}_\lambda(\pi) = d$. Let

$$L^k(\boldsymbol{x},t) = \sum_{\lambda\in\Lambda_k}\sum_{d\geq 0}|\mathcal{I}^\lambda(d)|x^\lambda t^d, \qquad L_j^k(\boldsymbol{x},t) = \sum_{\lambda\in\Lambda_k}\sum_{d\geq 0}|\mathcal{I}_j^\lambda(d)|x^\lambda t^d,$$

$$\text{and} \qquad D_i^k(\boldsymbol{x},t) = \sum_{\lambda\in\Lambda_k}\sum_{d\geq 0}|\mathcal{D}_i^\lambda(d)|x^\lambda t^d.$$

We can keep track of descents by paying attention to where we add new elements. For example, consider Figure 2. In the first three cases, no descents would be added. However, in the last case, we would get an extra two descents if the first segment (of length λ_1) were decreasing (one descent in position 1 and one descent in position $\lambda_1 - 1$). Now consider Figure 3. Cases 1 and 3 add no descents; cases 2, 3, and 4 add one descent;

and case 6 adds two descents. This gives us the formula:

$$\begin{aligned}L^k(\boldsymbol{x},t) = \frac{1}{1-x_1}\Big[&(x_1+x_1^2t)L^{k-1}(\boldsymbol{x},t;\hat{x}_1) + x_1^2t(L_1^k(\boldsymbol{x},t)+(t-1)D_1^k(\boldsymbol{x},t)) \\ &+\sum_{i=2}^{k} x_1x_i[(t+1)L^{k-1}(\boldsymbol{x},t;\hat{x}_1) + L_{i-1}^{k-1}(\boldsymbol{x},t;\hat{x}_i) + L_{i-1}^k(\boldsymbol{x},t) + (t-1)D_{i-1}^k(\boldsymbol{x},t) \\ &+(t-1)D_{i-1}^{k-1}(\boldsymbol{x},t;\hat{x}_i) + L^{k-2}(\boldsymbol{x},t;\hat{x}_1,\hat{x}_i) + t(L_i^k(\boldsymbol{x},t)+(t-1)D_i^k(\boldsymbol{x},t))]\Big];\end{aligned}$$

the formulas for $L_j^k(\boldsymbol{x},t)$ and $D_i^k(\boldsymbol{x},t)$ are computed in a similar fashion. Notice that by Equation (1.1),

$$L^k(\boldsymbol{x},-1) = \sum_{\lambda}\chi_\lambda^G x^\lambda = G^k(\boldsymbol{x}).$$

Thus, we can obtain the generating functions listed in the theorem. □

Below are the first few terms of $G^k(\boldsymbol{x})$ for $k \in \{1,2,3\}$ as computed from the formulas in Theorem 4.1.

$$\begin{aligned}G^1(x) &= x+x^3+x^5+x^7+x^9+x^{11}+x^{13}+x^{15}+x^{17}+x^{19}+x^{21}+x^{23}+x^{25}+\cdots \\ G^2(x,y) &= 2xy+xy^3+2x^2y^2+x^3y+xy^5+4x^3y^3+x^5y+xy^7+x^3y^5+4x^4y^4+\cdots \\ G^3(x,y,z) &= 4xyz+2xyz^3+2xy^2z^2+2x^2yz^2+2x^2y^2z+2x^3yz+2xy^3z+4z^3yz^3+\cdots\end{aligned}$$

Notice that for each $k \geq 1$, $G^k(\boldsymbol{x})$ is symmetric in its k variables. This must happen since Equation (1.1) holds for any ordering of the composition λ. Therefore, if we would like to compute $\chi_{(1,1,3)}^G$, we can take either the coefficient of xyz^3, xy^3z, or x^3yz in $G^k(\boldsymbol{x})$ as our answer. In each case, we find that $\chi_{(1,1,3)}^G = 2$.

5 Discussion

It remains to determine the computational complexity for computing the Gelfand character using this method and to compare it with known methods (as in [7, 8, 11]).

In addition, several other characters can be realized by studying certain properties of λ-unimodal permutations. In particular, if a set $B(n) \subseteq \mathcal{S}_n$ is a so-called *fine set* (see for example, [2]), then

$$\chi_\lambda = \sum_{\pi \in B(n)\cap\mathcal{L}^\lambda} (-1)^{\mathrm{des}_\lambda(\pi)}$$

where χ is a character of some representation of $\mathcal{S}_n$ and $\mathcal{L}^\lambda$ is the set of λ-unimodal permutations. Fine sets include conjugacy classes and their unions, Knuth classes, Coxeter length, and more [2]. It should be possible to employ techniques similar to the ones found in [3] and here to find combinatorial proofs for the formulas of these characters.

Acknowledgements

The authors would like to thank the University of Texas at Tyler's Office of Sponsored Research and Center for Excellence in Teaching and Learning for their support in conducting this research. The awards from these offices supported the research conducted for this paper by two faculty members, Kassie Archer and L.-K. Lauderdale, together with undergraduate students Marin King, Thomas Lupo, and Francesca Rossi and graduate students Angela Gay and Virginia Germany.

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