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The Canonical Join Complex of the Tamari lattice

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Abstract. In this paper, we study a simplicial complex on the elements of the Tamari lattice in types A and B called the canonical join complex. The canonical join representation of an element w in a lattice L is the unique lowest expression $\bigvee A$ for w. We abuse notation and also say that the set A is a canonical join representation (when we mean $\bigvee A$ is a canonical join representation). The collection of all such subsets is an abstract simplicial complex called the canonical join complex of L. We realize the canonical join complex of the Tamari lattice as a complex of noncrossing arc diagrams, give a shelling order on its facets, and show that it is homotopy equivalent to a wedge of Catalan-many spheres.

Résumé. Dans cet article, nous étudions un complexe simplicial sur les éléments du Treillis de Tamari en types A et B appelé complexe sup-canonique. Nous caractérisons le complexe sup-canonique du Treillis de Tamari comme un complexe de diagrammes d'arcs non croisés, donnons un ordre d'épluchage sur ses facettes, et montrons qu'il est homotope á un "wedge" de plusieurs sphéres de type Catalan.

1 Introduction

In this paper, we study a certain simplicial complex on the elements of the Tamari lattice arising from a lattice-theoretic "factorization" called the canonical join representation. Informally, the canonical join representation of an element w is the unique lowest irredundant expression $\bigvee A$ for w. An expression $\bigvee A$ is irredundant if for each $A' \subsetneq A$, the join $\bigvee A'$ is strictly smaller than $\bigvee A$. In Section 2.1, we make the notion of "lowest" precise by comparing the order ideal generated by A under containment, for each such expression. For example, the canonical join representation of an element in the boolean lattice is the join of the atoms below it. In the Tamari lattice shown in Figure 2, the top element $\hat{1}$ has three irredundant join representations: $\bigvee\{\hat{1}\}$, $\bigvee\{x, z\}$, and $\bigvee\{x, y\}$. The canonical join representation is the lowest among these, the join of the atoms $\{x, y\}$. When $\bigvee A$ is the canonical join representation for some element $w \in L$, we will abuse notation and say that the set A is a canonical join representation (when, more precisely, we mean that the expression $\bigvee A$ is a canonical join representation).

In a finite lattice L, each element admits a canonical join representation if and only if L satisfies a certain weakening of the distributive law called join-semidistributivity.

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(For a non-example, see Example 2.2 and Figure 4.) In this case, we say that L is join-semidistributive. We define the ***canonical join complex*** of L to be the abstract simplicial complex whose faces are the subsets A of L such that A is a canonical join representation. (By [11, Proposition 2.2] this is indeed a complex.) In general, the canonical join complex is not a pure complex. In particular, the canonical join complex of the Tamari lattice is very different from the associahedron.

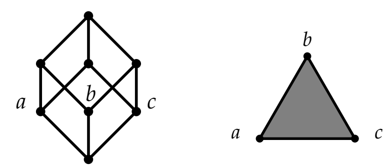

Figure 1: The canonical join complex of the Boolean lattice is a simplex on its atoms.

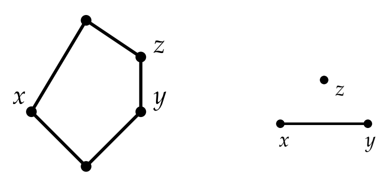

Figure 2: A Tamari lattice and its canonical join complex.

For each finite Coxeter group W and each orientation c of its associated Coxeter diagram, there is a lattice quotient of the weak order on W called the ***c-Cambrian lattice***. The canonical join representation of its elements is closely related to the associated cluster algebra and to the noncrossing partition lattice $NC(W, c)$ [12]. In type A, each c-Cambrian lattice is a lattice quotient of the weak order on S_n, consisting of certain pattern avoiding permutations. In particular, when c is a linear orientation – an orientation in which all of the arrows point in the same direction – the corresponding c-Cambrian lattice is a Tamari lattice. For one choice of linear orientation, the elements of this quotient are the 312-avoiding permutations. Throughout, we write T_n for this realization of the Tamari lattice. That is, T_n is the subset of the weak order on S_n induced by the set of 312-avoiding permutations. (For the opposite orientation, the elements of the corresponding c-Cambrian lattice avoid the pattern 231.)

As with the classical Tamari lattice, the type-B Tamari lattice can be realized as a partial order on certain triangulations of a fixed convex polygon or certain bracket vectors. We realize the type-B Tamari lattice T_n^s as a c-Cambrian lattice for the type-B Coxeter group B_n where c is a linear orientation for the type-B Coxeter diagram. See [10, Section 7] and [13].

In the following theorems and throughout this abstract, we do not distinguish between an abstract simplicial complex and its geometric realization. In the statements,

$\mathrm{Cat}(A_{r-1}) = \frac{1}{r+1}\binom{2r}{r}$ is the classical Catalan number, $\mathrm{Cat}(B_r) = \binom{2r}{r}$ is the type-B analogue, and $\mathrm{Cat}^+(B_r) = \binom{2r-1}{r-1}$ is the type-B positive Catalan number.

Theorem 1.1. *The canonical join complex of the Tamari lattice T_n is shellable. It is contractible when n is even and homotopy equivalent to a wedge of $\mathrm{Cat}(A_{r-1})$ many spheres, all of dimension $r-1$, when $n = 2r+1$.*

Theorem 1.2. *The canonical join complex of the type-B Tamari lattice T_n^s is shellable.*

1. *When $n = 2r$, the canonical join complex is homotopy equivalent to a wedge of $\mathrm{Cat}(B_r)$ many spheres all of dimension $r-1$.*
2. *When $n = 2r-1$ for $r > 1$, the canonical join complex is homotopy equivalent to a wedge of $\mathrm{Cat}^+(B_r) - \mathrm{Cat}(A_{r-2}) = 2\binom{2r-2}{r-2}$ many spheres, equally distributed in dimensions $r-1$ and $r-2$.*

Canonical join representations have played a key role in Coxeter–Catalan combinatorics [12, Section 8] and in Coxeter-biCatalan combinatorics [4]. More recently, canonical join representations have appeared in the study of the lattice of torsion classes over a finite dimensional associative algebra [3]. The topology of a join-semidistributive lattice is closely related the combinatorics of its canonical join complex. For example, see [1, Theorem 1.2 and Corollary 1.3].

The canonical join complex was first defined in [11], and studied in depth in [1]. In [11], Reading considered the canonical join complex of the symmetric group S_n (ordered according to the weak order) and its connections to enumerative problems involving pattern avoiding-permutations. The canonical join representation of a permutation is encoded by a noncrossing arc diagram, a generalization of the bump diagram for a noncrossing partition. Each diagram consists of a collection of curves, called arcs, that satisfy certain compatibility relations. For example, no two arcs may intersect in their interiors. (See Section 2.2 for the complete definition.) Each arc corresponds to a vertex of the canonical join complex, and a collection of arcs corresponds to a face if and only if each pair of arcs is compatible. (This is [11, Corollary 3.5].) Figure 3 shows the noncrossing arc diagrams that correspond to the faces in the canonical join complex of the weak order on S_3.

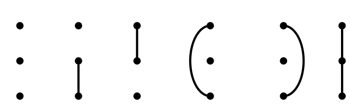

Figure 3: The faces in the canonical join complex of the weak order on S_3.

Like the h-complex of the Coxeter complex defined in [8], the entries of the f-vector of the canonical join complex of the weak order on the symmetric group are equal to the

Eulerian numbers. (However, in general, the canonical join complex of the symmetric group is not isomorphic, or even homotopy-equivalent, to the h-complex of the Coxeter complex.) Similar statements hold for the canonical join complex of the Tamari lattice and each c-Cambrian lattice: The entries of the f-vector of the canonical join complex of the Tamari lattice (in both types A and B) are equal to the Narayana numbers (of type A and B respectively). As an immediate consequence of Theorems 1.1 and 1.2, the alternating sum of the Narayana numbers is either zero or a signed Catalan number. For n even, the alternating sum of type-B Narayana numbers is the type-B Catalan number. These identities are well-known and also appear as specializations of Coker's identities. See [7], or [6, Equation 1.1] for the type-A case and [6, Equation 2.1] for the type-B case.

We conclude this introduction by considering the topology the canonical join complex of the more general c-Cambrian lattices in type A.

Theorem 1.3. *For each orientation c of the type-A Coxeter diagram, the canonical join complex of the corresponding c-Cambrian lattice is vertex decomposable.*

Since vertex decomposability implies shellability, and the Tamari lattice is an example of a c-Cambrian lattice, Theorem 1.3 implies the shellability assertion in Theorem 1.1. (We highlight Theorem 1.1 here because its proof is more approachable, and it will motivate the proof of the analogous type-B result.) The particularly nice topological results for the Tamari lattice and the Tamari-like c-Cambrian lattices in type A do not extend to other finite Coxeter groups. For example, for each orientation c, the c-Cambrian lattice in the type-D_5 Coxeter group is not shellable.

2 Background

2.1 Lattice-theoretic background

In this section, we briefly review the necessary lattice-theoretic terminology. Throughout, we assume that L is a finite lattice. A ***join representation*** for an element $w \in L$ is an expression $\bigvee A$ that evaluates to w, where A is a subset of L. A join representation $\bigvee A$ is a ***irredundant*** if $\bigvee A' < \bigvee A$ for each proper subset $A' \subsetneq A$. Observe that if $\bigvee A$ is irredundant, then A is an antichain. We write $\mathrm{ijr}(w)$ for the collection of irredundant join representations of w. We partially order $\mathrm{ijr}(w)$ as follows: $A \ll B$ whenever the order ideal generated by A is contained in the order ideal generated by B. (This relation is also sometimes called ***join-refinement*** [9, Section I.3].) The ***canonical join representation*** of w is the unique minimal element of $\mathrm{ijr}(w)$, when such an element exists.

Example 2.1. *Recall that j is join-irreducible if, whenever $j = \bigvee A$, we have $j \in A$. (Equivalently, j is join-irreducible if and only if it covers precisely one element in L.) Thus, if j is join-irreducible then $\bigvee\{j\}$ is its canonical join representation. On the other hand, if $\bigvee A$ is a canonical join representation, then each element $a \in A$ is join-irreducible.*

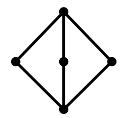

Figure 4: The canonical join representation of the top element does not exist.

Example 2.2. *Let w be the top element of the lattice shown in Figure 4. Each pair of atoms is a minimal join representation for w. Thus, w does not have a canonical join representation.*

When L is finite and each element admits a canonical join representation, we say that L is ***join-semidistributive***. If the dual lattice is also join-semidistributive, then we say that L is ***semidistributive***. (There is an equivalent definition that involves a weakening of the distributive law. See [9, Theorem 2.24].)

Suppose that L is a finite join-semidistributive lattice. We define the ***canonical join complex*** of L to be the collection of subsets A such that A is a canonical join representation. Observe that the vertex set for the canonical join complex is just the set of join-irreducible elements in L. Since L is join-semidistributive, the number of faces in the canonical join complex is equal to the number of elements in L. The next proposition is [11, Proposition 2.2], and it implies that the canonical join complex is indeed a simplicial complex.

Proposition 2.3. *Suppose L is a finite lattice and A is a canonical join representation in L. Then each subset of A is a canonical join representation.*

2.2 The noncrossing arc complex

In this section, we review the definition of a noncrossing arc diagram, establish some useful notation, and review the connection to canonical join representations. The definitions here are based on [11], where the reader will find additional examples. For the remainder of the paper, we write $[n]$ for the set $\{1, 2, \ldots, n\}$ and $[i, k]$ for the set $\{i, i+1, \ldots, k\}$ when $i < k$.

A ***noncrossing arc diagram*** consists of n nodes arranged vertically and labeled in increasing order from bottom to top, together with a (possibly empty) collection of curves called ***arcs***. Each arc connects two distinct nodes and travels monotonically upward from its lower endpoint to its higher endpoint, passing either to the left or to the right of each node in between. In addition, each pair of arcs α and α' must satisfy:

(C1) α and α' do not share the same top endpoint or the same bottom endpoint;

(C2) α and α' do not intersect in their interiors.

The ***support of an arc*** α, written $\mathrm{supp}(\alpha)$, with endpoints $i < l$ is the set of numbers $\{i, i+1, \ldots, l\}$. We write $\mathrm{supp}(\alpha)^\circ$ for the set $\{i+1, \ldots, l-1\}$. When $\mathrm{supp}(\alpha)^\circ$ is empty, we say that α is a ***simple arc***. We say that the arcs α and α' are ***combinatorially equivalent*** if α and α' have the same endpoints and for each $k \in \mathrm{supp}(\alpha)^\circ$, α and α' pass on the same side (either left or right) of k. Each arc is considered only up to combinatorial equivalence. Two arcs are ***compatible*** if there is a noncrossing arc diagram that contains them. The next proposition is [11, Proposition 3.2].

Proposition 2.4. *Given any collection of pairwise compatible arcs, there is a noncrossing arc diagram whose arcs are combinatorially equivalent to the given arcs.*

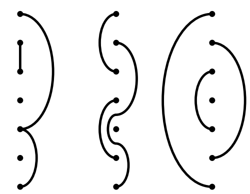

Figure 5: Nonempty faces of the noncrossing arc complex on seven nodes.

The ***noncrossing arc complex*** on n nodes is the simplicial complex whose faces are the collections of pairwise compatible arcs. We view each collection of compatible arcs as a noncrossing arc diagram. For example, Figure 5 depicts some of the nonempty faces in the noncrossing arc complex on seven nodes. To avoid confusion, we will only use the word *vertex* to refer to a vertex of the noncrossing arc complex; that is, a diagram that contains precisely one arc. The endpoint of an arc will always be referred to as a *node*.

The next theorem is [11, Corollary 3.4].

Theorem 2.5. *The canonical join complex of the weak order on S_n is isomorphic to the noncrossing arc complex on n nodes.*

Restricting to the set of 312-avoiding permutations, we obtain the canonical join complex of the Tamari lattice T_n. In the statement below, a ***right arc*** is an arc that does not pass to the left of any node between its endpoints. For example, the left-most noncrossing arc diagram in Figure 5 contains only right arcs. (See also [11, Example 4.9].)

Corollary 2.6. *The canonical join complex of the Tamari lattice T_n is isomorphic to the subcomplex of the noncrossing arc complex on n nodes induced by the set of right arcs.*

Recall that a complex is ***flag*** if each of its minimal non-faces has size 2. As an immediate consequence of Proposition 2.4 and Corollary 2.6, the canonical join complex of the Tamari lattice T_n is flag. (Indeed, the canonical join complex of any finite semidistributive lattice is flag. This is one direction of [1, Theorem 1.1].)

We write $\alpha_{i,k}$ for the right arc with endpoints $i < k$. (Observe that there is precisely one right arc for each pair of nodes $i, k \in [n]$.) Throughout the remainder of the paper, we write $\Delta(n)$ for the complex of compatible right arcs on n nodes. At times it is convenient to restrict the node set of an arc diagram to a contiguous subset of $[n]$. We write $\Delta([i,k])$ for the subcomplex of $\Delta(n)$ induced by restricting to the nodes $[i,k]$.

3 Shellability of the Tamari lattices

3.1 The Tamari lattice in type A

In this section we prove a more detailed version of Theorem 1.1. Before we begin, we recall some terminology. A ***d-complex*** is a simplicial complex in which the maximal dimension of the faces is equal to d. A d-complex is ***pure*** if each of its facets has dimension d. For each $n > 2$, the complex of compatible right arcs $\Delta(n)$ is not pure.

A (not necessarily pure) complex is ***shellable*** if its facets can be arranged in a linear order $F_1, \ldots, F_m$ so that the subcomplex $\left(\bigcup_{i=1}^{k-1} \overline{F_i}\right) \cap \overline{F_k}$ is a pure simplicial complex of dimension $\dim(F_k) - 1$ for all $k \in [2, m]$. (We write $\overline{F_k}$ for the collection of faces in F_k.) Such a linear order is called a ***shelling***. A facet F is a ***homology facet*** if $\left(\bigcup_{i=1}^{k-1} \overline{F_i}\right) \cap \overline{F_k}$ is equal to the entire boundary of F_k. The following theorem is a combination of [5, Theorem 3.4 and Theorem 4.1].

Theorem 3.1. *Suppose that Δ is a shellable complex. Then Δ is homotopy equivalent to a wedge of spheres where each r-dimensional sphere corresponds to an r-dimensional homology facet.*

Suppose that $\mathcal{L} = F_1, F_2, \ldots, F_m$ is a shelling of the facets for a non-pure simplicial complex. The ***rearrangement lemma*** [5, Lemma 2.6], says that $\mathcal{L}$ can be rearranged so that it satisfies the following condition. (We write (DD) for "decreasing dimension".)

$$\text{For facets } F \text{ and } F', \text{ if } |F| > |F'| \text{ then } F \text{ precedes } F' \text{ in } \mathcal{L}. \tag{DD}$$

We will see that this condition is sufficient for shelling the facets of $\Delta(n)$.

Fix some non-simple right arc $\alpha_{i,k} \in \Delta(n)$. Suppose that α' is a right arc that is compatible with $\alpha_{i,k}$. Note that α' does not have i as its bottom endpoint, nor $i+1$ as its top endpoint (otherwise the two arcs share bottom endpoints or they cross). Also, since α' is a right arc, it does not pass between i and $i+1$. Thus, $\{\alpha', \alpha_{i,i+1}\}$ is a face in $\Delta(n)$. Similarly, $\{\alpha', \alpha_{k-1,k}\} \in \Delta(n)$. Since $\Delta(n)$ is a flag complex, we obtain the following lemma.

Lemma 3.2. *Suppose that $\alpha_{i,k}$ is a right arc in $\Delta(n)$ with $1 \leqslant i < k-1 \leqslant n-1$. Then, for each face $F \cup \{\alpha_{i,k}\}$ in $\Delta(n)$, the set $F \cup \{\alpha_{i,i+1}, \alpha_{k-1,k}\}$ is in $\Delta(n)$.*

For each arc α in $\Delta(n)$ write $S(\alpha)$ for the set of simple arcs that are compatible with it. In the next lemma we show that the degree of a face J is determined by the set $\bigcap_{\alpha\in J} S(\alpha)$. Recall that the ***degree*** of F, denoted $\deg(F)$, is $\max\{|F'| : F' \supseteq F\}$.

Lemma 3.3. *Suppose that J is a face in $\Delta(n)$, and write $S' = \bigcap_{\alpha\in J} S(\alpha)$. Then, $S' \cup J$ is a facet of $\Delta(n)$, and every other face F that contains J has size strictly smaller than $|J \cup S'|$. In particular, $\deg(J) = |J \cup S'|$.*

Proof. Observe that S' is the unique maximal set of simple arcs that are compatible with each arc in J. Since any two simple arcs are compatible, $S' \cup J$ is in $\Delta(n)$. Suppose that $\alpha_{i,k}$ is a non-simple right arc satisfying: the set $J \cup S' \cup \{\alpha_{i,k}\}$ is in $\Delta(n)$. (In particular, $\alpha_{i,k}$ is compatible with each arc in S'.) Then Lemma 3.2 implies that $J \cup S' \cup \{\alpha_{i,i+1}, \alpha_{k-1,k}\}$ is also in $\Delta(n)$. The maximality of S' implies that $\{\alpha_{i,i+1}, \alpha_{k-1,k}\} \in S'$. But $\alpha_{i,k}$ is *not* compatible with either $\alpha_{i,i+1}$ or $\alpha_{k-1,k}$ because, for example, $\alpha_{i,k}$ and $\alpha_{i,i+1}$ share a bottom endpoint. We have reached a contradiction.

Suppose that F is a face in $\Delta(n)$ containing J, and $F \nsubseteq J \cup S'$. Thus, F contains some non-simple arc that does not belong to J. Applying Lemma 3.2, we replace each such non-simple arc (not in J) with a pair of simple arcs and obtain a chain of faces that is strictly increasing in size. This chain terminates in a face of the form $J \cup S''$, where S'' is a collection of simple arcs. Thus $S'' \subseteq S'$, and we conclude that $|F| < |J \cup S'|$. □

Finally, we prove a more detailed version of Theorem 1.1.

Theorem 3.4. *Let $\mathcal{L} = F_1, \ldots, F_m$ be a linear ordering of the facets of $\Delta(n)$ satisfying (DD). Then $\mathcal{L}$ is a shelling for $\Delta(n)$, and F_k is a homology facet if and only if it contains no simple arcs. Moreover,*

- *when $n = 2r$, each facet contains a simple arc;*
- *and when $n = 2r + 1$, each homology facet has precisely r arcs and maps bijectively to a noncrossing perfect matching on $[2r]$.*

Proof of Theorem 3.4 and Theorem 1.1. Let $F_1, \ldots, F_m$ be a linear ordering of the facets of $\Delta(n)$ satisfying (*DD*), and consider the complex $\overline{F_k} \bigcap \left(\bigcup_{i=1}^{k-1} \overline{F_i}\right)$, where k ranges over the set $[2, m]$. We write J for the set of non-simple arcs in F_k and S' for the set of simple arcs in F_k. Lemma 3.3 implies that every other facet containing J occurs after F_k in this linear ordering. So, each face of $\overline{F_k} \bigcap \left(\bigcup_{i=1}^{k-1} \overline{F_i}\right)$ is contained in $(J \cup S')\backslash\{\alpha\}$, for some α belonging to J. Lemma 3.2 says that we can swap out α in J for a pair of simple arcs, and obtain a face with strictly larger size. We conclude that $(J \cup S')\backslash\{\alpha\}$ is a facet of $\overline{F_k} \bigcap \left(\bigcup_{i=1}^{k-1} \overline{F_i}\right)$ for each $\alpha \in J$. We have proved that $F_1, \ldots, F_m$ is a shelling of $\Delta(n)$, and F_k is a homology facet if and only if it contains no simple arcs. We write $\mathcal{H}(n)$ for the set of noncrossing arc diagrams that are facets in $\Delta(n)$ and that do not contain any simple arcs. In general, we write $\mathcal{H}([i,k])$ for the set of noncrossing arc diagrams that are facets in $\Delta([i,k])$ and that do not contain any simple arcs.

Suppose that $n = 2r$, and F is a facet of $\Delta(n)$. We prove by induction on r that F contains a simple arc. Since F is a facet, there is some arc that has 1 as its bottom endpoint and $l \leqslant n$ as its top endpoint. If l is equal to 2, then we are done; assume that l is greater than 2. We remove this arc and both of its endpoints. If some other arc α' in F had l as its bottom endpoint, then we shift α' down so that it now has a bottom endpoint at the node $l-1$. (No other arc in F has $l-1$ as a bottom endpoint. Otherwise it would either cross the arc $\alpha_{1,l}$ or share a top endpoint with it.) We obtain a facet of $\Delta(n-2)$. Since this procedure preserves the size of the support of each arc in $F\backslash\{\alpha_{1,l}\}$, we are done by induction.

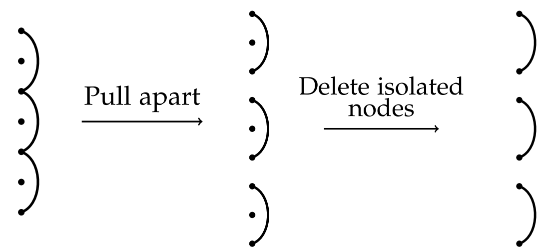

Figure 6: A demonstration of the map μ.

When $n = 2r+1$, we define a map μ from $\mathcal{H}(n)$ to the set of noncrossing perfect matchings on the set $[n-1]$ as follows: Each pair of arcs in a homology facet F that share an endpoint are pulled apart, and isolated nodes are deleted. See Figure 6. □

3.2 The Tamari lattice in type B

We now turn to the type-B Tamari lattice. Throughout, we write $[\pm n]$ for the set $\{-n,\ldots,-1,1,\ldots,n\}$ and $S_{\pm n}$ for the symmetric group on $[\pm n]$. A ***signed permutation*** (in full one-line notation) is a permutation $w_{-n}\ldots w_{-1}w_1\ldots w_n$ satisfying $w_{-i} = -w_i$. We realize B_n, the type-B Coxeter group of rank n, as the subposet of the weak order on $S_{\pm n}$ induced by the set of signed permutations. The ***type-B Tamari lattice*** T_n^s is the subposet of the weak order on B_n induced by the set of signed permutations that avoid the 312-pattern where the "2" is positive.

Consider the noncrossing arc diagram of a permutation $w \in S_{\pm n}$, with nodes labeled $-n,\ldots,-1,1,\ldots,n$ from bottom to top. A noncrossing arc diagram that is fixed by a half-turn rotation that sends each node i to $-i$ is called a ***symmetric noncrossing arc diagram***. A ***symmetric arc*** is either a pair of arcs that are related by this half-turn rotation or a single arc this is fixed by this rotation. See Figure 7 for some examples. The next corollary is [2, Proposition 3.2.10].

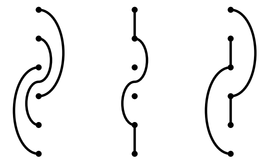

Figure 7: Each diagram contains two symmetric arcs.

Corollary 3.5. *The canonical join complex of the type-B Tamari lattice T_n^s is isomorphic to the subcomplex of symmetric noncrossing arc diagrams on $[\pm n]$ induced by set of symmetric arcs which do not pass to the left of any positive node or to the right of any negative node.*

We write $\Delta^s(n)$ for the canonical join complex of the type-B Tamari lattice T_n^s. There is precisely one symmetric arc for each pair of nodes in $[\pm n]$. Given a pair of arcs $\alpha_{i,k}$ and $\alpha_{-k,-i}$ that together comprise a symmetric arc in $\Delta^s(n)$, we write $\alpha^s_{i,k}$ for the corresponding symmetric arc, where $k > i$ and $k > -i$. When the endpoints of a symmetric arc are not specified, we simply write α^s. To distinguish the arc $\alpha_{i,k}$ from the symmetric arc $\alpha^s_{i,k}$, we sometimes refer to the former as an ***ordinary arc***. A ***simple symmetric arc*** is either a pair of simple arcs fixed by the half-turn rotation through the center of the diagram, or the ordinary simple arc with endpoints -1 and 1.

Theorem 3.6. *Let $\mathcal{L} = F_1, \ldots, F_m$ be a linear ordering of the facets of $\Delta^s(n)$ satisfying (DD) and the following condition: If F_i and F_k are facets with the same size and if the number of simple symmetric arcs in F_i is greater than the number of simple symmetric arcs in F_k, then $i < k$. Then $\mathcal{L}$ is a shelling of $\Delta^s(n)$, and F_i is a homology facet if and only if it does not contain any simple symmetric arcs.*

To conserve space, we will not prove Theorem 3.6. (See [2, Theorem 3.3.8] for the complete proof.) Instead, we sketch how to count the homology facets for $\Delta^s(n)$, when $n = 2r$. Let $\mathcal{H}^s(n)$ denote the set of symmetric noncrossing arc diagrams that are facets in $\Delta^s(n)$ and that contain no simple symmetric arcs. We define a map μ_s from $\mathcal{H}^s(n)$ to the set of symmetric noncrossing perfect matchings on $[\pm n]$. A ***symmetric noncrossing perfect matching*** on $[\pm n]$ is a noncrossing perfect matching M that satisfies $\{a, b\} \in M$ if and only if $\{-a, -b\} \in M$. The following proposition is [2, Proposition 3.2.13].

Proposition 3.7. *There are* $\mathrm{Cat}(B_r) = \binom{2r}{r}$ *many symmetric noncrossing perfect matchings on the set* $[\pm(2r)]$.

Suppose that $F \in \mathcal{H}^s(n)$. We would like to use the map μ (defined at the end of the proof for Theorem 3.4) whenever possible. To that end, we write $P(F)$ for the set of arcs $\alpha^s_{i,k} \in F$ with $0 < i < k$, and $N(F)$ for the set of arcs $\alpha^s_{i,k} \in F$ with $i < 0 < k$. Observe that the set $P(F)$ decomposes into a collection of smaller noncrossing arc diagrams, each of which is either a maximal collection of non-simple ordinary right arcs or, symmetrically, a maximal collection of non-simple ordinary "left arcs". We will apply the map μ to each collection of right arcs and, by symmetry, to each collection of left arcs. That leaves us with one main challenge: how to pair off the endpoints of the arcs in $N(F)$. We visualize a simplified version of this "pairing off" below. (The technical details can be found in [2, Section 3.3.2].)

First step. Cut every arc in $N(F)$ where it passes between -1 and 1. We call the resulting curves, each of which have precisely one endpoint, ***arc segments***. We write α_a for the arc segment whose endpoint is a. Reflect the negative half of the diagram about the vertical column of the nodes, so that each arc and arc segment passes to the right of each node.

Second step. Write $N(F) = \{\alpha^s_{i_1,k_1}, \alpha^s_{i_2,k_2}, \ldots, \alpha^s_{i_l,k_l}\}$ where $k_1 < \cdots < k_l$. After cutting the arcs in the step above, $-i_l$ is the top endpoint of the arc segment closest to the node 1. Anchor this arc segment to 1 and symmetrically anchor α_{i_l} to -1, unless $i_l = -1$. If $i_l = -1$, then we glue the segments α_{-i_l} and α_{i_l} together between -1 and 1. We glue each remaining arc segment α_a to the corresponding negative segment α_{-a}. See Figure 8 and Figure 9.

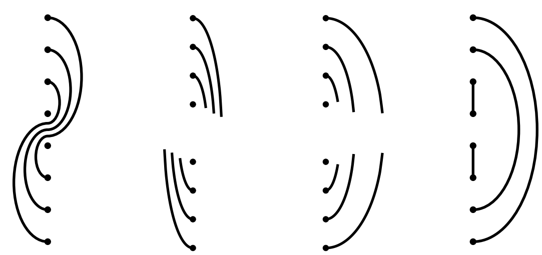

Figure 8: An illustration of the map μ_s when $i_l \neq -1$.

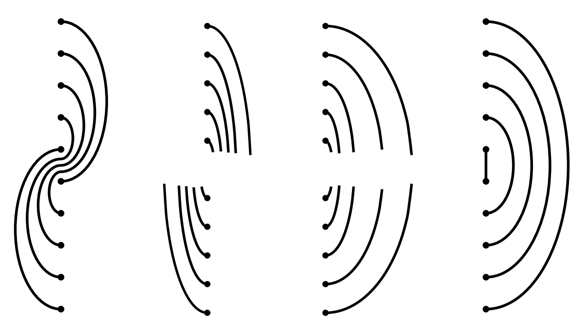

Figure 9: An illustration of the map μ_s when $i_l = -1$.

Proposition 3.8. *The map μ_s is a bijection from $\mathcal{H}^s(n)$ to the set of symmetric noncrossing perfect matchings on $[\pm n]$.*

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