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Combinatorics of orbit configuration spaces

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Abstract. From a group action on a variety, define a variant of the configuration space by insisting that no two points inhabit the same orbit. When the action is almost free, this "orbit configuration space" is the complement of an arrangement of subvarieties inside the cartesian product, and we use this structure to study its topology.

We give an abstract combinatorial description of its poset of layers (connected components of intersections from the arrangement) which turns out to be of much independent interest as a generalization of partition and Dowling lattices. The close relationship to these classical posets is then exploited to give explicit cohomological calculations akin to those of (Totaro '96).

Lastly, the wreath product of the group acts naturally. We study the induced action on cohomology using the language of representation stability: considering the sequence of all such arrangements and maps between them, the sequence of representations stabilizes in a precise sense. This is a consequence of combinatorial stability at the level of posets.

Keywords: orbit configuration space, Dowling lattice, subspace arrangement, representation stability

1 Orbit configuration spaces

A fundamental topological object attached to a topological space X is its ordered configuration space $\mathrm{Conf}_n(X)$ of n distinct points in X. Analogously, given a group G acting freely on X one defines the *orbit configuration space* by

$$\mathrm{Conf}_n^G(X) = \{(x_1, \ldots, x_n) \in X^n \mid Gx_i \cap Gx_j = \emptyset \text{ for } i \neq j\}.$$

These spaces were first defined in [18] and come up in many natural topological contexts, including:

- Universal covers of $\mathrm{Conf}_n(X)$ when X is a manifold with $\dim(X) > 2$ [18].

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- Classifying spaces of well studied groups, such as normal subgroups of surface braid groups with quotient G^n [18].
- Arrangements associated with root systems [2, 12].
- Equivariant loop spaces of X and $\mathrm{Conf}_n(X)$ [17].

A fundamental problem is thus to compute the cohomology $\mathrm{H}^*(\mathrm{Conf}_n^G(X))$. This has been previously studied e.g. by [3, 6, 9].

The current literature typically requires the action to be free, with main results relying on this assumption. For an action that is not free, one could simply throw out the set of "bad" points and consider $\mathrm{Conf}_n^G(X \setminus S)$, where

$$S = \bigcup_{g \in G \setminus \{e\}} X^g,$$

the set of points fixed by a nontrivial group element. However, the excision can create more harm than good: e.g. when X is a smooth projective variety, removing S destroys the projective structure and causes mixing of Hodge weights in cohomology. In particular, having a projective structure makes a spectral sequence calculation more manageable (see Theorem 3.4). Furthermore, one is often interested in allowing orbit configurations to inhabit S (see Remark 3.5).

We propose an alternative approach: observe that inside X^n, the space $\mathrm{Conf}_n^G(X \setminus S)$ is the complement of an arrangement $\mathcal{A}_n(G, X)$ consisting of the following subspaces:

- $H_{ij}(g) := \{(x_1, x_2, \ldots, x_n) \in X^n \mid g.x_i = x_j\}$, where $1 \leq i < j \leq n$ and $g \in G$, and
- $H_i^s := \{(x_1, x_2, \ldots, x_n) \in X^n \mid x_i = s\}$, where $1 \leq i \leq n$ and $s \in S$.

The cohomology $\mathrm{H}^*(\mathrm{Conf}_n^G(X \setminus S))$ can then be computed from the combinatorics of this arrangement and from $\mathrm{H}^*(X)$. Furthermore, the natural action of the wreath product $\mathcal{G}_n = G \wr \mathfrak{S}_n$ on the space X^n induces an action on the set $\mathcal{A}_n(G, X)$ and its complement $\mathrm{Conf}_n^G(X \setminus S)$. The induced action on $\mathrm{H}^*(\mathrm{Conf}_n^G(X \setminus S))$ can also be traced through the combinatorial computation.

Example 1.1. As a running example, let us consider the case of $G = \mathbb{Z}_2$ acting on $X = S^1$, where the nontrivial group element acts by $e^{\theta i} \mapsto e^{-\theta i}$. Here, the set of "bad" points, which are the points fixed by the nontrivial group element, is $S = \{1, -1\}$. The resulting arrangement $\mathcal{A}_n(\mathbb{Z}_2, S^1)$ corresponding to $n = 2$ is depicted in Figure 1. The orbit configuration space in question $\mathrm{Conf}_2^{\mathbb{Z}_2}(S^1 \setminus \{1, -1\})$ is the complement of the thickened lines inside the torus $(S^1)^2$.

The picture in Figure 1 is the real analog of the complex arrangement $\mathcal{A}_n(\mathbb{Z}_2, \mathbb{C}^\times)$ that arises by considering the $\mathbb{Z}_2$-complex inversion action on $S^1 \subseteq \mathbb{C}^\times$. While this particular

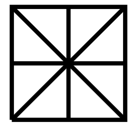


Figure 1: The arrangement $\mathcal{A}_2(\mathbb{Z}_2, S^1)$ inside the torus $(S^1)^2$.

real subspace may not be topologically interesting, it captures the combinatorics of the complex arrangement. Note, however, that the picture does not portray the topology of $\mathrm{Conf}_n^{\mathbb{Z}_2}(\mathbb{C}^\times \setminus \{1, -1\})$, since at the very least this orbit configuration space is connected.

This example is a special case of a more general type of arrangement: $\mathbb{Z}_2$ acting on an algebraic group using the group inversion, where the set of "bad" points is the set of two-torsion points. In the case that $X = \mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve E, the arrangement $\mathcal{A}_n(\mathbb{Z}_2, X)$ naturally arises from a type C root system. Their combinatorics and cohomology representations were studied by the first author in [2].

For our study, we from hereon assume that G and S are finite sets, so that the arrangement $\mathcal{A}_n(G, X)$ is finite. Moreover, we assume that X is a smooth connected oriented manifold of dimension $d > 1$. While much of what we say will hold for more general spaces, we make this last assumption for the purpose of simplifying the exposition.

2 Combinatorics

The combinatorics at play is the *poset of layers*: connected components of intersections from $\mathcal{A}_n(G, X)$, ordered by reverse inclusion. This poset admits an abstract combinatorial description, that does not in fact depend on X (only depending on the G-set S) and it is of much independent interest. For example,

- In the case of classical configuration spaces (G trivial), the poset is the lattice of set partitions of $\mathbf{n} = \{1, 2, \dots, n\}$.
- In the case that G is a cyclic group acting on $X = \mathbb{C}$ via multiplication by roots of unity, the poset is an instance of the Dowling lattice, described in [7] as an analogue of the partition lattice which consists of partial G-partitions of $\mathbf{n}$.

In what follows we define an abstract poset $\mathcal{D}_n(G, S)$ which specializes to these classical examples.

Definition 2.1 (see [7])**.** A *partial G-partition* $\tilde{\beta}$ of $\mathbf{n}$ consists of a partition β of some subset of $\mathbf{n}$, along with a *projective G-coloring* on each block $B \in \beta$, i.e. a function $b : B \to G$ defined up to the equivalence: $b \sim bg$ for every $g \in G$. The *zero block* of a partial G-partition $\tilde{\beta}$ of $\mathbf{n}$ is the set $Z := \mathbf{n} \setminus \cup_{B \in \beta} B$, i.e. the elements not included in β.

Definition 2.2. The poset $\mathcal{D}_n(G,S)$ consists of pairs $(\tilde{\beta}, z)$ where $\tilde{\beta}$ is a partial G-partition of $\mathbf{n}$ and $z : Z \to S$ is an S-coloring of its zero block. The following covering relations determine the partial order on $\mathcal{D}_n(G,S)$:

(merge): $(\tilde{\beta} \cup \{\tilde{A}, \tilde{B}\}, z) \prec (\tilde{\beta} \cup \{\tilde{C}\}, z)$ where $C = A \cup B$ with function $c = a \cup bg$ for some $g \in G$, and

(color): $(\tilde{\beta} \cup \{\tilde{B}\}, z) \prec (\tilde{\beta}, z')$ where z' is the extension of z to $Z' = B \cup Z$ given on B by a composition $f \circ b : B \to G \to S$ for some G-equivariant function f.

The poset $\mathcal{D}_n(G,S)$ has a natural action of the wreath product $\mathcal{G}_n = G \wr \mathfrak{S}_n$ given as follows: for $g = (g_1, \ldots, g_n, \sigma) \in \mathcal{G}_n$ and $(\tilde{\beta}, z) \in \mathcal{D}_n(G,S)$ define $g.(\tilde{\beta}, z) = (\tilde{\beta}', z')$ where

- $\beta' = \{\sigma B \mid B \in \beta\}$ with zero block σZ,
- $b' : \sigma B \to G$ is given by $b'(\sigma(j)) = g_j b(j)$, and
- $z' : \sigma Z \to S$ is given by $z'(\sigma(j)) = g_j.z(j)$.

Example 2.3. As said before, the classical examples are specializations of our poset. Indeed, the partition lattice arises by taking G to be trivial and S the empty set; the Dowling lattice arises by taking $S = \{0\}$. In fact, the classical terminology of the zero block is fitting in our context: it is the block colored by 0.

Example 2.4. Recall the arrangement $\mathcal{A}_n(\mathbb{Z}_2, \mathbb{C}^\times)$ from Example 1.1. Associated to this arrangement was the set $S = \{1, -1\}$, on which $G = \mathbb{Z}_2$ acts trivially. In Figure 2, we depict the poset $\mathcal{D}_2(\mathbb{Z}_2, S)$ consisting of partial $\mathbb{Z}_2$-partitions of $\mathbf{2} = \{1,2\}$ whose zero block (in red) is colored by S (the blue subscripts). The representative maps $b : B \to G$ for each block B in a partial $\mathbb{Z}_2$-partition are represented by the cyan subscripts.

Even though $\mathcal{D}_n(G,S)$ is not in general a lattice, it supports a myriad of properties that have been fundamental in the modern study of posets, since it is essentially built out of partition and Dowling lattices as indicated in the following theorem:

Theorem 2.5 (Local structure). *For any $A, B \in \mathcal{D}_n(G,S)$ with $A < B$, the interval $[A,B]$ is isomorphic to a product*

$$\Pi_{n_1} \times \ldots \times \Pi_{n_d} \times \mathcal{D}_{m_1}(G_1) \times \ldots \times \mathcal{D}_{m_k}(G_k)$$

where Π_{n_i} denotes a partition lattice and $\mathcal{D}_{m_j}(G_j)$ denotes a Dowling lattice. In particular, every interval is a geometric lattice and has the homology of a wedge of spheres.

Example 2.6. In Example 2.4 and Figure 2 one finds, for example,

$$\Big[\ 1_e|2_e|\varnothing\ ,\ \varnothing|1_1 2_1\ \Big] \cong \mathcal{D}_2(\mathbb{Z}_2)$$

and

$$\Big[\ 1_e|2_e|\varnothing\ ,\ 1_e|2_{-1}\ \Big] \cong \Pi_1 \times \mathcal{D}_1(\mathbb{Z}_2).$$

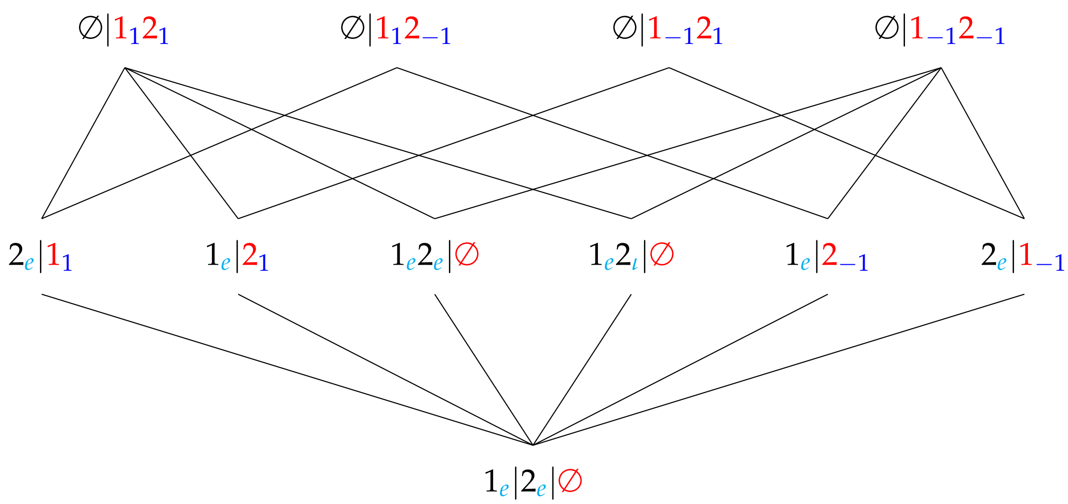


Figure 2: $\mathcal{D}_2(\mathbb{Z}_2, S)$ for $\mathbb{Z}_2 = \{e, \iota\}$ acting on $\mathbb{C}^\times$, so that the zero block is colored by $S = \{1, -1\}$.

3 Topology

As mentioned above, the poset $\mathcal{D}_n(G, S)$ defined in Definition 2.2 arises naturally in the study of orbit configuration spaces, when we take S to be the set of "bad" points for the action of G on X. Recall that the poset of layers of the arrangement $\mathcal{A}_n(G, X)$ is the collection of connected components of intersections from $\mathcal{A}_n(G, X)$, ordered by reverse inclusion. This poset encodes subtle aspects of the topology of $\mathrm{Conf}_n^G(X \setminus S)$, as we shall see below.

Theorem 3.1 (Poset of layers). *The poset of layers of the arrangement $\mathcal{A}_n(G, X)$ is naturally isomorphic to the poset $\mathcal{D}_n(G, S)$.*

Proof sketch. Every tuple of points $(x_1, \ldots, x_n) \in X^n$ gives a partition of $\mathbf{n}$ according to which points belong to the same orbits. If a point falls inside the set S then it is put in the zero block and colored by the corresponding element in S. Otherwise, the orbits disjoint from S are free, and on them a projective G-coloring is well defined by insisting that it describes the relations between any two points belonging to the orbit.

This association gives a function $X^n \to \mathcal{D}_n(G, S)$ (which is in fact continuous with respect to the natural poset topology), and it induces a bijection between the elements of $\mathcal{D}_n(G, S)$ and the layers of $\mathcal{A}_n(G, X)$. □

Example 3.2. Returning to our running example $\mathcal{A}_2(\mathbb{Z}_2, \mathbb{C}^*)$, consider its poset of layers. Figure 3 depicts the collection of layers in the real analogue $\mathcal{A}_2(\mathbb{Z}_2, S^1)$ and the inclusion relations between them. One can think of this illustration as a schematic depiction of the

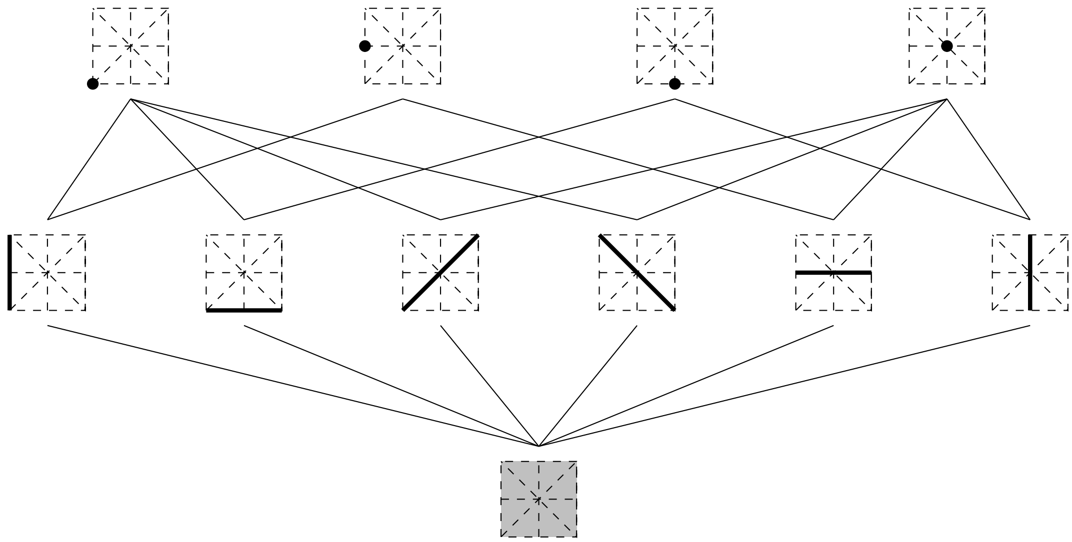
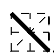


Figure 3: The poset of layers for $\mathcal{A}_2(\mathbb{Z}_2, S^1)$. Note the isomorphism with the poset in Figure 2.

layers in $\mathbb{C}^*$, whose descriptions and relations are completely identical. The isomorphism with $\mathcal{D}_2(\mathbb{Z}_2, \{1, -1\})$ of Figure 2 is apparent, with the natural bijection given e.g. by

$$1_e 2_\iota | \varnothing \longleftrightarrow$$  .

With this, the description of intervals in $\mathcal{D}_n(G,S)$ given by Theorem 2.5 translates back to topology, as we now explain. Since each element of $\mathcal{D}_n(G,S)$ could be thought of as a subspace of X^n, the incidence relation attaches to every point $p \in X^n$ the downward closed subposet

$$\mathcal{D}_n(G,S)_p = \{B \in \mathcal{D}_n(G,S) \mid p \in B\}$$

This subposet clearly has a maximum B_p and a minimum $\hat{0} = X^n$, and is thus the interval $[\hat{0}, B_p]$ described by Theorem 2.5. Geometrically, this gives the local picture of the arrangement in X^n: it is well known that since X is a smooth manifold, the restriction of the arrangement $\mathcal{A}_n(G,X)$ to a small ball centered at p is isomorphic to a linear subspace arrangement $\mathcal{A}_p$, whose intersection poset is the interval $[\hat{0}, B_p]$. Theorem 2.5 thus translates to the following.

Theorem 3.3 (Local arrangements). *For every $p \in X^n$ the complement of the local arrangement $\mathcal{A}_p$ is isomorphic to a product of (free) orbit configuration spaces of points in $\mathbb{R}^d$. Equivalently, the restriction of $\mathrm{Conf}_n^G(X \setminus S)$ to any sufficiently small ball is isomorphic to a product of such orbit configuration spaces.*

This observation opens the door to cohomology calculations: considering the Leray spectral sequence for the inclusion $\mathrm{Conf}_n^G(X \setminus S) \hookrightarrow X^n$, one obtains a description of the E_2 page in terms of the above mentioned local arrangements (see [15, 1, 8, 13]). The E_2 page decomposes as

$$E_2^{p,q} \cong \bigoplus \mathrm{H}^p(B;\mathbb{Q}) \otimes \mathrm{H}^q(\mathcal{M}_B;\mathbb{Q})$$

where the sum is over the layers B of codimension $qd/(d-1)$, and $\mathcal{M}_B$ is the complement of the local arrangement $\mathcal{A}_p$ near a generic point $p \in B$. In particular, one observes by the work of Goresky–MacPherson [11] that $\mathrm{H}^q(\mathcal{M}_B;\mathbb{Q})$ is trivial unless $(d-1)|q$, thus E_2^{pq} is trivial unless $(d-1)|q$. Furthermore, our work complements [11] and allows the substantial simplification, that for a layer B of codimension $qd/(d-1)$,

$$\mathrm{H}^q(\mathcal{M}_B;\mathbb{Q}) \cong \bigoplus_{A \in [\hat{0},B]} \widetilde{\mathrm{H}}_{\mathrm{codim}(A)-q-2}(\hat{0},A) \cong \widetilde{\mathrm{H}}_{\mathrm{rank}(B)-2}(\hat{0},B).$$

The latter isomorphism follows from the fact that $\mathrm{codim}(A) = d\,\mathrm{rank}(A)$ along with Theorem 2.5: $(\hat{0},A)$ has the homology of a wedge of spheres in degree $\mathrm{rank}(A)-2$, thus it is trivial unless $A = B$.

Lastly, when X is a smooth projective algebraic variety over $\mathbb{C}$, a Hodge theory argument guarantees that there could only be one non-zero differential. Thus, in this case one is closer to getting a hand on the rational cohomology. We summarize this explicit description of the spectral sequence machinery in the following theorem.

Theorem 3.4 (Leray spectral sequence description). *The Leray spectral sequence for the inclusion* $\mathrm{Conf}_n^G(X \setminus S) \hookrightarrow X^n$ *decomposes as*

$$E_2^{p,(d-1)q} = \bigoplus_{B \in \mathcal{D}_n(G,S)^q} \mathrm{H}^p(B;\mathbb{Q}) \otimes \widetilde{\mathrm{H}}_{q-2}(\hat{0},B) \implies \mathrm{H}^{p+(d-1)q}(\mathrm{Conf}_n^G(X \setminus S);\mathbb{Q})$$

while terms $E_2^{p,q}$ *with* $(d-1) \nmid q$ *vanish.*

Here, the summands are indexed by the layers B *of rank* q. *The* $\widetilde{\mathrm{H}}_{q-2}(\hat{0},B)$ *denotes the reduced homology of the order complex for the interval* $(\hat{0},B) \subset \mathcal{D}_n(G,S)$, *and is therefore described explicitly by Theorem 2.5.*

When X *is a smooth complex projective variety, there is at most one non-zero differential in the above spectral sequence, defined on the* d*'th page.*

Remark 3.5. Our handle on the combinatorics of these arrangements can be exploited to understand what happens when one removes from X a set T other than the set of bad points S. For example, let T be a G-invariant subset of S so that G acts (not freely) on $X \setminus T$, and consider the space $\mathrm{Conf}_n^G(X \setminus T)$. This is the complement in X^n of a subarrangement of $\mathcal{A}_n(G,X)$, defined by only including the subvarieties H_i^s when $s \in T$. The new poset of layers is a subposet of $\mathcal{D}_n(G,S)$ which inherits many properties from $\mathcal{D}_n(G,S)$ to which our study applies.

These types of arrangements arise naturally, as follows. Recall the example $\mathcal{A}_n(\mathbb{Z}_2, X)$ of a type C root system arrangement, when X is one of $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve. The type B and D root system arrangements manifest as subarrangements of this, by removing from X only a proper subset of bad points.

4 Representation stability

The wreath product $\mathcal{G}_n = G \wr \mathfrak{S}_n$ acts naturally on X^n, respecting the arrangement $\mathcal{A}_n(G, X)$, its complement $\mathrm{Conf}_n^G(X \setminus S)$, and its poset of layers $\mathcal{D}_n(G, S)$. It is therefore natural to study the rational cohomology $\mathrm{H}^*(\mathrm{Conf}_n^G(X \setminus S); \mathbb{Q})$ as a linear representation of $\mathcal{G}_n$. We go about this task using the following seemingly unrelated observation.

As the parameter n varies, the arrangements $\mathcal{A}_n(G, X)$ with their corresponding $\mathcal{G}_n$-actions assemble into a functorial sequence. Briefly, the projections $X^{n+k} \to X^n$ induce maps

$$\mathrm{Conf}_{n+k}^G(X \setminus S) \to \mathrm{Conf}_n^G(X \setminus S)$$

which suitably intertwine the $\mathcal{G}_{n+k}$ and $\mathcal{G}_n$ actions. This structure is succinctly characterized by saying that $\mathrm{Conf}_\bullet^G(X \setminus S)$ is a functor from a certain category $\mathtt{FI}_G$ to spaces, where $\mathtt{FI}_G$ serves as a means of collating the various $\mathcal{G}_n$ into one object. In the classical case that G is trivial, this is the category $\mathtt{FI}$ used by Church–Ellenberg–Farb [5] in their formulation of representation stability; the categories $\mathtt{FI}_G$ have been studied first for $G = \mathbb{Z}_2$ by Wilson [16] and later for general G in [14, 3].

The functoriality of the aforementioned spectral sequence makes the E_2 page a representation of $\mathtt{FI}_G$, and thus much of the actions on the sequence $n \mapsto \mathrm{H}^*(\mathrm{Conf}_n^G(X \setminus S))$ can be understood from the induced $\mathtt{FI}_G$ action on the posets of layers. There, it turns out that studying the entire sequence of $\mathcal{D}_n(G, S)$ along with the induced intertwining maps is easier and more informative than the study of any individual poset separately.

Theorem 4.1 (Combinatorial stability). *The sequence of posets $\mathcal{D}_\bullet(G, S)$ is combinatorially stable in the sense of [10]. Explicitly,*

- *Finite generation: in every fixed rank q there are finitely many elements $x_i \in \mathcal{D}_{n_i}(G, S)^q$ whose images under the intertwining maps cover $\mathcal{D}_m(G, S)^q$ for all m.*
- *Downward stability: for every intertwining map $f_* : \mathcal{D}_n(G, S) \to \mathcal{D}_{n+k}(G, S)$ and for every $x \in \mathcal{D}_n(G, S)$ the induced map on intervals*

$$f_* : [\hat{0}, x] \to [\hat{0}, f_*(x)]$$

is an isomorphism.

Qualitatively, the theorem states that the various $\mathcal{D}_n(G, S)$ are built from finitely many types of identical intervals, which are merely permuted by the intertwining maps.

Proof sketch. The poset maps $\mathcal{D}_n(G,S) \to \mathcal{D}_{n+k}(G,S)$ extend partitions of $\mathbf{n}$ to ones of $\mathbf{n}+\mathbf{k}$ by adding all remaining elements $(\mathbf{n}+\mathbf{k}) \setminus \mathbf{n}$ as singleton blocks. Therefore downward stability is trivial: every covering relation in $\mathcal{D}_{n+k}(G,S)$ involves a refinement of partitions, but when refining a given partition singleton blocks must be ignored since they can not be refined further. It follows that a refinement is possible in $\mathcal{D}_{n+k}(G,S)$ if and only if it was already possible in $\mathcal{D}_n(G,S)$.

As for finite generation, fixing a rank q has the effect of bounding the number of elements $i \in \mathbf{n}$ that are *not* in singleton non-zero blocks: there can be at most $2q$ such elements. It is therefore expected that for large n, almost all $i \in \mathbf{n}$ are partitioned into singletons. Once those are removed, one is left with a small partition that already appeared in $\mathcal{D}_r(G,S)$ for some $r \leq 2q$. Thus every element in $\mathcal{D}_n(G,S)^q$ is in the image of $\mathcal{D}_r(G,S)$ for a finite list of r's. □

Combinatorial stability implies that, for each i, understanding the sequence of $\mathcal{G}_n$-representations $H^i(\mathrm{Conf}_n^G(X \setminus S);\mathbb{Q})$ reduces to a finite problem, as the following shows.

Theorem 4.2 (Representation stability). *Let T be a G-invariant subset (possibly empty and possibly S). For every $i \geq 0$ the sequence $\mathrm{H}^i(\mathrm{Conf}_n^G(X \setminus T))$ of $\mathcal{G}_n$-representations exhibits representation stability in the sense of [3]. Explicitly, for all $n \gg 1$ and every intertwining map $f_* : \mathrm{H}^i(\mathrm{Conf}_n^G(X \setminus T)) \to \mathrm{H}^i(\mathrm{Conf}_{n+k}^G(X \setminus T))$ one finds,*

1. *Injectivity: f_* in injective.*
2. *Surjectivity: the image of f_* generates $\mathrm{H}^i(\mathrm{Conf}_{n+k}^G(X \setminus T))$ as a $\mathcal{G}_{n+k}$-representation.*
3. *Stable decomposition: the multiplicities of irreducible representations are the same[1] in both representations.*

This theorem generalizes many previous representation stability results:

- When $G = \{e\}$, one gets representation stability for cohomology of configuration spaces of manifolds, as proved by Church [4].
- When $G = \mathbb{Z}_2$, one recovers a result of Wilson [16] for complements of root systems in $X = \mathbb{C}$ and of Bibby [2] for the analogous complements in $X = \mathbb{C}^*$ and in a complex elliptic curve.
- When $T = S$, one recovers a theorem of Casto [3], which addressed the case of *free* G-manifolds. In contrast, our Theorem 4.2 applies to non-free G-spaces, and gives representation stability e.g. for $\mathrm{Conf}_n^G(X)$ for complete X.
- Lastly, it follows from Theorem 4.2 that the cohomology stabilizes as representations of $\mathfrak{S}_n$; this stability was shown by Petersen [13].

[1] There is a compatible way to name irreducible representations of $\mathcal{G}_n$ for all n. The claim is that the multiplicity of irreducibles with the same name does not depend on k.

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