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Bijective proof of the rationality of the generating series of higher-genus maps

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Abstract. Bender and Canfield proved in 1991 that the generating series of maps in higher genus is a rational function of the generating series of planar maps. In this paper, we give the first bijective proof of this result. Our approach starts with the introduction of a canonical orientation that enables us to construct a bijection between 4-valent bicolorable maps and a family of unicellular blossoming maps.

Keywords: bijective combinatorics, maps, higher genus, rational parametrization

1 Introduction

A *map* of genus g is a proper embedding of a graph in $\mathbb{S}_g$, the torus with g holes. Planar maps (or maps of genus 0) have been studied extensively since the pioneering work of Tutte in the sixties [23]. In a series of work, Tutte obtained remarkable formulas for many families of maps. His techniques rely on some recurrence relations for maps and some clever manipulations of generating series. They were extended in the late eighties to the case of maps of higher genus by Bender and Canfield, who first obtained the asymptotic number of maps on any orientable surface of genus g [3] and then obtained in [2] the following stronger result:

Theorem 1.1 (Bender and Canfield [2])**.** *For any $g \geq 0$, the generating series $M_g(z)$ of maps of genus g enumerated by edges is a rational function of z and $\sqrt{1-12z}$.*

Since then, many other approaches have been developed, illustrating deep connections of maps with various fields of algebra and mathematical physics (*e.g.* [22, 16, 13]).

The main purpose of this paper is to provide the first bijective proof of Theorem 1.1, for $g \geq 2$. Our proof starts with the well-known bijection between general maps and so-called *4-valent bicolorable maps*. In the planar case, Schaeffer exhibits in [20] a constructive bijection between 4-valent planar maps and some so-called *blossoming trees*. The blossoming tree associated to a map is one of its spanning trees, decorated by some stems that enable to reconstruct the "missing edges". In genus $g > 0$, the natural counterpart of trees are unicellular maps (*i.e.* maps with only one face) and we obtain in this work the following generalization of Schaeffer's result (see Section 3.1 for the terminology):

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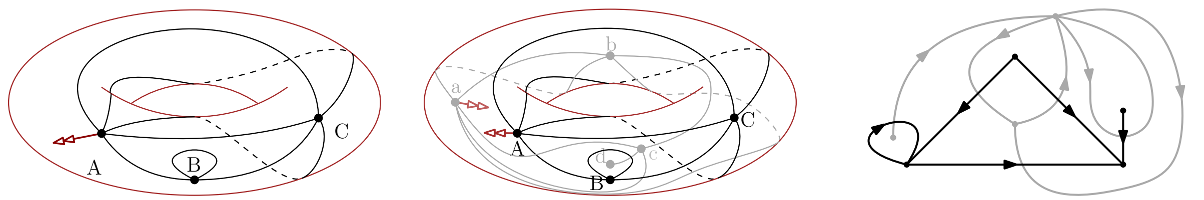
Theorem 1.2. *There exists a constructive bijection between rooted maps of genus g with n edges and well-rooted well-labeled well-oriented 4-valent unicellular blossoming maps with n vertices.*

The enumeration of maps now boils down to the much easier enumeration of this family of unicellular maps. As a byproduct we obtain a bijective proof of Theorem 1.1.

Let us now put our work in context of the existing literature. In the planar case, there are numerous bijections between maps and some families of decorated trees. Two main trends emerge in these bijections. Either the decorated trees are some blossoming trees as already described (*e.g.* [20, 8, 18]) or the trees are decorated by some integers that capture some metric properties of the maps (*e.g.* [21, 9]). Bijections of the latter type have been successfully extended to higher genus [11, 10, 17]. Part of our work is in fact directly inspired by the paper of Chapuy, Marcus and Schaeffer [11]. In particular, the analysis of the unicellular maps obtained by our bijection is very similar to theirs. Unfortunately, with their bijection the generating series of maps can be expressed as a rational function of some auxiliary functions whose degree of algebraicity is higher than the known enumerative results. The case of bijections with blossoming trees is much different and apart from the recent work [12] which treats simple triangulations of genus 1, there was, previously to our work, no other extension of the existing bijections in higher genus.

Let us end with an important connection to our work. As shown by Bernardi [4] in the planar case and generalized by Bernardi and Chapuy [5], a map endowed with a spanning unicellular embedded graph (whose genus can be smaller than the genus of the initial surface) can also be viewed as a map endowed with an orientation of its edges with specific properties. The general theory of α-orientations developed by Felsner in the planar case [14] has been successfully combined with the result of [4] to give general bijective schemes in the planar case [6, 7, 1], which enables to recover the previously known bijections. It would be highly desirable to obtain systematic bijective schemes in higher genus by combining Bernardi and Chapuy's result together with the theory of c-orientations introduced by Propp [19] or its extension by Felsner and Knauer [15]. The main difficulty to tackle is to characterize the orientations that produce spanning unicellular embedded graph whose genus matches the genus of the original surface. Our work, presented here only in the case of bicolorable 4-valent maps for lack of space, but which can be extended straightforwardly to all bicolorable maps, can also be seen as an important first step in that direction and we hope to be able to extend to other families of maps in some future work.

Notation: In this article, combinatorial families are named with calligraphic letters, their generating series is the corresponding capital letter, and an object of the family, is usually denoted by the corresponding lower case letter. The size being denoted by $|\centerdot|$, we therefore have for a combinatorial family $\mathcal{S}$: $S(z) = \sum_{s\in\mathcal{S}} z^{|s|}$.



(a) A rooted map of genus 1

(b) The same map (grey) and its dual (black)

(c) An oriented planar map (black) with its dual (grey).

Figure 1: Examples of maps. The root corner is indicated by a double-arrow.

2 Orientations in higher genus

2.1 General

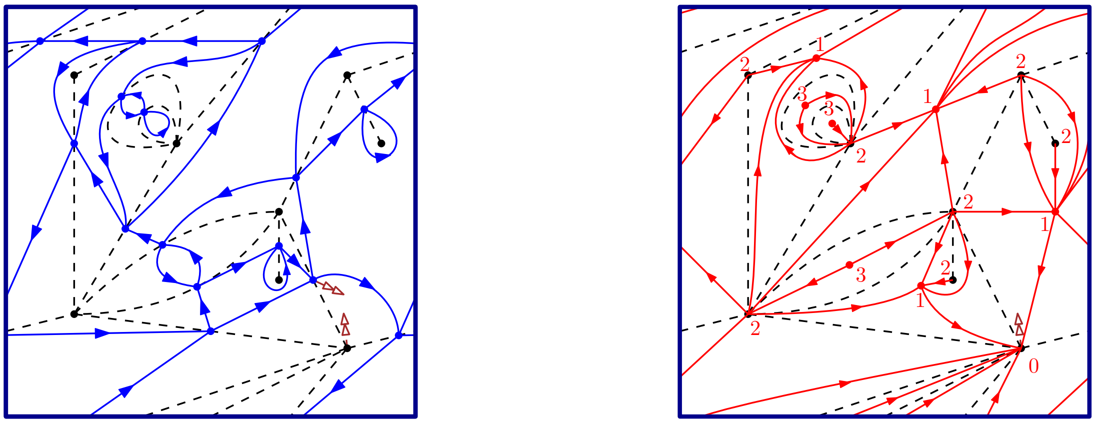
We begin with some definitions about maps. An *embedded graph* is an embedding of a connected graph into a given compact surface, taken up to orientation-preserving homeomorphism of the surface. An embedded graph is *cellularly embedded* if all its *faces* (connected component of the complement) are homeomorphic to discs. A *map* is a cellularly embedded graph. The set of maps, counted by number of edges, is denoted $\mathcal{M}$. In this paper we only consider maps embedded on orientable surfaces. *General maps* have no other restriction, and in particular, can have loops or multiple edges. The *genus* of a map is the genus of its underlying surface. All families of maps can be refined by their genus; we denote this refinement by an index indicating the genus, so that for instance $\mathcal{M}_0$ is the set of planar maps. See Figure 1a for an example of a map of genus 1.

An adjacency between a face and a vertex is called a *corner*. Note that a single pair vertex-face can give rise to several distinct corners. The *degree* of a face (resp. vertex) is the number of adjacent corners. To get rid of automorphisms, maps are *rooted* at a distinguished *root corner* (whose vertex and face are called *root vertex* and *root face*).

The set of vertices (resp. edges, faces) of m is denoted $\mathcal{V}_m$ (resp. $\mathcal{E}_m$, $\mathcal{F}_m$). The number of vertices (resp. edges, faces) of m is denoted v_m (resp. e_m, f_m). The genus of m is denoted g_m. We recall Euler's formula: $v_m - e_m + f_m = 2 - 2g_m$.

Since an edge connects two vertices and separates two faces, we can define the *dual map* $m^\star$ of m by exchanging the role of vertices and faces, and swapping the connection and separation induced by each edge (see Figure 1b). The root corner remains the same (but its vertex and its face are exchanged). Note that duality is involutive: $(m^\star)^\star = m$.

A map is *unicellular* if it has only one face. A *tree* is a map whose underlying graph has no cycle. A map is 4-*valent* if all its vertices are of degree 4. Dually, a map is a quadrangulation if all its faces are of degree 4. A map is *bipartite* if its underlying graph is bipartite, which means that its vertices can be properly colored black and white. In particular, a bipartite map has no loop. Dually, a map is *bicolorable* if its faces can be properly



(a) A map (dashed black) with its (4-valent bicolorable) radial map (full blue), with dual-geodesic orientation.

(b) A map (dashed black) with its (bipartite quadrangulation) quadrangulated map (full red), with geodesic orientation.

Figure 2: Classical constructions on a toroidal map.

colored black and white. The set of bipartite quadrangulations (resp. bicolorable 4-valent maps), counted by number of faces (resp. number of vertices), is denoted $\mathcal{BQ}$ (resp. $\mathcal{BF}$).

A map is *Eulerian* if all its vertices have even degree. In the sphere, Eulerian is equivalent to bicolorable. It is not the case anymore in higher genus, where bicolorability is more relevant. Indeed, in addition to having a nicer dual property, it also appears in the following well-known bijection, illustrated in Figure 2a:

Proposition 2.1. *Maps of genus g with n edges are in bijection with 4-valent bicolorable maps of genus g with n vertices, or dually, with bipartite quadrangulations of genus g with n faces.*

The 4-valent bicolorable map corresponding to a given map is called its *radial map*, while the bipartite quadrangulation is its *quadrangulated map*.

2.2 Structure of orientations of a graph

An *orientation* of a map is an orientation of each of its edges. The *dual orientation* of an orientation r of a map m is the orientation of $m^\star$ where all dual edges are oriented from the face to the right of the primal edge toward the face to its left (see Figure 1c). Note that applying duality twice reverses the orientation. A face is called *clockwise* (resp. *counterclockwise*) if it is to the right (resp. left) of all its adjacent edges.

Orientations provides additional structural properties to maps, useful for algorithmic purposes. However, since our final purpose is to study maps without an orientation, it is convenient to assign a canonical orientation to maps. Such a canonical orientation is obtained as the minimum of a lattice of orientations, as described below.

The *geodesic orientation* of a bipartite rooted map is the orientation whose edges are all oriented toward the root in terms of graph distance. Along any cycle, forward (resp. backward) edges in this orientation correspond to a distance to the root increasing (resp. decreasing) by 1. The geodesic orientation thus belongs to the set of *bipartite orientations*, in which there are as many forward edges as backward edges along any cycle. This set is endowed with the *vertex-push* operation, that changes a sink distinct from the root into a source, by reversing all adjacent edges. Dually, we call *bicolorable orientation* the dual of a bipartite orientation, and *face-flip* the dual of a vertex-push. The next result follows from [19, Theorem 1].

Theorem 2.2. *The set of bipartite orientations of a fixed map is a distributive lattice whose cover relation is the vertex-push operation, and whose minimum is the geodesic orientation.*

Dually, the set of bicolorable orientations of a fixed map is a distributive lattice whose cover relation is the face-flip operation, and whose minimum is the dual of the geodesic orientation.

Corollary 2.3. *The dual of the geodesic orientation (called* dual-geodesic orientation *for short, see Figure 3) is the unique bicolorable orientation with no clockwise face except for the root-face.*

3 Closing and opening maps

3.1 The closure of a blossoming map

A *blossoming* oriented map b is an oriented map with stems attached to its corners. These stems are oriented; an outgoing stem is called a *bud* and an ingoing stem is called a *leaf*. A blossoming map must have as many buds as leaves. The size of a blossoming map is the total number of stems. Blossoming maps are rooted on a bud. A stem is called *rootable* if it is either a leaf or the root bud.

In a blossoming oriented map, the *interior degree* (resp. *blossoming degree*, resp. *degree*) of a vertex is the number of edges adjacent to the vertex (resp. the number of half-edges attached to the vertex, resp. the sum of the interior and blossoming degrees). These can be refined into *ingoing* and *outgoing* degrees.

A blossoming oriented map is said to be *well-labeled* if its corners are labeled so that:

- the labels of two corners adjacent around a vertex differ by 1, in which case the higher label is to the left of the separating edge or stem,
- the labels of two corners adjacent along an edge coincide, and
- the root bud has labels 0 and 1.

In particular, the orientation of a well-labeled map is *Eulerian* (which means that the indegree of each vertex is equal to its outdegree).

Let b be a unicellular blossoming map. The *contour word* of b is defined as follows: when doing a clockwise tour of the unique face (which means that the face is to the

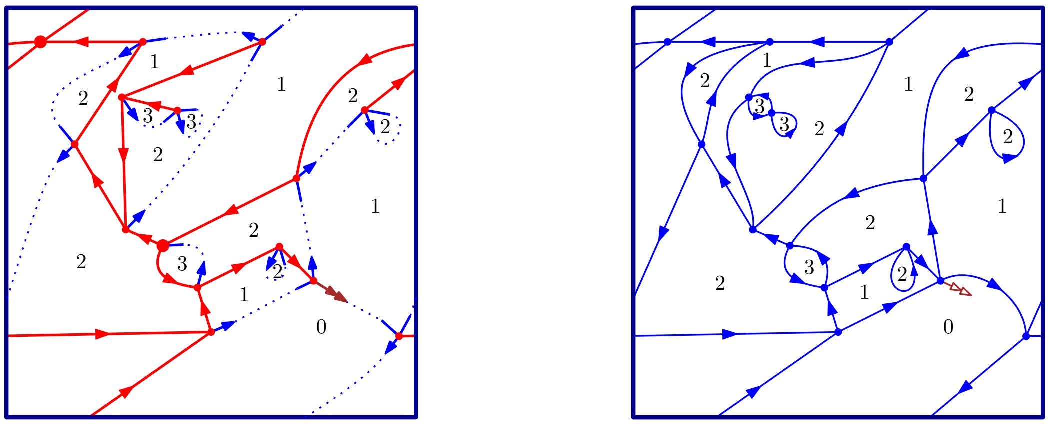


Figure 3: Example of the bijections between a unicellular blossoming map of $\mathcal{O}$ (left) and a 4-valent bicolorable map of $\mathcal{BF}$ with its dual-geodesic orientation (right).

right), starting from the root bud, write U (for up-step) for each bud and D (for down-step) for each leaf. We say that b is *well-rooted* if its contour word is a Dyck path.

The map b is *well-oriented* if in a tour of the face starting from the root, each edge is first followed backward and then forward. Note that this does not depend on the direction of the tour. In the case of a tree, this means that it is oriented toward the root.

Unicellular blossoming maps are interesting because they are easier to analyse than maps, but still can encode a map, thanks to the *closing algorithm*, defined hereafter. Let b be a well-rooted unicellular blossoming map. We write the contour word of b and match its steps into pairs up-step/down-step; each up-step U going from height i to $i+1$ is matched to the first down-step D after U going from height $i+1$ to i. The half-edges corresponding to matched steps are then merged into a single oriented edge. An example of the closing algorithm is given in Figure 3, from left to right.

Lemma 3.1. *The closure of a well-rooted well-labeled well-oriented* 4*-valent unicellular blossoming map of genus* g *is a rooted* 4*-valent bicolorable map of genus* g *with dual-geodesic orientation.*

To prove that the resulting orientation is indeed the dual-geodesic orientation, we prove that it is bicolorable and has no clockwise face (see Corollary 2.3). The set of well-rooted well-labeled well-oriented 4-valent unicellular blossoming maps is denoted $\mathcal{O}$.

3.2 The opening of a map

In this section, we prove that the closing operation can actually be turned into a bijection and describe its inverse: the opening algorithm.

Given a rooted oriented map, we define the *opening algorithm* as follows. We explore the corners and edges of the map, starting from the root ones, by following or crossing the edges. In particular, when we meet an edge for the first time:

- if the edge is ingoing, we follow it without changing side, clockwise around the adjacent face.
- if the edge is outgoing, we cut it and replace it by a bud and a leaf on the corners corresponding to the former ingoing and outgoing halfedges. We then cross the former edge and move to the next one in clockwise direction around the vertex.

When we meet an edge for the second time, we follow it if it was followed the first time, and cross it if it was crossed. An example of an execution of the opening algorithm is given in Figure 3, from right to left.

Theorem 3.2. *The opening algorithm for maps endowed with their (canonical) dual-geodesic orientation induces a bijection from $\mathcal{BF}_g$ to $\mathcal{O}_g$. Its inverse is the closing algorithm.*

Proof (sketch). Applying the opening algorithm on the geodesic orientation of a bipartite quadrangulation yields the rightmost breadth-first-search exploration tree, along with its buds and leaves. A closer look at the definition of the opening algorithm reveals a symmetry between the roles of faces and vertices, which implies that the opening algorithm applied to a 4-valent bicolorable map yields the complement of the dual of the leftmost breadth-first-search exploration tree, which, in particular, has the same genus as the original map. □

4 Enumeration and rationality

4.1 Reducing a map to a scheme

We saw that general maps are in bijection with well-rooted well-labeled well-oriented 4-regular Eulerian unicellular map. However the analysis of such objects is made difficult by the non-locality of a condition such as well-rootedness. The following lemma enables to ignore that condition in the rest of the analysis. The set of rooted well-labeled well-oriented 4-regular unicellular maps, counted by leaves, is denoted $\mathcal{U}$. We call *stem size* of an interior map (recall this means its stems and orientation are removed) the number of leaves contained in a blossoming map which has this specific interior map.

Theorem 4.1. *Let m° be an interior map of stem size n. There is a $(n+1)$-to-2 application from rooted well-labeled well-oriented 4-regular unicellular maps with interior map m, to well-rooted well-labeled well-oriented 4-regular unicellular maps with interior map m.*

Proof (sketch). A method similar to the cyclic lemma implies that there are exactly 2 cyclic permutations of the contour word among $n+1$ that yield a Dyck word. To prove Theorem 4.1, we hence need to reroot a non-well-rooted map on one of the 2 special stems corresponding to these permutations, by replacing the bud root by a leaf and the special stem by a root bud. The main difficulty is to prove that there exists a unique directed path from this special stem to the root, and that reversing it yields a well-labeled map. □

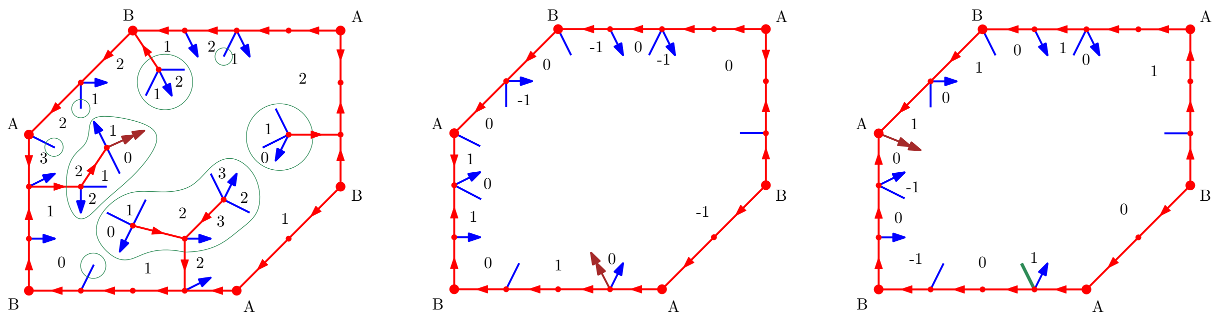


Figure 4: An example of the pruning of a opened map (whose treelike parts are encompassed) with *scheme vertices* A and B; and rerooting on the scheme of the opened map. Again the opposite sides are identified, so that the map is of genus 1.

Let u be a rooted well-labeled well-oriented 4-regular unicellular map. Define the *extended scheme* as the unicellular map of genus g obtained by iteratively removing all vertices of u with interior degree 1 along with all stems.

The map u is composed of an extended scheme upon which are attached some half-edges and treelike parts. These treelike parts, with their leaves, are binary trees, oriented towards the root of the map. Furthermore, on each interior vertex of these trees is attached a bud. The set of such trees, counted by leaves, is denoted $\mathcal{T}$. Its generating series satisfies the recurrence relation $T(z) = z + 3T(z)^2$.

The *pruning* procedure is defined as follows: each treelike part is replaced by a half-edge: a root bud if it contains the root, a leaf otherwise (see Figure 4 left and middle). The image of $\mathcal{U}$ by the pruning procedure, counted by leaves, is denoted $\mathcal{P}$.

Lemma 4.2. *The pruning algorithm yields:* $U(z) = \dfrac{\partial T}{\partial z} \cdot P(T(z))$.

All vertices of the pruned map are of interior degree 2, 3 or 4. We call v_2, v_3, and v_4 the number of such vertices. A quick calculation based on Euler formula gives: $v_3 + 2v_4 = 4g - 2$. There are thus a finite number of vertices of degree 3 or 4, the other ones being of degree 2. Vertices of interior degree at least 3 are called *scheme vertices*, and a stem (resp. bud, leaf) attached on a scheme vertex (of interior degree 3) is called a *scheme stem* (resp. *bud*, *leaf*). Another calculation using Euler formula gives:

Lemma 4.3. *Out of its $v_3 = 4g - 2v_4 - 2$ scheme stems, a pruned map p has exactly $2g - v_4$ rootable scheme stems. In particular $v_3 > 0$.*

We now proceed to reroot the pruned map on a rootable scheme stem. We therefore choose one rootable scheme stem out of $2g - v_4$ and mark it. The *rerooting* is defined similarly to the proof of Theorem 4.1 (see Figure 4 middle and right). The subset of $\mathcal{P}$ composed of *scheme-rooted* maps is denoted $\mathcal{R}$.

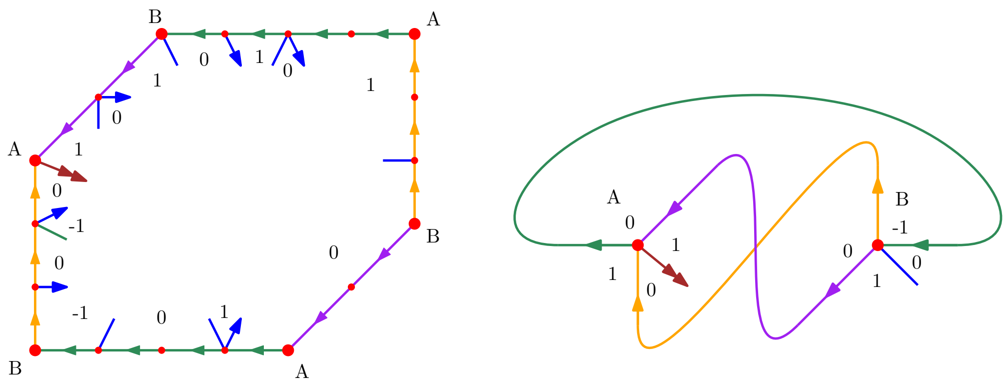


Figure 5: Reducing a map of $\mathcal{R}$ to a labeled scheme.

Lemma 4.4. *The rerooting-on-the-scheme algorithm yields:* $P(z) = \dfrac{1}{2g - v_4(e)} \cdot \dfrac{d(tR(t))}{dt}(z)$.

Let $r \in \mathcal{R}$. The sequence of edges encountered between two scheme vertices in a tour of the face of r all have the same orientation. Such a sequence is called a *branch*. We call *merging* the procedure that replaces each branch by a single edge with the same orientation (see Figure 5).

The map we obtain is called the *labeled scheme*. It is not well-labeled because corners adjacent along an edge do not necessarily have the same label, but the rule around a vertex is respected (see Figure 5 right). The set of labeled schemes is denoted $\mathcal{L}$.

4.2 Analyzing a scheme

For $l \in \mathcal{L}$, we now want to determine which maps have l as labeled scheme. Each edge of l should be replaced by a valid branch. However we need to be sure that after replacement, the map is well-labeled, and agrees with the labeling of the scheme.

There are 6 types of vertices of interior degree 2. If the bud and leaf are on opposite sides, the label of the corners either increases on both side or decreases on both sides. In the 4 other cases, the half-edges are on the same side, and the label remains the same before and after the vertex. Therefore each type of vertex of interior degree 2 can be represented by a step, depending on the variation of the labels around it: an up-step if the label increases, a down-step if it decreases, and 4 types of stay-steps if it stays the same. These steps are called *weighted Motzkin steps*, and together they form a *weighted Motzkin path*. An edge of the labeled scheme going from label i to label j can therefore be replaced by a weighted Motzkin path going from height i to height j.

We denote by $\mathcal{D}$ the set of weighted Motzkin paths going from 0 to -1, that remain non-negative before the last step, counted by length. It satisfies the decomposition equation: $D = z(1 + 4D + D^2)$. We denote by $\mathcal{B}$ the set of weighted Motzkin paths going from 0 to 0, counted by length. It satisfies the decomposition equation: $B = 1 + 4zB + 2zDB$.

After combination with the previous equation, this equation is rewritten as a function of D only: $B = \frac{1+4D+D^2}{1-D^2}$. The generating series of paths going from heigh i to j is: $B \cdot D^{|i-j|}$.

4.3 Rationality

We conclude by the bijective proof of Theorem 1.1 announced in the introduction. In fact, we prove a refinement by unlabeled schemes. An *unlabeled scheme* is a scheme where we forget all labels; their set is denoted $\mathcal{S}$. We specialize our classes of maps depending on their scheme, for instance by denoting $\mathcal{M}^s$ the set of maps that lead to the unlabeled scheme s through the successive operations described in Sections 3 and 4.

Theorem 4.5. *For any s in $\mathcal{S}$, the generating series $M^s(t)$ is a rational function of $T(t)$.*

Since $\mathcal{S}_g$ is finite for any fixed g, it implies that $M_g(t) = \sum_{s \in \mathcal{S}_g} M^s(t)$ is rational in $T(t)$, and Theorem 1.1 follows directly.

Proof (sketch). We derive from Theorems 3.2 and 4.1 and Lemmas 4.2 and 4.4, that:

$$M^s(t) = \frac{2t^{2g-2}}{2g - v_4(s)} \int_0^t \frac{d(uR^s(u))}{du}(T(z)) \cdot \frac{dT}{dz} \cdot dz = \frac{2t^{2g-2}}{2g - v_4(s)} \cdot T(t) \cdot R^s(T(t)).$$

In order to prove that M^s is rational in T, we prove that $R^s(z)$ is rational in z. We apply the same strategy as in [11]; since $z = \frac{1}{D^{-1}+4+D}$, it is enough for that to prove that R^s is rational and symmetric in D (a function Ψ is symmetric in D if $\Psi(D) = \Psi(D^{-1})$).

In a labeled scheme, the *offset label* of a corner is defined as the difference between its label and the minimal label of the corners incident to the same vertex. Note that offset labels belong to $\{0, 1, 2\}$. In fact, these labels only depend on the unlabeled scheme. Hence, an unlabeled scheme has 2 different types of edges: if the offset labels are the same (01 or 12) at both extremities, the edge is called *level*. If the offset labels are 01 at one extremity and 12 at the other, the edge is called *offset* toward the second one. For instance, in Figure 5, the orange and green edges are level, while the purple edge is offset toward B. We define the *offset graph* as the oriented sub-graph of the scheme where only the offset edges are kept, along with their orientation. The offset graph can be proved to be acyclic.

A first expression of R^s is obtained by summing the contribution of all labelled scheme leading to the unlabelled scheme s:

$$R^s = \sum_{\substack{h_1 \cdots h_{n_v} \in \mathbb{N} \\ \min(h_1, \cdots, h_{n_v}) = 0}} \prod_{\substack{e=(v_i,v_j) \in \mathcal{E} \\ i<j}} B \cdot D^{|h_i - h_j| + \epsilon_{\text{off}}(e)}, \tag{4.1}$$

where $\epsilon_{\text{off}}(e) \in \{-1, 0, 1\}$ is a correction term that takes the offset into account.

We look at the relative order of labels of the scheme vertices (defined as the minimum label of a corner adjacent to this vertex), and encode this by an element o of $S(n_v)$, the set of surjections from $[1;n_v]$ to an interval of the form $[1;k]$, where $k \leq n_v$ is the number of distinct labels. We define the cut of a subgraph $S \subset \mathcal{V}$ as follows: $Cut(S) = |\{(u,v) \in \mathcal{E} \text{ s.t. } u \in S, v \notin S\}|$, and $\Phi_o(i) = \frac{D^{Cut(o^{-1}([i]))}}{1-D^{Cut(o^{-1}([i]))}}$. A change of variable leads to:

$$R^s = B^{n_e} \cdot \sum_{o \in S(n_v)} \left(\prod_{i=1}^{k(o)-1} \Phi_o(i) \right) \cdot D^{n_{\text{off}}(o)}, \tag{4.2}$$

where $n_{\text{off}}(o)$ is an integer expressed as a sum of the $\epsilon_{\text{off}}(e)$, that is equal to 0 if the offset graph is empty. Using that $\Phi_0(i)(D^{-1}) = -(1 + \Phi_0(i)(D))$, we obtain:

$$R^s(D^{-1}) = (-1)^{n_e} \cdot B^{n_e} \cdot \sum_{p \in S(n_v)} \prod_{i=1}^{k(p)-1} \Phi_p(i) \cdot \sum_{o \preceq p} \left((-1)^{k(o)-1} \cdot D^{-n_{\text{off}}(o)} \right), \tag{4.3}$$

where $o \preceq p$ means that o refines p, or in other words: $\forall x, y; o(x) = o(y) \Rightarrow p(x) = p(y)$.

Using an inclusion-exclusion argument, based on standard properties of surjections and relying on the acyclicity of the offset graph, we obtain:

$$\sum_{o \preceq p} \left((-1)^{k(o)-1} \cdot D^{-n_{\text{off}}(o)} \right) = (-1)^{n_e} \cdot D^{n_{\text{off}}(p)}.$$

Note that when the offset graph is empty, this formula can be obtained as a direct byproduct of the Euler–Poincaré formula applied to the permutahedron.

Combining Equations (4.1) to (4.3), we obtain that $R^s(D) = R^s(D^{-1})$, which concludes the proof that $M^s(t)$ is rational in $T(t)$. □

References

[1] M. Albenque and D. Poulalhon. "A Generic Method for Bijections between Blossoming Trees and Planar Maps". *Electron. J. Combin.* **22**.2 (2015), P2.38, 44 pp. URL.

[2] E.A. Bender and E.R. Canfield. "The number of rooted maps on an orientable surface". *J. Combin. Theory Ser. B.* **53**.2 (1991), pp. 293–299. DOI: 10.1016/0095-8956(91)90079-Y.

[3] E.A. Bender, E.R. Canfield, and R.W. Robinson. "The asymptotic number of tree-rooted maps on a surface". *J. Combin. Theory Ser. A* **48**.2 (1988), pp. 156–164. DOI: 10.1016/0097-3165(88)90002-7.

[4] O. Bernardi. "Bijective counting of tree-rooted maps and shuffles of parenthesis systems". *Electron. J. Combin.* **14**.1 R (2006), pp. 1–36.

[5] O. Bernardi and G. Chapuy. "A bijection for covered maps, or a shortcut between Harer–Zagier's and Jackson's formulas". *J. Combin. Theory Ser. A* **118**.6 (2011), pp. 1718–1748. URL.

[6] O. Bernardi and E. Fusy. "A bijection for triangulations, quadrangulations, pentagulations, etc". *J. Combin. Theory Ser. A* **119**.1 (2012), pp. 218–244. DOI: 10.1016/j.jcta.2011.08.006.

[7] O. Bernardi and E. Fusy. "Unified bijections for maps with prescribed degrees and girth". *J. Combin. Theory Ser. A* **119**.6 (2012), pp. 1351–1387. DOI: 10.1016/j.jcta.2012.03.007.

[8] J. Bouttier, P. Di Francesco, and E. Guitter. "Census of planar maps: from the one-matrix model solution to a combinatorial proof". *Nucl. Phys. B* **645**.3 (2002), pp. 477–499. DOI: 10.1016/S0550-3213(02)00813-1.

[9] J. Bouttier, P. Di Francesco, and E Guitter. "Planar maps as labeled mobiles". *Electron. J. Combin.* **11**.1 R (2004), pp. 1–27.

[10] G. Chapuy. "Asymptotic enumeration of constellations and related families of maps on orientable surfaces". *Combin. Probab. Comput.* **18**.4 (2009), pp. 477–516. URL.

[11] G. Chapuy, M. Marcus, and G. Schaeffer. "A bijection for rooted maps on orientable surfaces". *SIAM J. Discrete Math.* **23**.3 (2009), pp. 1587–1611. DOI: 10.1137/080720097.

[12] V. Despré, D. Gonçalves, and B. Lévêque. "Encoding toroidal triangulations". *Discrete & Computational Geometry* **57**.3 (2017), pp. 507–544. DOI: 10.1007/s00454-016-9832-0.

[13] B. Eynard. *Counting surfaces*. Vol. 70. Progress in Mathematical Physics. Springer, 2011.

[14] S. Felsner. "Lattice structures from planar graphs". *Electron. J. Combin.* **11**.1 (2004), 15 pp. URL.

[15] S. Felsner and K.B. Knauer. "ULD-lattices and Δ-bonds". *Combin. Probab. Comput.* **18**.5 (2009), pp. 707–724. DOI: 10.1017/S0963548309010001.

[16] S.K. Lando and A.K. Zvonkin. *Graphs on surfaces and their applications. Appendix by Don B. Zagier.* 2004, pp. xv + 455.

[17] G. Miermont. "Tessellations of random maps of arbitrary genus". *Ann. Sci. Ecole Norm. Sup.* **42**.5 (2009), pp. 725–781. DOI: 10.24033/asens.2108.

[18] D. Poulalhon and G. Schaeffer. "Optimal coding and sampling of triangulations". *Algorithmica* **46**.3 (2006), pp. 505–527. DOI: 10.1007/s00453-006-0114-8. URL.

[19] J. Propp. "Lattice structure for orientations of graphs". 2002. arXiv: math/0209005.

[20] G. Schaeffer. "Bijective census and random generation of Eulerian planar maps with prescribed vertex degrees". *Electron. J. Combin.* **4**.1 R (1997), pp. 1–14. URL.

[21] G. Schaeffer. "Conjugaison d'arbres et cartes combinatoires aléatoires". PhD thesis. Université Bordeaux I, 1998. URL.

[22] G. Schaeffer. "Planar Maps". *Handbook of enumerative combinatorics*. Ed. by M. Bóna. Vol. 87. CRC Press, 2015. Chap. 5.

[23] W.T. Tutte. "A census of planar maps". *Canadian Journal of Mathematics* **15** (1963), pp. 249–271. DOI: 10.4153/CJM-1963-029-x.