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Crystal graphs, key tabloids, and nonsymmetric Macdonald polynomials

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Abstract. We construct a Demazure crystal for nonsymmetric Macdonald polynomials specialized at $t = 0$, giving a new proof that these specialized nonsymmetric Macdonald polynomials are graded sums of Demazure characters.

Keywords: Macdonald polynomials, Demazure characters, crystal graphs

1 Introduction

Macdonald [20] defined symmetric functions with two parameters $P_\lambda(X;q,t)$ indexed by partitions as the unique symmetric function basis satisfying certain triangularity (with respect to monomials in the variables $X = x_1, x_2, \ldots$) and orthogonality (with respect to a generalized Hall inner product) conditions. The coefficients of $P_\lambda(X;q,t)$ when written as a sum of monomials are rational functions in the parameters q and t. Based on hand computations for partitions up to size 8, Macdonald conjectured that the *Kostka–Macdonald coefficients* $K_{\lambda,\mu}(q,t)$ defined by expanding the *integral form* $J_\mu(X;q,t)$, a scalar multiple of the original $P_\lambda(X;q,t)$, into the plethystic Schur basis,

$$J_\mu(X;q,t) = \sum_\lambda K_{\lambda,\mu}(q,t) s_\lambda[X(1-t)], \tag{1.1}$$

are polynomials in q and t with nonnegative integer coefficients.

Inspired by Garsia and Procesi [10], Garsia and Haiman [9] constructed a bi-graded module for the symmetric group and conjectured that the Frobenius character is

$$H_\mu(X;q,t) = J_\mu[X/(1-t);q,t], \tag{1.2}$$

thus the Kostka–Macdonald coefficients give the Schur function expansion of $H_\mu(X;q,t)$. This conjecture gives a representation theoretic interpretation for the Kostka–Macdonald polynomials as the graded coefficients of the irreducible decomposition of these modules. Haiman [14] resolved both conjectures by analyzing the isospectral Hilbert scheme of points in a plane, ultimately showing that it is Cohen–Macaulay (and Gorenstein).

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The nonsymmetric Macdonald polynomials $E_a(X;q,t)$, introduced by Opdam [23] and Macdonald [21], are indexed by weak compositions and form a basis for the full polynomial ring. They generalize Macdonald polynomials in the sense that

$$\begin{aligned} E_{\lambda_n,\lambda_{n-1},\ldots,\lambda_1}(x_1,\ldots,x_n;q,t) &= P_\lambda(x_1,\ldots,x_n;q,t), \\ E_{0^m\times a}(x_1,\ldots,x_m,0,\ldots,0;q,t) &= P_{\mathrm{sort}(a)}(x_1,\ldots,x_m;q,t), \end{aligned}$$

where $0^m \times a$ denotes the composition obtained by prepending m 0's to a.

Generalizing Haglund's elegant combinatorial formula for $H_\mu(X;q,t)$ [11, 12], Haglund, Haiman, and Loehr [13] gave a combinatorial formula for $E_a(X;q,t)$ as

$$E_a(X;q,t) = \sum_{\substack{T:a\to[n] \\ \text{non-attacking}}} q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} X^{\mathrm{wt}(T)} \prod_{c\neq \mathrm{left}(c)} \frac{1-t}{1-q^{\mathrm{leg}(c)+1}t^{\mathrm{arm}(c)+1}}, \tag{1.3}$$

where the sum is over certain positive integer fillings T of the diagram of the composition a and coinv and maj are nonnegative integer statistics. In stark contrast with the symmetric case, there are no known (or even conjectured) positivity results for the nonsymmetric Macdonald polynomials.

Demazure [8] generalized the Weyl character formula to certain submodules generated by extremal weight spaces under the action of a Borel subalgebra of a Lie algebra. These *Demazure characters* κ_a, where $a = w\cdot\lambda$, for w a permutation acting on the coordinates of a partition λ, arose in connection with Schubert calculus [7], and, in type A, also form a basis of the polynomial ring. Recent work of Assaf and Searles [5] indicates that the type A Demazure characters are the most natural pull backs of Schur functions to the polynomial ring, in the sense that the combinatorics of the former stabilizes to that of the latter. Therefore, in the search for polynomial analogs of Schur positivity statements for nonsymmetric Macdonald polynomials, the natural basis for comparison is the Demazure characters. Recently, Assaf [1] proved that the specialization $E_a(X;q,0)$ is a nonnegative sum of Demazure characters, and that this nonnegativity precisely parallels the Schur expansion of $P_\lambda(X;0,t)$. While the proof is purely combinatorial, the resulting formula is difficult to work with and, in practice, requires computing the fundamental slide polynomial [6] expansion of $E_a(X;q,0)$.

In this abstract, we present a new combinatorial proof of the Demazure positivity of $E_a(X;q,0)$ that provides a representation theoretic interpretation for $E_a(X;q,0)$ as the character of a graded Demazure module and results in an explicit combinatorial formula for the Demazure coefficients. Our proof proceeds by constructing crystal operators [15] on fillings of a diagram that we prove generate a Demazure crystal, generalizing a recent result of Assaf and Schilling [4]. In Section 2, we review crystal graphs and Demazure truncations of them. In Section 3, we review the combinatorial model for nonsymmetric Macdonald polynomials. Bringing these two ideas together, in Section 4, we define our Demazure crystal and, in Section 5, deduce from it a formula for the Demazure expansion of $E_a(X;q,0)$. For further details, see [3].

2 Crystal graphs

Irreducible polynomial representation of the general linear group GL_n are naturally indexed by partitions λ of length n with a basis naturally indexed by *semistandard Young tableaux of shape* λ. These are fillings of the diagram of λ from the alphabet $\{1, 2, \ldots, n\}$ such that entries weakly increase along rows and strictly increase up columns. For example, the semi-standard Young tableaux of shape $(2, 2, 1, 0)$ are shown in Figure 1.

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Figure 1: The semi-standard Young tableaux of shape $(2, 2, 1, 0)$.

Let $\mathrm{SSYT}_n(\lambda)$ denote the set of semi-standard Young tableaux of shape λ over the alphabet $\{1, 2, \ldots, n\}$. To each semi-standard Young tableau T, we associate the weak composition $\mathrm{wt}(T)$ whose ith part is the number of entries of T equal to i.

The *Schur polynomials* are the characters of these irreducible representations. They may be defined as the generating polynomial for semi-standard Young tableaux,

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}_n(\lambda)} x_1^{\mathrm{wt}(T)_1} \cdots x_n^{\mathrm{wt}(T)_n}. \tag{2.1}$$

Kashiwara [15] introduced the notion of crystal bases in his study of the representation theory of quantized universal enveloping algebras at $q = 0$. A *crystal graph* is a directed, colored graph with vertex set given by the crystal basis of a quantum group and directed edges given by deformations of the Chevalley generators. There is an explicit combinatorial construction of the crystal graph, with raising and lowering operators denoted by e_i and f_i, respectively, that act on tableaux [17, 19].

For a semi-standard Young tableau T, let $T|_{\leq c}$ denote the first c columns of T. Given a positive integer $1 \leq i < n$, define indices:

$$m_i(T, c) = \mathrm{wt}(T|_{\leq c})_i - \mathrm{wt}(T|_{\leq c})_{i+1}, \qquad m_i(T) = \max_{c>0} \left(m_i(T, c) \right).$$

Observe that if $m_i(T) > 0$ and c is the leftmost column that attains this maximum, then there is an i and no $i + 1$ in column c of T.

Definition 2.1. Given an integer $1 \leq i < n$, define the *lowering operator* f_i on semi-standard Young tableaux as follows: if $m_i(T) \leq 0$, then $f_i(T) = 0$; otherwise, for c the smallest index such that $m_i(T, c) = m_i(T)$, $f_i(T)$ changes the i in column c of T to $i + 1$.

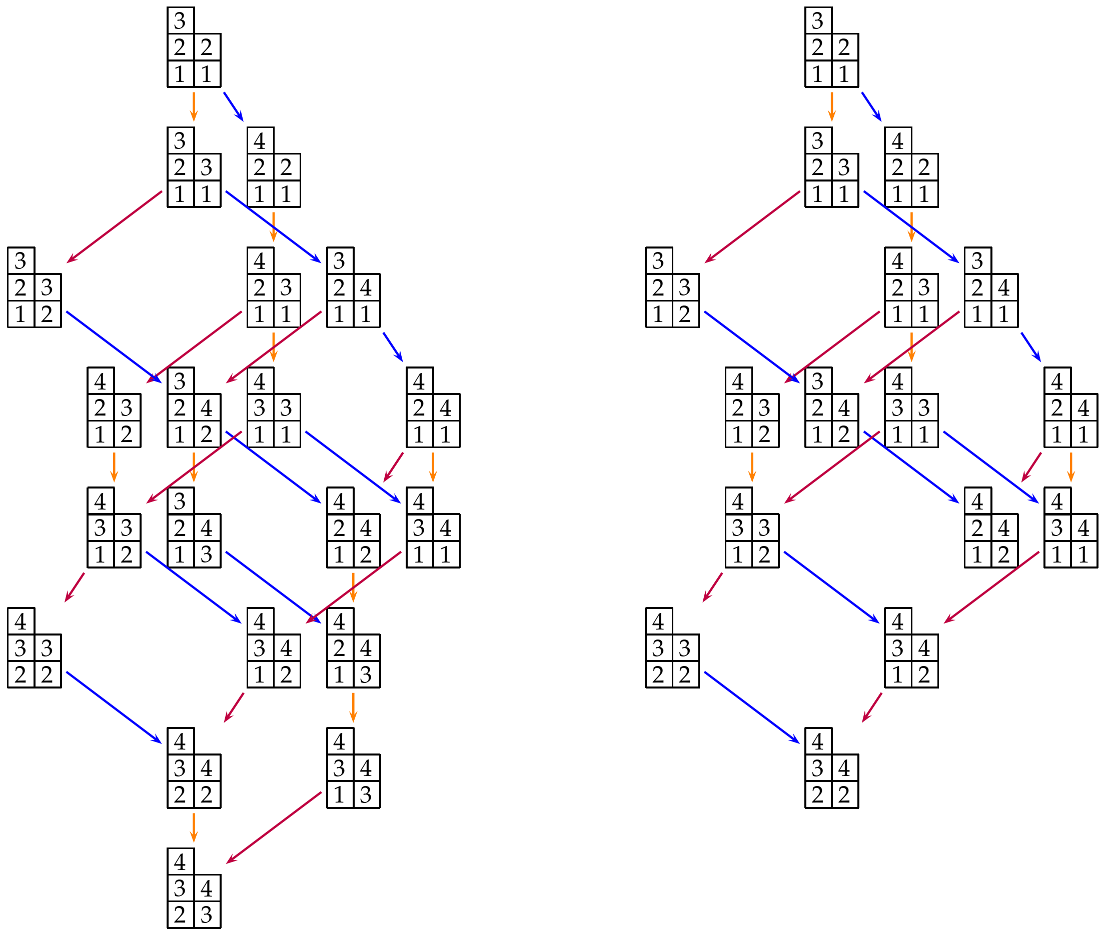

Figure 2: The crystal $\mathcal{B}(2,2,1,0)$ (left) and the Demazure crystal $\mathcal{B}_{2431}(2,2,1,0)$ (right), with edges $f_1 \swarrow$, $f_2 \downarrow$, $f_3 \searrow$ defined by the lowering operators.

For example, Figure 2 shows the lowering operators on $\mathrm{SSYT}_4(2,2,1,0)$.

Denote the highest weight crystal for a partition λ by $\mathcal{B}(\lambda)$. Then

$$\mathrm{ch}\mathcal{B}(\lambda) = \sum_{b \in \mathcal{B}(\lambda)} x_1^{\mathrm{wt}(b)_1} \cdots x_n^{\mathrm{wt}(b)_n} = s_\lambda(x_1, \ldots, x_n), \tag{2.2}$$

which is the character of the irreducible representation of highest weight λ.

Demazure [8] generalized the Weyl character formula to certain submodules generated by extremal weight spaces under the action of a Borel subalgebra of a Lie algebra. These *Demazure characters* arose in connection with Schubert calculus [7]. *Demazure crystals* are certain truncations conjectured by Littelmann [19] and proved by Kashiwara [16] to generalize Demazure characters. Given a subset $X \subseteq \mathcal{B}(\lambda)$, we define $\mathfrak{D}_i$ by

$$\mathfrak{D}_i X = \{b \in \mathcal{B}(\lambda) \mid e_i^k(b) \in X \text{ for some } k \geq 0\}, \tag{2.3}$$

where e_i denotes the *raising operator* satisfying $e_i(b) = b'$ if and only if $f_i(b') = b$ for all $b, b' \in \mathcal{B}(\lambda)$.

These operators satisfy the braid relations for the symmetric group, and so for a permutation w with reduced expression $w = s_{i_k} \cdots s_{i_1}$, we may define

$$\mathcal{B}_w(\lambda) = \mathfrak{D}_{i_k} \cdots \mathfrak{D}_{i_1}\{u_\lambda\}, \tag{2.4}$$

where u_λ is the highest weight element in $\mathcal{B}(\lambda)$; that is, $e_i(u_\lambda) = 0$ for all i. For example, Figure 2 shows the Demazure crystal for $w = 2431$ and $\lambda = (2, 2, 1, 0)$.

Theorem 2.2 ([16])**.** *Given a weak composition a, the Demazure character κ_a is given by*

$$\kappa_a = \sum_{b \in \mathcal{B}_{w(a)}(\mathrm{sort}(a))} x_1^{\mathrm{wt}(b)_1} \cdots x_n^{\mathrm{wt}(b)_n} = \mathrm{ch}\mathcal{B}_{w(a)}(\mathrm{sort}(a)), \tag{2.5}$$

where $\mathrm{sort}(a)$ *is the partition sorting of a and $w(a)$ is the shortest permutation that sorts a.*

3 Macdonald polynomials

Macdonald's symmetric functions $P_\lambda(X; q, t)$ [20] are two parameter generalizations of classical symmetric functions that simultaneously generalize the Hall–Littlewood symmetric functions $H_\lambda(X; t) = P_\lambda(X; 0, t)$ and the Jack symmetric functions $J_{\lambda,\alpha}(X) = \lim_{t \to 1} P_\lambda(X; t^\alpha, t)$. The nonsymmetric Macdonald polynomials $E_a(x_1, \ldots, x_n; q, t)$ were introduced by Opdam [23] and Macdonald [21]. The symmetric Macdonald polynomials are a special case of their nonsymmetric analogs,

$$E_{\lambda_n, \lambda_{n-1}, \ldots, \lambda_1}(x_1, \ldots, x_n; q, t) = P_\lambda(x_1, \ldots, x_n; q, t). \tag{3.1}$$

Haglund, Haiman and Loehr [13] gave a combinatorial formula for the monomial expansion of nonsymmetric Macdonald polynomials as follows.

Two cells of a diagram are *attacking* if they lie in the same column or if they lie in adjacent columns with the cell on the left strictly higher than the cell on the right. A filling is *non-attacking* if no two attacking cells have the same value.

Given a non-attacking filling T, define the major index of T, denoted by $\mathrm{maj}(T)$, to be the sum of the legs of all cells c such that the entry in c is strictly greater than the entry immediately to its left. A *triple* is a collection of three cells with two row adjacent and either (Type I) the third cell is above the left and the lower row is strictly longer, or (Type II) the third cell is below the right and the higher row is weakly longer. The *orientation* of a triple is determined by reading the entries of the cells from smallest to largest. A *co-inversion triple* is a Type I triple oriented counterclockwise or a Type II triple oriented clockwise. For example, the filling T in Figure 3 is non-attacking, with $\mathrm{maj}(T) = 2 + 1 = 3$ and $\mathrm{coinv}(T) = 2$.

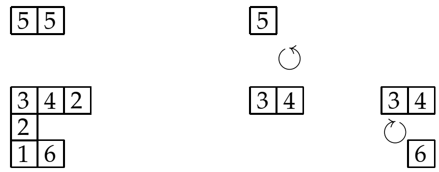

Figure 3: A non-attacking filling of the diagram for $(2,1,3,0,0,2)$ with the two co-inversion triples noted to the right.

Theorem 3.1 ([13]). *The nonsymmetric Macdonald polynomial $E_a(x_1,\ldots,x_n;q,t)$ is given by*

$$E_a(X;q,t) = \sum_{\substack{T:a\to[n] \\ \text{non-attacking}}} q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} X^{\mathrm{wt}(T)} \prod_{c\neq \mathrm{left}(c)} \frac{1-t}{1-q^{\mathrm{leg}(c)+1}t^{\mathrm{arm}(c)+1}}. \tag{3.2}$$

We consider the specialization at $t=0$, in which case the product becomes 1, and the only terms that survive are those T with $\mathrm{coinv}(T)=0$.

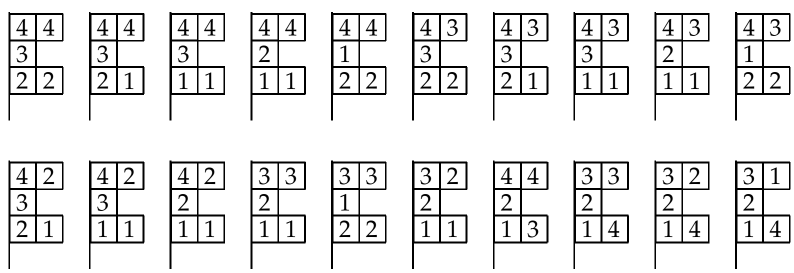

Figure 4: The semi-standard key tabloids of shape $(0,2,1,2)$.

Definition 3.2 ([1]). The *semi-standard key tabloids of shape a*, denoted by $\mathrm{SSKD}(a)$, are the non-attacking fillings of the diagram of a with no co-inversion triples.

For example, the 20 semi-standard key tabloids of shape $(0,2,1,2)$ are given in Figure 4. With this terminology, the specialized nonsymmetric Macdonald polynomial is

$$E_a(x_1,\ldots,x_n;q,0) = \sum_{T\in\mathrm{SSKD}(a)} q^{\mathrm{maj}(T)} x_1^{\mathrm{wt}(T)_1}\cdots x_n^{\mathrm{wt}(T)_n}. \tag{3.3}$$

Define the *nonsymmetric Kostka–Foulkes polynomial* $K_{a,b}(q)$ by the expansion

$$E_b(X;q,0) = \sum_a K_{a,b}(q)\kappa_a(X). \tag{3.4}$$

In [1], Assaf uses weak dual equivalence [2] to prove $K_{a,b}(q)\in\mathbb{N}[q]$.

Theorem 3.3 ([1]). *The specialized nonsymmetric Macdonald polynomial $E_a(x_1, \ldots, x_n; q, 0)$ is a positive graded sum of Demazure characters.*

While the proof in [1] is combinatorial, the interpretation for $K_{a,b}(q)$ comes as the maj-weighted number of weak dual equivalence classes on standard key tabloids of shape b that have type a. This requires computing each equivalence class in its entirety. Thus we desire a direct formula analogous to that for the Kostka–Foulkes polynomials $K_{\lambda,\mu}(0,t)$.

4 Demazure crystal on key tabloids

Generalizing the crystal constructions on Young tableaux, we give a new proof of Theorem 3.3 by constructing an explicit Demazure crystal on semi-standard key tabloids as follows. For a semi-standard key tabloid T, let $w(T)$ denote the word obtained by reading columns of T right to left, and reading each column top to bottom. Given a positive integer $1 \leq i < n$ and a word w of length k, define indices:

$$M_i(w,j) = \mathrm{wt}(w_j \cdots w_k)_{i+1} - \mathrm{wt}(w_j \cdots w_k)_i, \qquad M_i(w) = \max_{j>0} \left(M_i(w,j) \right).$$

Definition 4.1. Given an integer $1 \leq i < n$, define the *raising operator* e_i on $\mathrm{SSKD}(a)$ as follows: if $M_i(w(T)) \leq 0$, then $e_i(T) = 0$; otherwise, for j the largest index such that $M_i(w(T), j) = M_i(w(T))$, and letting c be the column in which the letter of T corresponding to w_j lies, $e_i(T)$ changes all column-consecutive $i+1$s weakly right of column c of T to i and simultaneously changes all is in the affected columns to $i+1$s.

For example, Figure 5 shows the raising operators on $\mathrm{SSKD}(0,2,1,2)$. Note that, unlike the crystal operators on semi-standard Young tableaux, the raising operator e_i on semi-standard key tabloids might change the relative order of i and $i+1$ within a given column. For e_i acting on T, we might change i above $i+1$ in some column to $i+1$ above i. Whenever this happens for a semi-standard key tabloid T, we say that e_i *flips* T.

Theorem 4.2. *The raising operators are well-defined,* maj*-preserving maps* $e_i : \mathrm{SSKD}(a) \to \mathrm{SSKD}(a) \cup \{0\}$ *that are invertible on the subset of tabloids T such that $e_i(T) \neq 0$.*

We also consider the *lowering operators*, denoted by f_i, satisfying $f_i(T') = T$ if and only if $e_i(T) = T'$ for all $T, T' \in \mathrm{SSKD}(a)$. We similarly say that f_i *flips* T if within some column f_i changes $i+1$ above i to become i above $i+1$.

In the case of semi-standard key tabloids with maj $= 0$, i.e. the semi-standard key *tableaux* [1] that correspond to Mason's semi-skyline augmented fillings [22], the raising operators in Definition 4.1 are precisely those defined by Assaf and Schilling [4] in connection with Schubert polynomials.

Identifying each semi-standard key tabloid of shape a with the diagram in the plane obtained by placing all cells with entry i into row i, the semi-standard key tableaux are

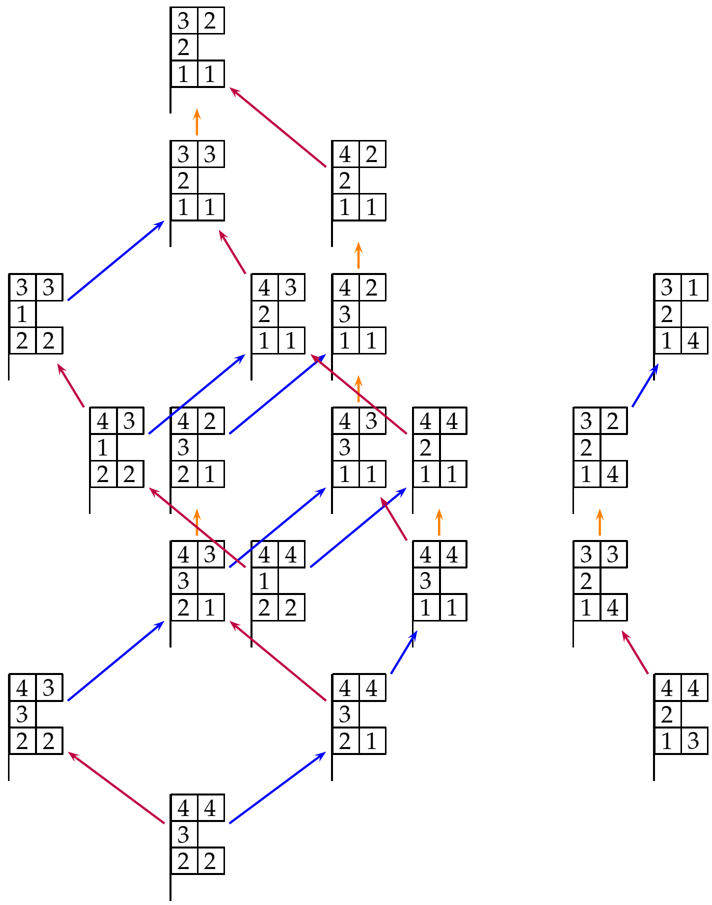

Figure 5: The Demazure crystal for $E_{(0,2,1,2)}(X;q,0)$ on $\mathrm{SSKD}(0,2,1,2)$, with edges e_1 ↗, e_2 ↑, e_3 ↖ defined by raising operators.

precisely the diagrams that arise by applying Kohnert's algorithm [18] to the composition diagram for a [2]. We define a *rectification map* that sends an arbitrary diagram to a Kohnert diagram, which back at the level of tabloids, sends a semi-standard key tabloid to a semi-standard key tableau. This rectification map, illustrated in Figure 6, commutes with the raising operators, allowing us to give a bijective proof of the following using the Demazure crystal of Assaf and Schilling [4] on semi-standard key tableaux.

Theorem 4.3. *The raising operators define a Demazure crystal graph on* $\mathrm{SSKD}(a)$*. In particular, their generating polynomial* $E_a(x_1, \ldots, x_n; q, 0)$ *is a graded sum of Demazure characters.*

For example, from Figure 5, we see that

$$E_{(0,2,1,2)}(X;q,0) = \kappa_{(0,2,1,2)}(X) + q\kappa_{(1,1,1,2)}(X).$$

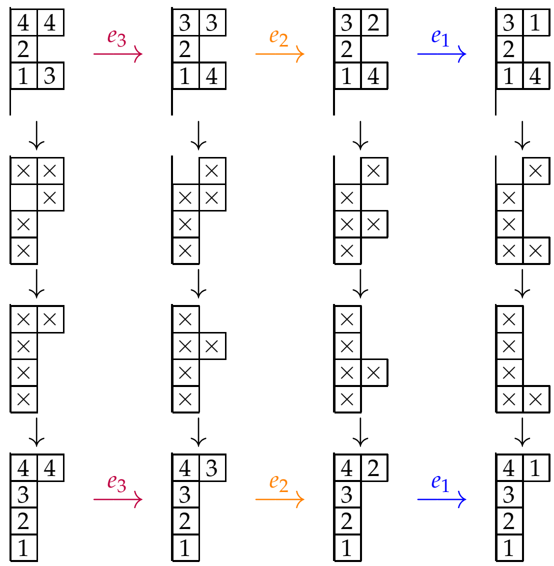

Figure 6: Examples of the rectification algorithm from semi-standard key tabloids, to diagrams, to Kohnert diagram (via rectification), to semi-standard key tableaux.

5 Highest weights

A connected crystal graph $\mathcal{B}$ has a unique highest weight element Y characterized by the property $e_i(Y) = 0$ for all i. Moreover, a component with highest weight element Y is isomorphic to the highest weight crystal $\mathcal{B}(\lambda)$. In terms a characters, this becomes

$$\mathrm{ch}\mathcal{B} = \sum_{b \in \mathcal{B}} x_1^{\mathrm{wt}(b)_1} \cdots x_n^{\mathrm{wt}(b)_n} = \sum_{\substack{Y \in \mathcal{B} \\ e_i(Y)=0\ \forall\ i}} s_{\mathrm{wt}(Y)}(x_1, \ldots, x_n). \tag{5.1}$$

For Demazure crystal graphs, each connected component still has a unique highest weight element characterized as the unique element killed by all raising operators. Given our explicit raising operators on semi-standard key tabloid, these are easy to compute. For example, the six highest weight semi-standard key tabloids of shape $(0,3,0,2)$ are shown in Figure 7, indicating that the Demazure crystal has six connected components and the Demazure expansion of $E_{(0,3,0,2)}(X;q,0)$ has six terms. Moreover, the q-weight of these terms is easily determined by the simple major index statistic.

As powerful as highest weight elements are, they fall short of determining the truncation permutation w for the given Demazure crystal. In particular, every Demazure truncation of $\mathcal{B}(\lambda)$ has the same highest weight. However, each $\mathcal{B}_w(\lambda)$ has a unique *Demazure lowest weight element* Z determined by $\mathrm{wt}(Z) \leq \mathrm{wt}(T)$ for all T on the same connected component. While it is true that $f_i(Z) = 0$ for all i, this property alone does not uniquely characterize it. For example, the tableau with reading word 42412 in the

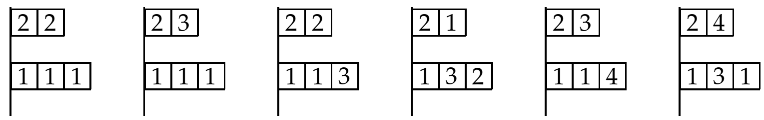

Figure 7: The highest weights for the Demazure crystal for $E_{(0,3,0,2)}(X; q, 0)$.

Demazure crystal $\mathcal{B}_{2431}(2, 2, 1, 0)$ shown in Figure 2 is killed by all of the lowering operators, but it is not the global lowest weight element. In order to determine the *Demazure lowest weight* a, given by $w \cdot \lambda$, we have the following algorithm to determine the Demazure lowest weight element from a given highest weight element.

For the following definition, given a semi-standard key tabloid T and an index i, set $s_i(T) = f_i^k(T)$ where k is maximal such that $f_i^k(T) \neq 0$.

Definition 5.1. Given a highest weight element Y of the Demazure crystal on $\mathrm{SSKD}(a)$, the corresponding *Demazure lowest weight element* Z is constructed by letting $Z_0 = Y$, and for $m \geq 0$ defining Z_{m+1} as follows. Find indices $i \leq j \leq k$ such that

- j is minimal such that $f_j(Z_m) \neq 0$ and not a flip;
- k is maximal and then i is minimal such that $s_i s_{i+1} \cdots s_k(Z_m)$ acts nontrivially and has no flips at each step.

Then set $Z_{m+1} = s_i s_{i+1} \cdots s_k(Z_m)$. Stop when no such j exists.

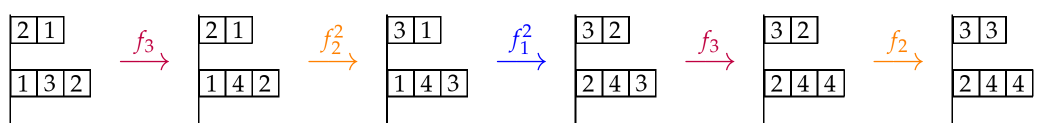

Figure 8: Computing the Demazure lowest weight for the given highest weight.

For example, in Figure 8, we take Z_0 to be the leftmost tabloid. Then $j = 2$, and so $k = 3$ and $i = 1$, giving $Z_1 = s_1 s_2 s_3(Z_0)$, which is the fourth from the left tabloid. Continuing the algorithm, we have $j = 3$, and so $k = 3$ and $i = 2$, giving $Z_2 = s_2 s_3(Z_1)$, which is the rightmost tabloid and also the Demazure lowest weight.

Figure 9 shows the six Demazure lowest weights of the Demazure crystal on $\mathrm{SSKD}(0, 3, 0, 2)$ corresponding to the highest weights in Figure 7.

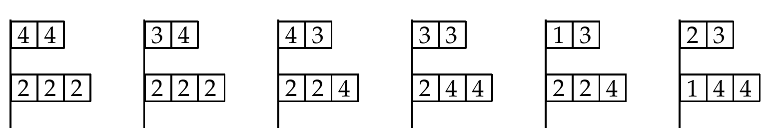

Figure 9: The Demazure lowest weights for the Demazure crystal for $E_{(0,3,0,2)}(X; q, 0)$.

Using this notion, we have the following explicit expansion.

Theorem 5.2. *The specialized nonsymmetric Macdonald polynomial $E_a(X;q,0)$ is given by*

$$E_a(X;q,0) = \sum_{\substack{Z \in \mathrm{SSKD}(a) \\ Z \text{ Demazure lowest weight}}} q^{\mathrm{maj}(Z)} \kappa_{\mathrm{wt}(Z)}(X). \tag{5.2}$$

In particular, the nonsymmetric Kostka–Foulkes polynomial $K_{a,b}(q)$ is given by

$$K_{a,b}(q) = \#\{Z \in \mathrm{SSKD}(b) \mid Z \text{ Demazure lowest weight with } \mathrm{wt}(Z) = a\}. \tag{5.3}$$

For example, from Figure 9, we see that

$$\begin{aligned} E_{(0,3,0,2)}(X;q,0) &= \kappa_{(0,3,0,2)}(X) + q\kappa_{(0,3,1,1)}(X) + q\kappa_{(0,2,1,2)}(X) \\ &\quad + q^2\kappa_{(0,1,2,2)}(X) + q^2\kappa_{(1,2,1,1)}(X) + q^3\kappa_{(1,1,1,2)}(X). \end{aligned}$$

References

[1] S. Assaf. "Nonsymmetric Macdonald polynomials and a refinement of Kostka-Foulkes polynomials". *Trans. Amer. Math. Soc.* **370**.12 (2018), pp. 8777–8796. DOI: 10.1090/tran/7374.

[2] S. Assaf. "Weak dual equivalence for polynomials". 2017. arXiv: 1702.04051.

[3] S. Assaf and N.S. González. "Demazure crystals for specialized nonsymmetric Macdonald polynomials". Preprint, forthcoming.

[4] S. Assaf and A. Schilling. "A Demazure crystal construction for Schubert polynomials". *Algebr. Comb.* **1**.2 (2018), pp. 225–247.

[5] S. Assaf and D. Searles. "Kohnert tableaux and a lifting of quasi-Schur functions". *J. Combin. Theory Ser. A* **156** (2018), pp. 85–118. DOI: 10.1016/j.jcta.2018.01.001.

[6] S. Assaf and D. Searles. "Schubert polynomials, slide polynomials, Stanley symmetric functions and quasi-Yamanouchi pipe dreams". *Adv. in Math.* **306** (2017), pp. 89–122. DOI: 10.1016/j.aim.2016.10.015.

[7] M. Demazure. "Désingularisation des variétés de Schubert généralisées". *Ann. Sci. École Norm. Sup. (4)* **7** (1974). Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I, pp. 53–88. URL.

[8] M. Demazure. "Une nouvelle formule des caractères". *Bull. Sci. Math. (2)* **98**.3 (1974), pp. 163–172.

[9] A.M. Garsia and M. Haiman. "A graded representation model for Macdonald's polynomials". *Proc. Nat. Acad. Sci. U.S.A.* **90**.8 (1993), pp. 3607–3610. DOI: 10.1073/pnas.90.8.3607.

[10] A.M. Garsia and C. Procesi. "On certain graded S_n-modules and the q-Kostka polynomials". *Adv. Math.* **94**.1 (1992), pp. 82–138. DOI: 10.1016/0001-8708(92)90034-I.

[11] J. Haglund. "A combinatorial model for the Macdonald polynomials". *Proc. Natl. Acad. Sci. USA* **101**.46 (2004), pp. 16127–16131. DOI: 10.1073/pnas.0405567101.

[12] J. Haglund, M. Haiman, and N. Loehr. "A combinatorial formula for Macdonald polynomials". *J. Amer. Math. Soc.* **18**.3 (2005), pp. 735–761. DOI: 10.1090/S0894-0347-05-00485-6.

[13] J. Haglund, M. Haiman, and N. Loehr. "A combinatorial formula for nonsymmetric Macdonald polynomials". *Amer. J. Math.* **130**.2 (2008), pp. 359–383. DOI: 10.1353/ajm.2008.0015.

[14] M. Haiman. "Hilbert schemes, polygraphs and the Macdonald positivity conjecture". *J. Amer. Math. Soc.* **14**.4 (2001), pp. 941–1006. DOI: 10.1090/S0894-0347-01-00373-3.

[15] M. Kashiwara. "On crystal bases of the Q-analogue of universal enveloping algebras". *Duke Math. J.* **63**.2 (1991), pp. 465–516. DOI: 10.1215/S0012-7094-91-06321-0.

[16] M. Kashiwara. "The crystal base and Littelmann's refined Demazure character formula". *Duke Math. J.* **71**.3 (1993), pp. 839–858. DOI: 10.1215/S0012-7094-93-07131-1.

[17] M. Kashiwara and T. Nakashima. "Crystal graphs for representations of the q-analogue of classical Lie algebras". *J. Algebra* **165**.2 (1994), pp. 295–345. DOI: 10.1006/jabr.1994.1114.

[18] A. Kohnert. "Weintrauben, Polynome, Tableaux". *Bayreuth. Math. Schr.* 38 (1991). Dissertation, Universität Bayreuth, Bayreuth, 1990, pp. 1–97.

[19] P. Littelmann. "Crystal graphs and Young tableaux". *J. Algebra* **175**.1 (1995), pp. 65–87. DOI: 10.1006/jabr.1995.1175.

[20] I.G. Macdonald. "A new class of symmetric functions". *Actes du 20e Seminaire Lotharingien* **372** (1988), pp. 131–171.

[21] I.G. Macdonald. "Affine Hecke algebras and orthogonal polynomials". *Astérisque* 237 (1996). Séminaire Bourbaki, Vol. 1994/95, Exp. No. 797, 4, 189–207.

[22] S. Mason. "An explicit construction of type A Demazure atoms". *J. Algebraic Combin.* **29**.3 (2009), pp. 295–313. DOI: 10.1007/s10801-008-0133-4.

[23] E.M. Opdam. "Harmonic analysis for certain representations of graded Hecke algebras". *Acta Math.* **175**.1 (1995), pp. 75–121. DOI: 10.1007/BF02392487.