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Bases of the quantum matrix bialgebra and induced sign characters of the Hecke algebra

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Abstract. We combinatorially describe the transition matrices which relate monomial bases of the zero-weight space of the quantum matrix bialgebra. This description leads to a combinatorial rule for evaluating induced sign characters of the (type A) Hecke algebra $H_n(q)$ at all elements of the form $(1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})$, including the Kazhdan–Lusztig basis elements indexed by 321-hexagon-avoiding permutations. This result is the first subtraction-free rule for evaluating any character at all elements of a basis of $H_n(q)$.

Résumé. Nous décrivons les matrices de transition entre quelques bases de la bialgèbre quantique des matrices. Cette description donne une formule combinatoire pour l'évaluation des caractères induits signés de l'algèbre de Hecke (type A) $H_n(q)$ sur chacque élément de la forme $(1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})$. Cet ensemble inclut les éléments de la base de Kazhdan–Lusztig indexés par les permutations qui évitent les motifs 321-hexagonaux. Ce résultat est la première formule sans soustraction pour l'évaluation d'un caractére dans tous les éléments d'une base de $H_n(q)$.

1 Introduction

The *symmetric group algebra* $\mathbb{Z}[\mathfrak{S}_n]$ and the *(Iwahori–) Hecke algebra (of type A)* $H_n(q)$ have similar presentations as algebras over $\mathbb{Z}$ and $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ respectively, with multiplicative identity elements e and T_e, generators $s_1, \ldots, s_{n-1}$ and $T_{s_1}, \ldots, T_{s_{n-1}}$, and relations

$$\begin{aligned} s_i^2 &= e & T_{s_i}^2 &= (q-1)T_{s_i} + qT_e & &\text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j & T_{s_i} T_{s_j} T_{s_i} &= T_{s_j} T_{s_i} T_{s_j} & &\text{for } |i-j| = 1, \\ s_i s_j &= s_j s_i & T_{s_i} T_{s_j} &= T_{s_j} T_{s_i} & &\text{for } |i-j| \geq 2. \end{aligned} \tag{1.1}$$

Analogous to the natural basis $\{w \,|\, w \in \mathfrak{S}_n\}$ of $\mathbb{Z}[\mathfrak{S}_n]$ is the natural basis $\{T_w \,|\, w \in \mathfrak{S}_n\}$ of $H_n(q)$, where we define $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$ whenever $s_{i_1} \cdots s_{i_\ell}$ is a reduced (short as

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possible) expression for w in $\mathfrak{S}_n$. We call ℓ the *length* of w and write $\ell = \ell(w)$. The specialization of $H_n(q)$ at $q^{\frac{1}{2}} = 1$ is isomorphic to $\mathbb{Z}[\mathfrak{S}_n]$. The *Bruhat order* on $\mathfrak{S}_n$ is defined by $u \leq v$ if every reduced expression for v contains a reduced expression for u. In addition to the natural basis of $H_n(q)$, we have the (modified) *Kazhdan–Lusztig basis* $\{q_{e,w}C'_w(q) \,|\, w \in \mathfrak{S}_n\}$, where we define $q_{e,w} := q^{\frac{\ell(w)}{2}}$. (See [5] for definitions.)

Representations of $H_n(q)$ are often studied in terms of $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-linear functionals called *characters*. The $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-span of these characters is called the space of $H_n(q)$-*traces* and has dimension equal to the number of integer partitions of n. Two well-studied bases are the irreducible characters $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$, and induced sign characters $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$, related to one another by the *Kostka numbers*, i.e., $\epsilon_q^\lambda = \sum_{\mu \vdash n} K_{\mu^\top,\lambda}\chi_q^\mu$, where $\mu^\top$ denotes the transpose or *conjugate* of the partition μ.

The characters $\theta_q \in \{\chi_q^\lambda, \epsilon_q^\lambda\}$ satisfy $\theta_q(z) \in \mathbb{Z}[q]$ for all $z \in H_n(q)$ and $\lambda \vdash n$. An ideal combinatorial formula for such evaluations would define sequences $(S_k)_{k\geq 0}$, $(R_k)_{k\geq 0}$ of sets so that we have $\theta_q(z) = \sum_k (-1)^{|S_k|}|R_k|q^k$, or simply $\theta_q(z) = \sum_k |R_k|q^k$ if $\theta_q(z) \in \mathbb{N}[q]$. For z in the natural basis or modified Kazhdan–Lusztig basis of $H_n(q)$ we have the following results and open problems.

θ_q	Do we have $\theta_q(T_w) \in \mathbb{N}[q]$ for all $w \in \mathfrak{S}_n$?	Can we interpret $\theta_q(T_w)$ as $\sum_k (-1)^{\lvert S_k\rvert}\lvert R_k\rvert q^k$ for all $w \in \mathfrak{S}_n$?	Do we have $\theta_q(q_{e,w}C'_w(q)) \in \mathbb{N}[q]$ for all $w \in \mathfrak{S}_n$?	Can we interpret $\theta_q(q_{e,w}C'_w(q))$ as $\sum_k \lvert R_k\rvert q^k$ for all $w \in \mathfrak{S}_n$?
ϵ_q^λ	no	open	yes	open
χ_q^λ	no	open	yes	open

The polynomials $\chi_q^\lambda(T_w)$ and $\epsilon_q^\lambda(T_w)$ may be computed via a q-extension of the Murnaghan–Nakayama algorithm but neither has a conjectured expression of the type asked for above. Interpretations of $\epsilon_q^\lambda(q_{e,w}C'_w(q))$ and $\chi_q^\lambda(q_{e,w}C'_w(q))$ are not known for general $w \in \mathfrak{S}_n$, but nonnegativity is due to Haiman [4]. In the special case that w avoids the patterns 3412 and 4231, interpretations are given in [2, Thms. 6.4, 8.1].

To obtain ideal combinatorial interpretations analogous to those asked for above, we will consider ϵ_q^λ and the infinite spanning set of $H_n(q)$ of all elements of the form

$$(1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}}) = q^{\frac{m}{2}} C'_{s_{i_1}}(q) \cdots C'_{s_{i_m}}(q), \tag{1.2}$$

where the index sequence $s_{i_1} \cdots s_{i_m}$ varies over all products of generators of $\mathfrak{S}_n$. We have $\epsilon_q^\lambda((1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}}) \in \mathbb{N}[q]$ since each product $q_{e,u}C'_u(q)q_{e,v}C'_v(q)$ belongs to $\mathrm{span}_{\mathbb{N}[q]}\{q_{e,w}C'_w(q) \,|\, w \in \mathfrak{S}_n\}$. (See [4, Appendix].) Interpretation of these polynomials is new, and is the first result of its kind to include evaluation of an $H_n(q)$-character at all elements of a basis of $H_n(q)$.

In Section 2 we introduce the quantum matrix bialgebra $\mathcal{A}$ and prove combinatorial formulas for the entries of transition matrices that relate monomial bases of the zero-weight space of $\mathcal{A}$. In Sections 3–4 we define a function $\sigma : \mathcal{A} \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ which allows us to compute $\theta_q((1+T_{s_{i_1}})\cdots(1+T_{s_{i_m}}))$ for any linear function $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ in terms of a generating function in $\mathcal{A}$ for θ_q. Finally, in Section 5 we use the map σ to combinatorially evaluate induced sign characters of $H_n(q)$ at all elements of the spanning set (1.2).

2 The quantum matrix bialgebra

The *quantum matrix bialgebra* $\mathcal{A} = \mathcal{A}(n,q)$ is the associative algebra with unit 1 generated over $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ by n^2 variables $x = (x_{1,1}, \ldots, x_{n,n})$, subject to the relations

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{i,\ell}, \qquad & x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k}, \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{j,k}, \qquad & x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k}, \end{aligned} \tag{2.1}$$

for all $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$. The counit and coproduct maps $\varepsilon(x_{i,j}) = \delta_{i,j}$, $\Delta(x_{i,j}) = \sum_{k=1}^{n} x_{i,k} \otimes x_{k,j}$ give $\mathcal{A}$ a bialgebra structure. Two closely related Hopf algebras are the quantum coordinate rings of $SL_n(\mathbb{C})$ and $GL_n(\mathbb{C})$,

$$\mathcal{O}_q(SL_n(\mathbb{C})) \cong \mathbb{C} \otimes \mathcal{A}/(\det_q(x) - 1), \qquad \mathcal{O}_q(GL_n(\mathbb{C})) \cong \mathbb{C} \otimes \mathcal{A}[t]/(\det_q(x)t - 1),$$

where $\det_q(x) := \sum_{v \in \mathfrak{S}_n} (-q^{\frac{-1}{2}})^{\ell(v)} x_{1,v_1} \cdots x_{n,v_n}$ is the *quantum determinant* of the matrix $x = (x_{i,j})$. For k-element subsets $I, J \subset [n] := \{1, \ldots, n\}$, the submatrix $x_{I,J} = (x_{i,j})_{i \in I, j \in J}$ of x and the quantum minor $\det_q(x_{I,J})$ are defined in the obvious ways. Specializing $\mathcal{A}$ at $q^{\frac{1}{2}} = 1$, we obtain the commutative ring $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$. The submodule

$$\mathcal{A}_{[n],[n]} = \operatorname{span}_{\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]} \{x^{u,v} := x_{u_1,v_1} \cdots x_{u_n,v_n} \mid u, v \in \mathfrak{S}_n\} \tag{2.2}$$

is called the *zero weight space* of $\mathcal{A}$ and has the natural basis $\{x^{e,w} \mid w \in \mathfrak{S}_n\}$. Expanding other monomials in the basis, we have the following.

Proposition 2.1. *There are uniquely defined polynomials $\{r_{u,v,w}(q_1) \mid w \in \mathfrak{S}_n\}$ in $\mathbb{N}[q_1]$ which satisfy*

$$x^{u,v} = \sum_{w \in \mathfrak{S}_n} r_{u,v,w}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x^{e,w}. \tag{2.3}$$

Moreover, we have $r_{u,v,u^{-1}v}(q_1) = 1$ and $r_{u,v,w}(q_1) = 0$ unless $w \geq u^{-1}v$.

Proof of Proposition 2.1. Omitted. □

Corollary 2.2. *For each fixed $u \in \mathfrak{S}_n$, the set $\{x^{u,v} \,|\, v \in \mathfrak{S}_n\}$ is a basis for $\mathcal{A}_{[n],[n]}$.*

To combinatorially interpret coefficients of the polynomials $\{r_{u,v,w}(q_1) \,|\, u, v, w \in \mathfrak{S}_n\}$, we consider a seemingly unrelated problem concerning sequences of permutations.

Definition 2.3. *Fix permutations $u, v, w \in \mathfrak{S}_n$ and a reduced expression $s_{i_1} \cdots s_{i_k}$ for u. Define $C^b_{u,v,w}(s_{i_1} \cdots s_{i_k})$ to be the set of sequences $\pi = (\pi^{(0)}, \dots, \pi^{(k)})$ satisfying*

1. $\pi^{(0)} = v$, $\pi^{(k)} = w$,
2. $\pi^{(j)} \in \{s_{i_j}\pi^{(j-1)}, \pi^{(j-1)}\}$ *for* $j = 1, \dots, k$,
3. $\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ *if* $s_{i_j}\pi^{(j-1)} > \pi^{(j-1)}$ *for* $j = 1, \dots, k$,
4. $\pi^{(j)} = \pi^{(j-1)}$ *for exactly b values of j for* $j = 1, \dots, k$,

and define the polynomial $p_{u,v,w}(q_1; s_{i_1} \cdots s_{i_k}) = \sum_b |C^b_{u,v,w}(s_{i_1} \cdots s_{i_k})| q_1^b \in \mathbb{N}[q_1]$.

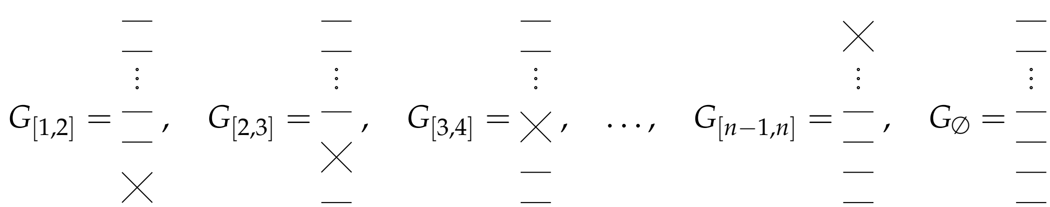
Surprisingly, these polynomials *do not* depend upon the choice of a reduced expression for u, although each set $C^b_{u,v,w}(s_{i_1} \cdots s_{i_k})$ does depend upon such a choice.

Theorem 2.4. *For u in $\mathfrak{S}_n$, the polynomials $\{r_{u,v,w}(q_1) \,|\, v, w \in \mathfrak{S}_n\}$ defined in (2.3) satisfy $r_{u,v,w}(q_1) = p_{u,v,w}(q_1, s_{i_1} \cdots s_{i_k})$, where $s_{i_1} \cdots s_{i_k}$ is any reduced expression for u.*

Proof of Theorem 2.4. (Omitted.) □

3 Wiring diagrams and path matrices

To combinatorially evaluate induced sign characters at elements $(1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})$ of $H_n(q)$, we will use standard $\mathfrak{S}_n$ wiring diagrams, concatenations of the diagrams



(3.1)

representing the elements $s_1, s_2, s_3, \dots, s_{n-1}, e$ of $\mathfrak{S}_n$, respectively. Each wiring diagram has n implicit vertices on the left and right, labeled *source* $1, \dots,$ *source* n and *sink* $1, \dots,$ *sink* n, respectively, from bottom to top. Edges are implicitly oriented from left to right. Let $\pi = (\pi_1, \dots, \pi_n)$ be a sequence of source-to-sink paths in an $\mathfrak{S}_n$ wiring diagram G. We call π a (bijective) *path family* if there exists a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ such that π_i is a path from source i to sink w_i. In this case, we say more specifically that π has *type* w. We say that the path family *covers* G if it contains every edge exactly once.

It is easy to see that the number of path families covering the wiring diagram

$$G = G_{[i_1,i_1+1]} \circ \cdots \circ G_{[i_m,i_m+1]} \tag{3.2}$$

of $s_{i_1} \cdots s_{i_m}$ is 2^m: for $j = 1, \ldots, m$, the two paths intersecting at the central vertex of $G_{[i_j,i_j+1]}$ either cross or do not cross at that vertex. In these two cases, we call the index j a *crossing* or *noncrossing*, respectively.

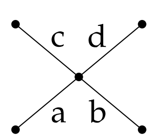
The diagram (3.2) may be used to encode $(1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}}) = \sum_{v \in \mathfrak{S}_n} a_v T_v$ in $H_n(q)$. Specifically, each coefficient $a_v \in \mathbb{N}[q]$ may be interpreted in terms of a path family statistic called *defects*. Call index j a *defect* of path family π if the two paths containing the central vertex of $G_{[i_j,i_j+1]}$ have previously crossed an odd number of times. We will call index j a *proper* crossing or noncrossing if it is not defective. Letting $\mathrm{D}(\pi)$ denote the number of defects in π we have $a_v = \sum_\pi q^{\mathrm{D}(\pi)}$, where the sum is over path families of type v which cover G [3, Proposition 3.5]. Billey and Warrington [1, Theorem 1] showed that if $s_{i_1} \cdots s_{i_m}$ is a reduced expression for w avoiding 321 and the *hexagon* patterns 56781234, 56718234, 46781235, 46718235, then $q_{e,w} C'_w(q) = (1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})$.

One often enhances a planar network by associating to each edge a *weight* belonging to some ring R, and by defining the *weight of a path* to be the product of its edge weights. If R is noncommutative, then weights are multiplied in the order that the corresponding edges appear in the path. For a *family* $\pi = (\pi_1, \ldots, \pi_n)$ of n paths in a planar network, one defines $\mathrm{wgt}(\pi) = \mathrm{wgt}(\pi_1) \cdots \mathrm{wgt}(\pi_n)$. The *(weighted) path matrix* $B = B(G) = (b_{i,j})$ of G is defined by letting $b_{i,j}$ be the sum of weights of all paths in G from source i to sink j. Thus the product $b_{1,w_1} \cdots b_{n,w_n}$ is equal to the sum of weights of all path families of type w in G.

Assigning weights to the edges of G (3.2) can aid in the evaluation of a linear function $\theta : \mathbb{Z}[\mathfrak{S}_n] \to \mathbb{Z}$ at $(1 + s_{i_1}) \cdots (1 + s_{i_m})$ by relating it to the generating function

$$\mathrm{Imm}_\theta(x) := \sum_{w \in \mathfrak{S}_n} \theta(w) x_{1,w_1} \cdots x_{n,w_n} \in \mathbb{Z}[x_{1,1}, \ldots, x_{n,n}], \tag{3.3}$$

called the *θ-immanant* in [7, Section 3]. In particular, for $j = 1, \ldots, m$, we assign weight 1 to the $n-2$ horizontal edges of $G_{[i_j,i_j+1]}$, and we assign (commuting) indeterminate weights $z_{i_j,j,1}$, $z_{i_j,j,2}$, $z_{i_j+1,j,1}$, $z_{i_j+1,j,2}$ to the remaining nonhorizontal edges a, b, c, d, respectively,



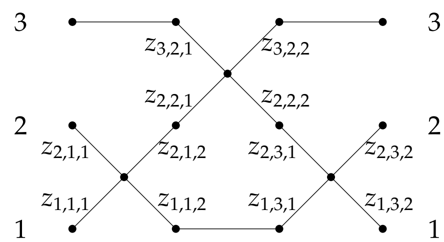
. (3.4)

Thus networks corresponding to expressions $s_{i_1} s_{i_2} s_{i_3} = s_1 s_2 s_1$ and $s_{i_4} s_{i_5} s_{i_6} = s_1 s_2 s_1$ are weighted differently because of the different indexing of the generators. Let z_G be the product of all $4m$ indeterminates $z_{i,j,k}$, and for $f \in \mathbb{Z}[z_{1,1,1}, \ldots, z_{i_m,m,2}]$, let $[z_G]f$ denote the coefficient of z_G in f. Then we have the following.

Proposition 3.1. *Assign weights to the edges of* G *(3.2) as above and let* B *be the resulting path matrix. Then for any linear function* $\theta : \mathbb{Z}[\mathfrak{S}_n] \to \mathbb{Z}$ *we have*

$$\theta((1 + s_{i_1}) \cdots (1 + s_{i_m})) = [z_G]\mathrm{Imm}_\theta(B).$$

Example 3.2. *Consider* $(1 + s_1)(1 + s_2)(1 + s_1) = 2 + 2s_1 + s_2 + s_1s_2 + s_2s_1 + s_1s_2s_1$ *in* $\mathbb{Z}[\mathfrak{S}_n]$ *and its wiring diagram* $G = G_{[i_1,i_1+1]} \circ G_{[i_2,i_2+1]} \circ G_{[i_3,i_3+1]} = G_{[1,2]} \circ G_{[2,3]} \circ G_{[1,2]}$. *Assigning weights to the edges of* G *we have*



(3.5)

and $z_G = z_{1,1,1} \cdots z_{3,2,2}$. *The weighted path matrix of* G *is*

$$B = \begin{bmatrix} z_{1,1,1}z_Uz_{1,3,2} + z_{1,1,1}z_Dz_{1,3,2} & z_{1,1,1}z_Uz_{2,3,2} + z_{1,1,1}z_Dz_{2,3,2} & z_{1,1,1}z_{2,1,2}z_{2,2,1}z_{3,2,2} \\ z_{2,1,1}z_Uz_{1,3,2} + z_{2,1,1}z_Dz_{1,3,2} & z_{2,1,1}z_Uz_{2,3,2} + z_{2,1,1}z_Dz_{2,3,2} & z_{2,1,1}z_{2,1,2}z_{2,2,1}z_{3,2,2} \\ z_{3,2,1}z_{2,2,2}z_{2,3,1}z_{1,3,2} & z_{3,2,1}z_{2,2,2}z_{2,3,1}z_{2,3,2} & z_{3,2,1}z_{3,2,2} \end{bmatrix} \qquad (3.6)$$

where $z_U = z_{2,1,2}z_{2,2,1}z_{2,2,2}z_{2,3,1}$, $z_D = z_{1,1,2}z_{1,3,1}$. *Now define* $\theta : \mathbb{Z}[\mathfrak{S}_n] \to \mathbb{Z}$ *by* $\theta(e) = 1$, $\theta(s_1s_2s_1) = -1$, $\theta(w) = 0$ *otherwise. We have* $\theta((1 + s_1)(1 + s_2)(1 + s_1)) = 1$. *Applying Lindström's Lemma to the wiring diagram (3.5), one sees that the corresponding immanant* $\mathrm{Imm}_\theta(x) = x_{1,1}x_{2,2}x_{3,3} - x_{1,3}x_{2,2}x_{3,1} = \det(x_{13,13})x_{2,2}$ *satisfies* $[z_G]\mathrm{Imm}_\theta(B) = 1$, *since exactly one family of paths* $\pi = (\pi_1, \pi_2, \pi_3)$ *from all sources to the corresponding sinks covers* G *(has weight* z_G*) and satisfies* $\pi_1 \cap \pi_3 = \emptyset$. *(See (5.9).)*

It is natural to ask for a q-analog of Proposition 3.1 which applies to the computation of $\theta_q((1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}}))$ for a linear function $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$. While the most naive approach will not suffice, we will state and prove the desired q-analog in Section 4.

4 The q-immanant evaluation theorem for wiring diagrams

In order to evaluate a linear function $\theta_q : H_n(q) \to \mathbb{Z}$ at $(1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})$ by relating this evaluation to the generating function

$$\mathrm{Imm}_{\theta_q}(x) := \sum_{w \in \mathfrak{S}_n} \theta_q(T_w) q_{e,w}^{-1} x_{1,w_1} \cdots x_{n,w_n} \in \mathcal{A}, \qquad (4.1)$$

we begin by assigning weights to the edges of the wiring diagram G (3.2) exactly as in (3.4). But now we define two indeterminates $z_{h,j,k}$, $z_{h',j',k'}$ to commute only if $j \neq j'$ or $k \neq k'$; otherwise we impose the relation

$$z_{i_j+1,j,k} z_{i_j,j,k} = q^{\frac{1}{2}} z_{i_j,j,k} z_{i_j+1,j,k}. \qquad (4.2)$$

Let Z_G be the quotient of $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]\langle z_{i_j,j,1}, z_{i_j,j,2}, z_{i_j+1,j,1}, z_{i_j+1,j,2} \,|\, j = 1, \ldots, m, \rangle$ modulo the ideal generated by the above commuting and quasicommuting relations, and assume that $q^{\frac{1}{2}}$, $q^{\frac{-1}{2}}$ commute with all other indeterminates. Let z_G be the product of all $4m$ indeterminates $z_{i,j,k}$, in lexicographic order.

This small change in the indeterminates $z_{1,1,1}, \ldots, z_{i_m,m,2}$ does not imply that the most naive q-analog of Proposition 3.1 holds, however. Indeed, the evaluation of an element of $\mathcal{A}$ at a matrix is not well defined unless the entries of that matrix satisfy the relations (2.1). We therefore define the $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-linear map

$$\begin{aligned} \sigma_B : \mathcal{A}_{[n],[n]} &\to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}] \\ x_{1,v_1} \cdots x_{n,v_n} &\mapsto [z_G] b_{1,v_1} \cdots b_{n,v_n}, \end{aligned} \tag{4.3}$$

where $[z_G] b_{1,v_1} \cdots b_{n,v_n}$ is the coefficient of z_G in $b_{1,v_1} \cdots b_{n,v_n}$, taken after $b_{1,v_1} \cdots b_{n,v_n}$ is expanded in the lexicographic basis of Z_G. Note that the substitution $x_{i,j} \mapsto b_{i,j}$ is performed only for monomials of the form $x^{e,v}$ in $\mathcal{A}_{[n],[n]}$; we define $\sigma_B(x^{u,w})$ by first expanding $x^{u,w}$ in the basis $\{x^{e,v} \,|\, v \in \mathfrak{S}_n\}$, and *then* performing the substitution.

Example 4.1. *Let us compute $\sigma_B(x_{2,2}x_{1,1}x_{3,3})$ for the path matrix B (3.6) of the wiring diagram in (3.5). Using (2.1) and linearity of σ_B, we write*

$$\begin{aligned} \sigma_B(x_{2,2}x_{1,1}x_{3,3}) &= \sigma_B(x_{1,1}x_{2,2}x_{3,3}) + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\sigma_B(x_{1,2}x_{2,1}x_{3,3}) \\ &= [z_G] b_{1,1}b_{2,2}b_{3,3} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})[z_G] b_{1,2}b_{2,1}b_{3,3}. \end{aligned} \tag{4.4}$$

Expanding $b_{1,1}b_{2,2}b_{3,3}$ and omitting terms with repeated indeterminates, we have

$$\begin{aligned} [z_G] b_{1,1}b_{2,2}b_{3,3} = [z_G](&z_{1,1,1}z_{1,1,2}z_{1,3,1}z_{1,3,2}z_{2,1,1}z_{2,1,2}z_{2,2,1}z_{2,2,2}z_{2,3,1}z_{2,3,2}z_{3,2,1}z_{3,2,2} \\ &+ z_{1,1,1}z_{2,1,2}z_{2,2,1}z_{2,2,2}z_{2,3,1}z_{1,3,2}z_{2,1,1}z_{1,1,2}z_{1,3,1}z_{2,3,2}z_{3,2,1}z_{3,2,2}). \end{aligned}$$

Sorting indeterminates into lexicographic order and using (4.2), we see that this is

$$[z_G](z_G + qz_G) = 1 + q.$$

Similarly computing $[z_G] b_{1,2}b_{2,1}b_{3,3}$, we obtain $(q^{\frac{1}{2}} + q^{\frac{3}{2}})$. Thus Equation (4.4) gives

$$\sigma_B(x_{2,2}x_{1,1}x_{3,3}) = (1 + q) + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(q^{\frac{1}{2}} + q^{\frac{3}{2}}) = q + q^2.$$

The map σ_B behaves well with respect to concatenation of star networks.

Proposition 4.2. *Let G and H be wiring diagrams corresponding to expressions $s_{i_1} \cdots s_{i_k}$ and $s_{i_{k+1}} \cdots s_{i_m}$, with weighted path matrices B and C respectively, so that $G \circ H$ corresponds to expression $s_{i_1} \cdots s_{i_m}$ and has weighted path matrix BC. Then for all $u, w \in \mathfrak{S}_n$ we have*

$$\sigma_{BC}(x^{u,w}) = \sum_{v \in \mathfrak{S}_n} \sigma_B(x^{u,v})\sigma_C(x^{v,w}). \tag{4.5}$$

Proof of Proposition 4.2. (Omitted.) □

The special case of Proposition 4.2 in which we have $m = k+1$ (so that H is the wiring diagram of the single generator $s_{i_{k+1}}$) leads to a simple formula for $\sigma_{BC}(x^{u,w})$ in terms of B.

Corollary 4.3. *Let wiring diagrams G, H of expressions $s_{i_1} \cdots s_{i_k}$, $s_{i_{k+1}}$ have weighted path matrices B, C, respectively. Then for all $w \in \mathfrak{S}_n$ we have*

$$\sigma_{BC}(x^{u,w}) = q^{\frac{1}{2}} \sigma_B(x^{u,ws_{i_{k+1}}}) + \begin{cases} q\sigma_B(x^{u,w}) & \text{if } ws_{i_{k+1}} < w, \\ \sigma_B(x^{u,w}) & \text{if } ws_{i_{k+1}} > w. \end{cases}$$

Proof of Corollary 4.3. Use the combinatorial interpretation in Theorem 2.4, and the relations (4.2). □

An important property of the map σ_B is that its evaluation at monomial basis elements of $\mathcal{A}_{[n],[n]}$ is related to coefficients in the expansion of $(1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})$.

Proposition 4.4. *Let G be the wiring diagram in (3.2) with weighted path matrix B, and fix $w \in \mathfrak{S}_n$. Then $\sigma_B(x^{e,w})$ is equal to $q_{e,w}$ times the coefficient of T_w in $(1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})$.*

Proof of Proposition 4.4. (Idea.) Use induction, Proposition 4.2, Corollary 4.3, and the relations (1.1). □

As a consequence of Proposition 4.4, we now have a q-analog of Proposition 3.1. Namely, when B is the path matrix of a wiring diagram, and $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ is linear, we may "evaluate" $\mathrm{Imm}_{\theta_q}(x)$ at B by applying the map σ_B to $\mathrm{Imm}_{\theta_q}(x)$.

Theorem 4.5. *Let $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ be linear, and let wiring diagram G of $s_{i_1} \cdots s_{i_m}$ have weighted path matrix B. Then we have*

$$\theta_q((1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})) = \sigma_B(\mathrm{Imm}_{\theta_q}(x)). \tag{4.6}$$

Proof of Theorem 4.5. Write $(1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}}) = \sum_{v \in \mathfrak{S}_n} a_v T_v$. Then the right-hand side of (4.6) is

$$\sigma_B\Big(\sum_{v \in \mathfrak{S}_n} \theta_q(T_v) q_{e,v}^{-1} x^{e,v}\Big) = \sum_{v \in \mathfrak{S}_n} \theta_q(T_v) q_{e,v}^{-1} \sigma_B(x^{e,v}) = \sum_{v \in \mathfrak{S}_n} \theta_q(T_v) q_{e,v}^{-1} q_{e,v} a_v = \theta_q\Big(\sum_{v \in \mathfrak{S}_n} a_v T_v\Big),$$

by Proposition 4.4. But this is precisely the left-hand side of (4.6). □

Now observe that if one fixes a reduced expression $s_{i_1} \cdots s_{i_m}$ for each $w \in \mathfrak{S}_n$ and uses each such expression to define an element

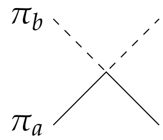
$$D_w := (1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}}) \in H_n(q),$$

then $\{D_w \,|\, w \in \mathfrak{S}_n\}$ forms a basis of $H_n(q)$: we have $D_w \in T_w + \mathrm{span}_{\mathbb{Z}[q]}\{T_v \,|\, v < w\}$. (See also [3, Corollary 3.6].) Thus we can evaluate $\theta_q(g)$ for every $g \in H_n(q)$, provided that we can expand g in this basis.

5 G-tableaux and the evaluation of induced sign characters

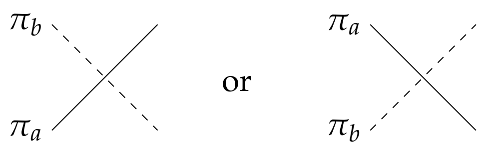
Given a path family π covering a wiring diagram G, we will find it convenient to arrange the paths of π into a Young diagram. We will call the resulting structure U a *G-tableau*, or more specifically a *π-tableau*. If $\mathrm{type}(\pi) = w$, we say that U has type w. We call U *left column-strict* if π_b appears above π_a in a column only when $a < b$. We call U *column-strict* it is left column-strict and no two paths in a column intersect. We call U *column-closed* if in each column, the sets of source and sink indices of paths there are equal. We define $L(U)$ and $R(U)$ to be the Young tableaux in which paths are replaced by their source indices and sink indices, respectively.

Let U be a π-tableau of any shape $\lambda \vdash n$. Define $\textsc{invnc}(U)$ to be the number of inverted noncrossings of U, i.e., the number of occurrences of



, (5.1)

where π_b appears in an earlier column of U than π_a (whether or not $b > a$). Thus inverted noncrossings may be proper or defective. Define $\textsc{c}(U) = \textsc{c}(\pi)$ to be the number of crossings of π, i.e., the number of occurrences of



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This depends only upon π; not upon the locations of π_a and π_b in U. Define $\textsc{cdnc}(U)$ to be the number of defective noncrossings of pairs of paths appearing in the same column of U, i.e., the number of occurrences of (5.1) where $b < a$ and π_b, π_a appear in the same column of U.

Proposition 5.1. *Let wiring diagram G have weighted path matrix B. For $u, w \in \mathfrak{S}_n$ we have*

$$\sigma_B(x^{u,w}) = \sum_{\pi} q^{\frac{\textsc{c}(\pi)}{2}} q^{\textsc{invnc}(U)}, \tag{5.2}$$

where the sum is over path families π of type $u^{-1}w$ covering G, and $U = U(\pi, u, w)$ is the unique π-tableau of shape (n) satisfying $L(U) = u$, $R(U) = w$.

Proof of Proposition 5.1. (Omitted.) □

By Theorem 4.5, the map σ_B (4.3) can be used to evaluate $\epsilon_q^\lambda(D_w)$ when one has a simple expression for the generating function $\mathrm{Imm}_{\epsilon_q^\lambda}(x)$ and can evaluate $\sigma_B(\mathrm{Imm}_{\epsilon_q^\lambda}(x))$. Such an expression was given by Konvalinka and the third author in [6, Theorem 5.4]:

$$\mathrm{Imm}_{\epsilon_q^\lambda}(x) = \sum_{(I_1,\dots,I_r)} \det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r}), \tag{5.3}$$

where $\lambda = (\lambda_1, \ldots, \lambda_r)$, $\det_q$, $(x_{I,I})$ are defined as in Section 2, and the sum is over all ordered set partitions of $\{1, \ldots, n\}$ satisfying $|I_j| = \lambda_j$. We will say that such an ordered set partition has *type* λ. To evaluate $\sigma_B(\mathrm{Imm}_{\epsilon_q^\lambda}(x))$, we will expand the right-hand side of (5.3) and recognize the resulting monomials as elements of bases $\{x^{u,v} \,|\, v \in \mathfrak{S}_n\}$ of $\mathcal{A}_{[n],[n]}$. These bases are best described in terms of the *Young subgroup* $\mathfrak{S}_\lambda$ of $\mathfrak{S}_n$, the subgroup generated by $\{s_1, \ldots, s_{n-1}\} \smallsetminus \{s_{\lambda_1}, s_{\lambda_1+\lambda_2}, s_{\lambda_1+\lambda_2+\lambda_3}, \ldots, s_{n-\lambda_r}\}$. Let $\mathfrak{S}_\lambda^-$ be the set of Bruhat-minimal representatives of cosets of the form $\mathfrak{S}_\lambda u$, i.e., the elements $u \in \mathfrak{S}_n$ for which each of the subwords

$$u_1 \cdots u_{\lambda_1}, \qquad u_{\lambda_1+1} \cdots u_{\lambda_1+\lambda_2}, \quad \ldots, \quad u_{n-\lambda_r+1} \cdots u_n \tag{5.4}$$

is strictly increasing. Such elements correspond bijectively to ordered set partitions $(I_1, \ldots, I_r)$ of $[n]$ of type λ. Specifically, the correspondence $u \mapsto I(u)$ is given by

$$I_j = \{u_{\lambda_1+\cdots+\lambda_{j-1}+1}, \ldots, u_{\lambda_1+\cdots+\lambda_j}\} \text{ for } j = 1, \ldots, r. \tag{5.5}$$

We will let $u(I)$ denote the permutation in $\mathfrak{S}_\lambda^-$ which corresponds to I.

To interpret $\epsilon_q^\lambda((1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})) = \sigma_B(\mathrm{Imm}_{\epsilon_q^\lambda}(x))$, we refer to the treatment of the formula (5.3) which appears in [2, Equation (6.1)]:

$$\sum_{(I_1,\ldots,I_r)} \sigma_B(\det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r})) = \sum_{u \in \mathfrak{S}_\lambda^-} \sum_{y \in \mathfrak{S}_\lambda} (-1)^{\ell(y)} q_{e,y}^{-1} \sigma_B(x^{u,yu}), \tag{5.6}$$

where the first sum is over all ordered set partitions $(I_1, \ldots, I_r)$ of $[n]$ of type λ. To interpret the final sum in (5.6) we need to consider evaluations of the form (5.2) only for $u \in \mathfrak{S}_\lambda^-$ for some partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, and $v = yu$ for $y \in \mathfrak{S}_\lambda^-$. Since elements of $\mathfrak{S}_\lambda^-$ correspond bijectively to ordered set partitions I of $[n]$ of type λ, and $\sigma_B(x^{u,yu})$ has the interpretation given in Proposition 5.1, it will be convenient to define

$$\mathcal{U}_I = \mathcal{U}_I(G) = \{U(\pi, u, yu) \,|\, \pi \text{ covers } G, u = u(I), \mathrm{type}(\pi) = y \in \mathfrak{S}_\lambda\}.$$

Note that our restriction on y forces the sink indices of paths located in the λ_j entries $(\lambda_1 + \cdots + \lambda_{j-1} + 1), \ldots, (\lambda_1 + \cdots + \lambda_j)$ of $U(\pi, u, yu)$ to be a permutation of the source indices of the same paths.

On the other hand, the final sum in (5.6) has both positive and negative signs. We will obtain a subtraction-free expression for this sum by defining a sign-reversing involution $\zeta = \zeta_I$ for each ordered set partition I of type $\lambda = (\lambda_1, \ldots, \lambda_r)$. It will be convenient to define this involution on a second set of tableaux, in obvious bijection with the first set. For a wiring diagram G, let $\mathcal{T}_I = \mathcal{T}_I(G)$ be the set of all column-closed, left column-strict G-tableaux W of shape $\lambda^\top$ such that $L(W^\top)_j = I_j$ (as sets) for $j = 1, \ldots, r$. The bijection $\delta_I : \mathcal{U}_I \to \mathcal{T}_I$ maps $U \in \mathcal{U}_I$ to the left column-strict tableau W of shape $\lambda^\top$ whose jth column consists of entries $(\lambda_1 + \cdots + \lambda_{j-1} + 1), \ldots, (\lambda_1 + \cdots + \lambda_j)$ of U.

Since U and $\delta_I(U)$ contain the same path family, it is easy to see that δ_I does not affect the statistic C. It changes the statistic INVNC in a very simple way.

Lemma 5.2. *Let $W = \delta_I(U)$. Then we have $\textsc{invnc}(U) = \textsc{invnc}(W) + \textsc{cdnc}(W)$.*

Proof of Lemma 5.2. (Omitted.) □

Fixing a wiring diagram G and ordered set partition I, we define a sign-reversing involution $\zeta : \mathcal{T}_I \to \mathcal{T}_I$ as follows.

1. If W is a column-strict tableau of type e, then define $\zeta(W) = W$.
2. Otherwise,
   (a) Let t be the greatest index such that column t of W is not column-strict.
   (b) Let k be the greatest index such that two paths π_j, $\pi_{j'}$ with $j, j' \in I_t$ both pass through the central vertex of the factor network $G_{[i_k,i_{k+1}]}$, and let π'_j, $\pi'_{j'}$ be the paths obtained by swapping the terminal subpaths of π_j and $\pi_{j'}$, beginning at the central vertex of $G_{[i_k,i_{k+1}]}$.
   (c) Using the above indices j, j', define $\zeta(W)$ to be the tableau obtained from W by replacing π_j, $\pi_{j'}$ by π'_j, $\pi'_{j'}$, respectively.

Proposition 5.3. *Fix $W \in \mathcal{T}_I$ containing path family $\pi = (\pi_1, \ldots, \pi_n)$ and satisfying $\zeta(W) \neq W$. Let π' be the path family in $\zeta(W)$ and let $v = R(\delta_I^{-1}(W))$, $v' = R(\delta_I^{-1}(\zeta(W)))$. Then we have $\textsc{invnc}(\zeta(W)) = \textsc{invnc}(W)$ and*

$$\textsc{cdnc}(\zeta(W)) + \frac{\mathrm{c}(\pi')}{2} = \begin{cases} \textsc{cdnc}(W) + \frac{\mathrm{c}(\pi)+1}{2} & \text{if } v < v', \\ \textsc{cdnc}(W) + \frac{\mathrm{c}(\pi)-1}{2} & \text{if } v > v'. \end{cases} \tag{5.7}$$

Proof of Proposition 5.3. (Omitted.) □

Now we can state and prove our main result.

Theorem 5.4. *Let G be the wiring diagram of $s_{i_1} \cdots s_{i_m}$. Then for $\lambda \vdash n$ we have*

$$\epsilon_q^\lambda((1 + T_{s_{i_1}}) \cdots (1 + T_{s_{i_m}})) = \sum_U q^{\textsc{invnc}(U) + \mathrm{c}(U)/2},$$

where the sum is over all column-strict G-tableaux of type e and shape $\lambda^\top$.

Proof of Theorem 5.4. (Idea.) Apply Proposition 5.1 and Lemma 5.2 to all monomials in (5.6) to obtain

$$\sum_{(y,\pi)} (-1)^{\ell(y)} q_{e,y}^{-1} q^{\frac{\mathrm{c}(\pi)}{2}} q^{\textsc{invnc}(W) + \textsc{cdnc}(W)}, \tag{5.8}$$

where the sum is over pairs (y, π) with $y \in \mathfrak{S}_\lambda$ and π a path family of type $u^{-1}yu$ which covers G. $W = \delta_I(U(\pi, u, yu))$ is then uniquely determined by (y, π) and varies over all tableaux in $\mathcal{T}_I$. Contributions from tableaux W, $\zeta(W) \neq W$ cancel one another, and by Proposition 5.3 the remaining tableaux contribute the claimed powers of q. □

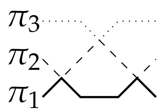
Corollary 5.5. *Let G be the wiring diagram of a reduced expression for a* 321*-hexagon-avoiding permutation w. Then we have*

$$\epsilon_q^\lambda(q_{e,w}C'_w(q)) = \sum_U q^{\mathrm{INVNC}(U)+\mathrm{C}(U)/2},$$

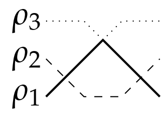
where the sum is over all column-strict G-tableaux of type e and shape $\lambda^\top$.

Proof of Corollary 5.5 . Billey and Warrington [1, Theorem 1] showed that if this reduced expression is $s_{i_1}\cdots s_{i_m}$ then we have $q_{e,w}C'_w(q) = (1+T_{s_{i_1}})\cdots(1+T_{s_{i_m}})$. (See page 5.) $\square$

To illustrate the theorem, we compute $\epsilon_q^{21}((1+T_{s_1})(1+T_{s_2})(1+T_{s_1}))$ using the wiring diagram (3.5). There are two path families of type e which cover G, and one column-strict G-tableau of shape $21^\top = 21$ for each:



$$, \quad U_\pi = \begin{array}{|c|c} \hline \pi_3 \\ \hline \pi_1 & \pi_2 \\ \hline \end{array}\,; \qquad \tag{5.9}$$



$$, \quad U_\rho = \begin{array}{|c|c} \hline \rho_3 \\ \hline \rho_2 & \rho_1 \\ \hline \end{array}\,.$$

Tableau U_π contributes $q^{\mathrm{INVNC}(U_\pi)}q^{\mathrm{C}(U_\pi)/2} = q^1q^{0/2} = q$, since π has no crossings, and only one of its noncrossings is inverted in U_π: π_3 intersects π_2 from above and appears in an earlier column of U_π. Tableau U_ρ contributes $q^{\mathrm{INVNC}(U_\rho)}q^{\mathrm{C}(U_\rho)/2} = q^1q^{2/2} = q^2$, since ρ has two crossings, and its unique noncrossing is inverted in U_ρ: ρ_3 intersects ρ_1 from above and appears in an earlier column of U_ρ. Adding the two contributions together, we have $\epsilon_q^{21}((1+T_{s_1})(1+T_{s_2})(1+T_{s_1})) = q + q^2$.

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