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Product formulas for standard tableaux of a family of skew shapes

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Abstract. We give new product formulas for the number of standard Young tableaux of a six parameter family of skew shapes generalizing a formula by DeWitt and a formula of Kim and Oh. These are proved by utilizing symmetries for evaluations of factorial Schur functions and the Naruse hook length formula for skew shapes.

Keywords: standard tableaux, skew shapes, factorial Schur functions

1 Introduction

1.1 Foreword

It is a truth universally acknowledged, that a combinatorial theory is often judged not by its intrinsic beauty but by the examples and applications. Fair or not, this attitude is historically grounded and generally accepted. While eternally challenging, this helps to keep the area lively, widely accessible, and growing in unexpected directions.

The full version [12] of this extended abstract is a third in a series and continues our study of the *Naruse hook-length formula* (NHLF), its generalizations and applications. In the first paper [10], we introduced two q-analogues of the NHLF and gave their (difficult) bijective proofs. In the second paper [11], we investigated the special case of *ribbon hooks*, which were used to obtain two new elementary proofs of NHLF in full generality, as well as various new mysterious summation and determinant formulas.

In this extended abstract we exploit the symmetry in a multivariate identity of evaluations of factorial Schur functions to give new product formulas for the number of *standard Young tableaux* of certain skew shapes.

As an immediate consequence of our results, we obtain the exact asymptotic formulas which were unreachable until now (see [12, Sections 6,8]). Below we illustrate our results one by one, leaving full statements and generalizations for later.

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1.2 Number of SYT of skew shape

Standard Young tableaux are fundamental objects in enumerative and algebraic combinatorics and their enumeration is central to the area. The number $f^\lambda = |\mathrm{SYT}(\lambda)|$ of standard Young tableaux of shape λ of size n, is given by the classical *hook-length formula*:

$$f^\lambda = n! \prod_{u\in[\lambda]} \frac{1}{h(u)}. \tag{HLF}$$

Famously, there is no general product formula for the number $f^{\lambda/\mu} = |\mathrm{SYT}(\lambda/\mu)|$ of standard Young tableaux of skew shape λ/μ.[1] However, such formulas do exist for a few sporadic families of skew shapes and truncated shapes (see [1]).

In this paper we give a six-parameter family of skew shapes (see Figure 1 (iv)) with product formulas for the number of their SYT. This product formula is given in our main result: Theorem 4.1. The three corollaries below showcase the most elegant special cases. We single out two especially interesting special cases: Corollary 1.1 due to its connection to the *Selberg integral*, and Corollary 1.2 due to its connection to the shifted shapes and a potential for a bijective proof. Both special cases are known, but their proofs do not generalize in this paper's direction.

The formulas below are written in terms of *superfactorials* $\Phi(n)$, *double superfactorials* $\daleth(n)$, *super doublefactorials* $\Psi(n)$, and *shifted super doublefactorials* $\Psi(n;k)$ defined as:

$$\begin{aligned} \Phi(n) &:= 1!\cdot 2!\cdots(n-1)!, & \daleth(n) &:= (n-2)!(n-4)!\cdots, \\ \Psi(n) &:= 1!!\cdot 3!!\cdots(2n-3)!!, & \Psi(n;k) &:= (k+1)!!\cdots(k+3)!!\cdots(k+2n-3)!! \end{aligned} \tag{1.1}$$

Corollary 1.1 (Kim–Oh [5]). *For all $a,b,c,d,e\in\mathbb{N}$, let λ/μ be the skew shape in* Figure 1 (i). *Then the number $f^{\lambda/\mu} = |\mathrm{SYT}(\lambda/\mu)|$ is equal to*

$$n!\,\frac{\Phi(a)\,\Phi(b)\,\Phi(c)\,\Phi(d)\,\Phi(e)\,\Phi(a+b+c)\,\Phi(c+d+e)\,\Phi(a+b+c+d+e)}{\Phi(a+b)\,\Phi(d+e)\,\Phi(a+c+d)\,\Phi(b+c+e)\,\Phi(a+b+2c+d+e)},$$

where $n=|\lambda/\mu| = (a+c+e)(b+c+d)-ab-ed$.

Note that in [5, Corollary 4.7], the product formula is equivalent, but stated differently.

Corollary 1.2 (DeWitt [3]). *For all $a,b,c\in\mathbb{N}$, let λ/μ be the skew shape in* Figure 1 (ii). *Then the number $f^{\lambda/\mu} = |\mathrm{SYT}(\lambda/\mu)|$ is equal to*

$$n!\,\frac{\Phi(a)\,\Phi(b)\,\Phi(c)\,\Phi(a+b+c)\cdot\Psi(c)\,\Psi(a+b+c)}{\Phi(a+b)\,\Phi(b+c)\,\Phi(a+c)\cdot\Psi(a+c)\,\Psi(b+c)\,\Psi(a+b+2c)},$$

where $n=|\lambda/\mu| = \binom{a+b+2c}{2}-ab$.

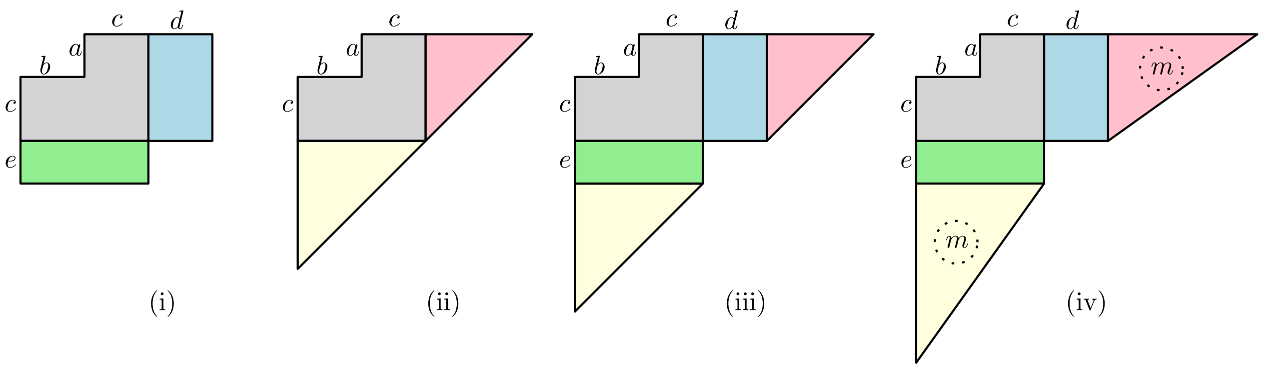

Figure 1: Skew shapes with product formulas for the number of SYT. The symbol m in the last shape indicates slope.

Corollary 1.3. *For all $a, b, c, d, e \in \mathbb{N}$, let λ/μ be the skew shape in* Figure 1 (iii)*. Then the number $f^{\lambda/\mu} = \left|\mathrm{SYT}(\lambda/\mu)\right|$ is equal to*

$$\frac{n! \cdot \Phi(a)\Phi(b)\Phi(c)\Phi(a+b+c) \cdot \Psi(c;d+e)\Psi(a+b+c;d+e) \cdot \gimel(2a+2c)\gimel(2b+2c)}{\Phi(a+b)\Phi(b+c)\Phi(a+c) \cdot \Psi(a+c)\Psi(b+c)\Psi(a+b+2c;d+e) \cdot \gimel(2a+2c+d)\gimel(2b+2c+e)},$$

where $n = |\lambda/\mu| = (a+b+c+e)(b+c+d) + \binom{a+c}{2} + \binom{b+c}{2} - ab - ed$.

Let us emphasize that the proofs of corollaries 1.1–1.3 are quite technical in nature. Here is a brief non-technical explanation. Fundamentally, the Naruse hook-length formula (NHLF) provides a new way to understand SYT of skew shape, coming from geometry rather than representation theory. What we show in this paper is that the proof of the NHLF has “hidden symmetries” which can be turned into product formulas. We refer to Section 4 for the complete proofs and common generalizations of these results.

1.3 Structure of the paper

We begin with Section 2 which summarizes both the notation and gives a brief review of the earlier work. In the next Section 3, we develop the technology of multivariate formulas including the key identity (Theorem 3.7). We use this identity to prove the product formulas for the number $f^{\lambda/\mu}$ of SYT of skew shape in Section 4, including generalization of corollaries 1.1–1.3.

[1] In fact, even for small zigzag shapes $\pi = (k+1, k, \ldots, 1)/(k-1, k-2, \ldots, 1)$, the number f^{π} can have large prime divisors.

2 Notation and Background

2.1 Young diagrams and skew shapes

Let $\lambda = (\lambda_1, \ldots, \lambda_r), \mu = (\mu_1, \ldots, \mu_s)$ denote integer partitions of length $\ell(\lambda) = r$ and $\ell(\mu) = s$. The *size* of the partition is denoted by $|\lambda|$ and λ' denotes the *conjugate partition* of λ. We use $[\lambda]$ to denote the *Young diagram* of the partition λ. The *hook length* $h_\lambda(i,j) = \lambda_i - i + \lambda'_j - j + 1$ of a square $u = (i,j) \in [\lambda]$ is the number of squares directly to the right and directly below u in $[\lambda]$ including u. The *content* of a square $u = (i,j)$ is $c(u) = j - i$.

A *skew shape* is the set difference of the Young diagrams of two partitions $[\mu] \subseteq [\lambda]$ and we denote it by λ/μ. The *staircase* shape is denoted by $\delta_n = (n-1, n-2, \ldots, 2, 1)$.

2.2 Plane partitions

Let $\mathrm{RPP}(a,b,c)$ denote the set of reverse plane partitions that fit into an $[a \times b \times c]$ box. Recall the *MacMahon box formula* for the number of such (reverse) plane partitions, which also can be written as follows:

$$\big|\mathrm{RPP}(a,b,c)\big| = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2} = \frac{\Phi(a+b+c)\,\Phi(a)\,\Phi(b)\,\Phi(c)}{\Phi(a+b)\,\Phi(b+c)\,\Phi(a+c)}\,. \tag{2.1}$$

2.3 Factorial Schur functions

The *factorial Schur function* (e.g., see [9]) is defined as

$$s_\mu^{(d)}(\mathbf{x} \mid \mathbf{a}) := \sum_T \prod_{u \in \mu} \big(x_{T(u)} - a_{T(u)+c(u)}\big), \tag{2.2}$$

where $\mathbf{x} = x_1, \ldots, x_d$ are variables, $\mathbf{a} = a_1, a_2, \ldots$ are parameters, and where the sum is over semistandard Young tableaux T of shape μ with entries in $\{1, \ldots, d\}$. Moreover, in addition, $s_\mu^{(d)}(\mathbf{x} \,|\, \mathbf{a})$ is symmetric in $x_1, \ldots, x_d$.

2.4 Excited diagrams

Let λ/μ be a skew partition and D be a subset of the Young diagram of λ. A cell $u = (i,j) \in D$ is called *active* if $(i+1,j)$, $(i,j+1)$ and $(i+1,j+1)$ are all in $[\lambda] \setminus D$. Let u be an active cell of D, define $\alpha_u(D)$ to be the set obtained by replacing $(i,j) \in D$ by $(i+1,j+1)$. We call this procedure an *excited move*. An *excited diagram* of λ/μ is a subdiagram of λ obtained from the Young diagram of μ after a sequence of excited moves on active cells. Let $\mathcal{E}(\lambda/\mu)$ be the set of excited diagrams of λ/μ. See Figure 2 for examples.

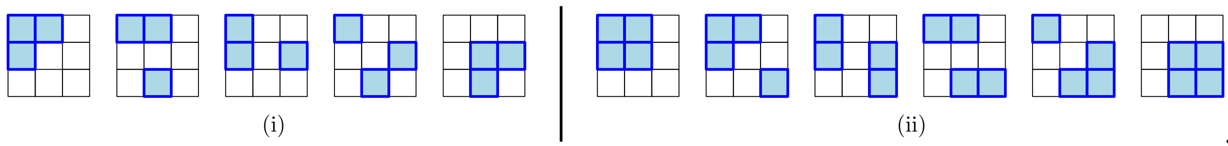

Figure 2: The excited diagrams of the shapes (i) 333/21 and (ii) 333/22.

2.5 Non-intersecting paths

Excited diagrams of λ/μ are in bijection with families of non-intersecting grid paths $\gamma_1, \ldots, \gamma_k$ with a fixed set of start and end points, which depend only on λ/μ. A variant of this was proved by Kreiman [8, Sections 5–6] (see also [11, Section 3]).

Formally, given a connected skew shape λ/μ, there is unique family of non-intersecting paths $\gamma_1^*, \ldots, \gamma_k^*$ in λ with support λ/μ, where each border strip γ_i^* begins at the southern box (a_i, b_i) of a column and ends at the eastern box (c_i, d_i) of a row [8, Lemma 5.3]. Let $\mathcal{NIP}(\lambda/\mu)$ be the set of k-tuples $\Gamma := (\gamma_1, \ldots, \gamma_k)$ of non-intersecting paths contained in $[\lambda]$ with $\gamma_i : (a_i, b_i) \to (c_i, d_i)$.

Proposition 2.1 (Kreiman [8], see also [11])**.** *Non-intersecting paths in $\mathcal{NIP}(\lambda/\mu)$ are uniquely determined by their* support, *i.e. set of squares. Moreover, the set of such supports is exactly the set of complements $[\lambda] \setminus D$ to excited diagrams $D \in \mathcal{E}(\lambda/\mu)$.*

2.6 The Naruse hook-length formula

Recall the formula of Naruse for $f^{\lambda/\mu}$ as a sum of product of hook-lengths (see [10]).

Theorem 2.2 (NHLF; Naruse [13])**.** *Let λ, μ be partitions, such that $\mu \subset \lambda$. We have:*

$$f^{\lambda/\mu} = n! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)}\,, \qquad \text{(NHLF)}$$

where the sum is over all excited diagrams D of λ/μ.

In [10] we gave two q-analogues of Naruse's formula in terms of SSYT of skew shape, as an identity for $s_{\lambda/\mu}(1, q, q^2, \ldots)$, and for RPP of skew shape.

3 Multivariate path identity

3.1 Multivariate sums of excited diagrams

For the skew shape $\lambda/\mu \subseteq d \times (n-d)$ we define $F_{\lambda/\mu}(\mathbf{x} \,|\, \mathbf{y})$ and $G_{\lambda/\mu}(\mathbf{x} \,|\, \mathbf{y})$ to be the multivariate sums of excited diagrams

$$G_{\lambda/\mu}(\mathbf{x}\,|\,\mathbf{y}) \;:=\; \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} (x_i - y_j), \qquad F_{\lambda/\mu}(\mathbf{x}\,|\,\mathbf{y}) \;:=\; \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in[\lambda]\setminus D} \frac{1}{x_i - y_j}\,.$$

By Proposition 2.1, the sum $F_{\lambda/\mu}(\mathbf{x}\,|\,\mathbf{y})$ can be written as a multivariate sum of non-intersecting paths.

Corollary 3.1. *In the notation above, we have:*

$$F_{\lambda/\mu}(\mathbf{x}\,|\,\mathbf{y}) \;=\; \sum_{\Gamma\in\mathcal{NIP}(\lambda/\mu)} \prod_{(i,j)\in\Gamma} \frac{1}{x_i - y_j}\,.$$

Note that by evaluating $(-1)^{|\lambda/\mu|}F_{\lambda/\mu}(\mathbf{x}\,|\,\mathbf{y})$ at $x_i = \lambda_i - i + 1$ and $y_j = -\lambda'_j + j$ and multiplying by $|\lambda/\mu|!$ we obtain the right hand side of (NHLF).

$$(-1)^{|\lambda/\mu|}F_{\lambda/\mu}(\mathbf{x}\,|\,\mathbf{y})\Big|_{\substack{x_i=\lambda_i-i+1\\ y_j=-\lambda'_j+j}} \;=\; \frac{f^{\lambda/\mu}}{|\lambda/\mu|!}\,. \tag{3.1}$$

Let $\mathbf{z}^{\langle\lambda\rangle}$ be the tuple of length n of x's and y's by reading the horizontal and vertical steps of λ from $(d,1)$ to $(1,n-d)$: i.e. $z_{\lambda_i+d-i+1} = x_i$ and $z_{\lambda'_j+n-d-j+1} = y_j$. For example, for $d = 4$, $n = 9$ and $\lambda = (5533)$, we have $\mathbf{z}^{\langle\lambda\rangle} = (y_1, y_2, y_3, x_4, x_3, y_4, y_5, x_2, x_1)$ (see Figure 3 (i))

Combining results of Ikeda–Naruse [4], Knutson–Tao [6], and Lakshmibai–Raghavan–Sankaran, one obtains the following formula for an evaluation of factorial Schur functions.

Lemma 3.2 (Theorem 2 in [4])**.** *For every skew shape $\lambda/\mu \subseteq d\times(n-d)$, we have:*

$$G_{\lambda/\mu}(\mathbf{x}\,|\,\mathbf{y}) \;=\; s^{(d)}_{\mu}(\mathbf{x}\,|\,\mathbf{z}^{\langle\lambda\rangle})\,. \tag{3.2}$$

Corollary 3.3. *We have $F_{\lambda/\mu}(\mathbf{x}\,|\,\mathbf{y}) \;=\; s^{(d)}_{\mu}(\mathbf{x}\,|\,\mathbf{z}^{\langle\lambda\rangle})/s^{(d)}_{\lambda}(\mathbf{x}\,|\,\mathbf{z}^{\langle\lambda\rangle})\,.$*

3.2 Symmetries

The factorial Schur function $s^{(d)}_{\mu}(\mathbf{x}\,|\,\mathbf{y})$ is symmetric in $\mathbf{x}$. By Lemma 3.2, the multivariate sum $G_{\lambda/\mu}(\mathbf{x}\,|\,\mathbf{y})$ is an evaluation of a certain factorial Schur function, which in general is not symmetric in $\mathbf{x}$.

Example 3.4. *The shape $\lambda/\mu = 332/21$ from Figure 2 (i) has five excited diagrams. One can check that the multivariate polynomial*

$$G_{332/21}(x_1,x_2,x_3\,|\,y_1,y_2,y_3) \;=\; (x_1-y_1)(x_1-y_2)(x_2-y_1) + (x_1-y_1)(x_1-y_2)(x_3-y_2) \\ + (x_1-y_1)(x_2-y_3)(x_2-y_1) + (x_1-y_1)(x_2-y_3)(x_3-y_2) + (x_2-y_2)(x_2-y_3)(x_3-y_2),$$

is not symmetric in $\mathbf{x} = (x_1,x_2,x_3)$.

Now, below we present two cases when the sum $G_{\lambda/\mu}(\mathbf{x}\,|\,\mathbf{y})$ is in fact symmetric in $\mathbf{x}$. The first case is when μ is a rectangle contained in λ.

Proposition 3.5. *Let $\mu = p^k$ be a rectangle, $p \geq k$, and let λ be arbitrary partition containing μ. Denote $\ell := \max\{i : \lambda_i - i \geq p-k\}$. Then:*

$$G_{\lambda/p^k}(\mathbf{x}\,|\,\mathbf{y}) \;=\; s^{(\ell)}_{p^k}(x_1,\ldots,x_\ell\,|\,y_1,\ldots,y_{p+\ell-k}).$$

In particular, the polynomial $G_{\lambda/p^k}(\mathbf{x}\,|\,\mathbf{y})$ is symmetric in $(x_1,\ldots,x_\ell)$.

Proof sketch. First, observe that $\mathcal{E}(\lambda/p^k) = \mathcal{E}((p+\ell-k)^\ell/p^k)$ since the movement of the excited boxes is limited by the position of the corner box of p^k, which moves along the diagonal $j-i = p-k$ up to the boundary of λ, at position $(\ell, p+\ell-k)$. Then:

$$G_{\lambda/p^k}(\mathbf{x}\,|\,\mathbf{y}) \;=\; \sum_{D\in\mathcal{E}((p+\ell-k)^\ell/p^k)} \prod_{(i,j)\in D} (x_i - y_j) \;=\; G_{(p+\ell-k)^\ell/p^k}(\mathbf{x}\,|\,\mathbf{y}) \\ = s^{\ell}_{p^k}(x_1,\ldots,x_\ell\,|\,y_1,\ldots,y_{p+\ell-k},x_\ell,\ldots,x_1)\,.$$

Let us now invoke the original combinatorial formula for the factorial Schur functions, equation (2.2), with $a_j = y_j$ for $j \leq p+\ell-k$ and $a_{p+\ell-k+j} = x_{\ell+1-j}$ otherwise. Note also that when T is an SSYT of shape p^k and entries at most ℓ, by the strictness of columns we have $T(i,j) \leq \ell-(k-i)$ for all entries in row i. We conclude that $T(i,j)+c(i,j) \leq \ell-k+p$. Therefore, $a_{T(u)+c(u)} = y_{T(u)+c(u)}$, where only the first $p+\ell-k$ parameters a_i are involved in the formula. Then:

$$s^{(\ell)}_{\mu}(x_\ell,\ldots,x_1\,|\,a_1,\ldots,a_{p+\ell-k},a_{p+\ell-k+1},\ldots) \;=\; s^{(\ell)}_{\mu}(x_\ell,\ldots,x_1\,|\,a_1,\ldots,a_{p+\ell-k}),$$

since now the parameters of the factorial Schur are independent of the variables $\mathbf{x}$ and the function is also symmetric in $\mathbf{x}$. □

Example 3.6. *For $\lambda/\mu = 333/22$, the multivariate sum $G_{333/22}(x_1,x_2,x_3\,|\,y_1,y_2,y_3)$ of the six excited diagrams in Figure 2 (ii) is symmetric in x_1,x_2,x_3.*

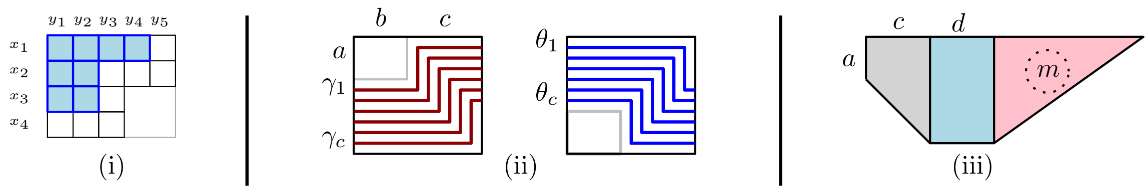

Figure 3: (i) Example of how to read off $z^{\langle\lambda\rangle}$, (ii) paths in proof of Theorem 3.7, and (iii) the shifted hook shape $\Lambda^{\nabla}(a,c,d,m)$ appearing in Conjecture 5.1.

3.3 Multivariate path identity

We give an identity for a multivariate sum over non-intersecting paths as applications of Propositions 3.5.

Theorem 3.7. *We have the following identity for multivariate rational functions:*

$$\sum_{\substack{\Gamma=(\gamma_1,\ldots,\gamma_c)\\ \gamma_p:(a+p,1)\to(p,b+c)}} \prod_{(i,j)\in\Gamma} \frac{1}{x_i-y_j} \;=\; \sum_{\substack{\Theta=(\theta_1,\ldots,\theta_c)\\ \theta_p:(p,1)\to(a+p,b+c)}} \prod_{(i,j)\in\Theta} \frac{1}{x_i-y_j}, \tag{3.3}$$

where the sums are over non-intersecting lattice paths as above. Note that the left hand side is equal to $F_{(b+c)^{a+c}/b^a}(\mathbf{x}\,|\,\mathbf{y})$ defined above.

In the next section we use this identity to obtain product formulas for $f^{\lambda/\mu}$ for certain families of shapes λ/μ. In the case $c=1$, we evaluate (3.3) at $x_i=i$ and $y_j=-j+1$ obtain the following corollary.

Corollary 3.8 ([10]). *We have:*

$$\sum_{\gamma:(a,1)\to(1,b)} \prod_{(i,j)\in\gamma} \frac{1}{i+j-1} \;=\; \sum_{\gamma:(1,1)\to(a,b)} \prod_{(i,j)\in\gamma} \frac{1}{i+j-1}. \tag{3.4}$$

Equation (3.3) is a special case of (NHLF) for the skew shape $(b+1)^{a+1}/b^a$ [10, Section 3.1]. This equation is also a special case of *Racah formulas* in [2, Section 10] (see [12, Section 9]).

Proof of Theorem 3.7. By Proposition 3.5 for the shape $(b+c)^{a+c}/b^a$, we have:

$$G_{(b+c)^{a+c}/b^a}(\mathbf{x}\,|\,\mathbf{y}) \;=\; s_{b^a}(x_1,\ldots,x_{a+c}\;|\;y_1,\ldots,y_{b+c}).$$

Divide the left hand side by $\prod_{(i,j)\in(b+c)^{a+c}}(x_i-y_j)$ to obtain $F_{(b+c)^{a+c}/b^a}(\mathbf{x}\,|\,\mathbf{y})$, the multivariate sum over excited diagrams. By Corollary 3.1, this is also a multivariate sum over tuples of non-intersecting paths in $\mathcal{NIP}((b+c)^{a+c}/b^a)$:

$$\sum_{\substack{\Gamma=(\gamma_1,\ldots,\gamma_c)\\ \gamma_p:(a+p,1)\to(p,b+c)}} \prod_{(i,j)\in\Gamma} \frac{1}{x_i-y_j} \;=\; s_{b^a}(x_1,\ldots,x_{a+c}\mid y_1,\ldots,y_{b+c}) \prod_{(i,j)\in(b+c)^{a+c}} \frac{1}{x_i-y_j}\,. \tag{3.5}$$

Finally, the symmetry in $x_1,\ldots,x_{a+c}$ of the right hand side above implies that we can flip these variables and consequently the paths γ'_p to paths $\theta_p : (p,1) \to (a+p, b+c)$ (see Figure 3 (ii)), and obtain the needed expression. □

4 Skew shapes with product formulas

In this section we use Theorem 3.7 to obtain product formulas for a family of skew shapes.

4.1 Six-parameter family of skew shapes

For all $a,b,c,d,e,m \in \mathbb{N}$, let $\Lambda(a,b,c,d,e,m)$ denote the skew shape λ/b^a, where λ is

$$\lambda \;:=\; (b+c)^{a+c} \,+\, \big(\nu\cup\theta'\big)\,, \tag{4.1}$$

and where $\nu = (d+(a+c-1)m, d+(a+c-2)m,\ldots,d)$, $\theta = (e+(b+c-1)m, e+(b+c-2)m,\ldots,e)$; see Figure 1 (iv). This shape satisfies two key properties:

$$\lambda_{a+c+1} \le b+c, \tag{P1}$$
$$\lambda_i + \lambda'_j = \lambda_r + \lambda'_s, \quad \text{if } i+j=r+s \text{ and } (i,j)\in(b+c)^{a+c}. \tag{P2}$$

The second property is equivalent to saying that the antidiagonals in $((b+c)^{a+c})$ inside λ have the same hook-lengths, or that $\lambda_i - \lambda_{i+1} = \lambda_{i+1} - \lambda_{i+2}$, i.e. the parts of λ are given by an arithmetic progression. Here are two extreme special cases:

$$\Lambda(a,b,c,0,0,1) \;=\; \delta_{a+b+2c}/b^a\,, \qquad \Lambda(a,b,c,d,e,0) \;=\; (b+c+d)^{a+c}(b+c)^e/b^a\,.$$

Next, we give a product formula for f^π where $\pi = \Lambda(a,b,c,d,e,m)$ in terms of falling superfactorials $\Psi^{(m)}(n) := \prod_{i=1}^{n-1}\prod_{j=1}^{i}(jm+j-1)_m$ where $(k)_r = k(k-1)\cdots(k-r+1)$.

Theorem 4.1. *Let $\pi = \Lambda(a,b,c,d,e,m)$ be as above. Then f^π is given by the following product:*

$$f^\pi \;=\; n!\cdot\frac{\Phi(a+b+c)\,\Phi(a)\,\Phi(b)\,\Phi(c)}{\Phi(a+b)\,\Phi(b+c)\,\Phi(a+c)\,\Psi^{(m)}(a+c)\,\Psi^{(m)}(b+c)}\times$$
$$\times\prod_{i=0}^{a+c-1}\frac{(i(m+1))!}{(d+i(m+1))!}\prod_{i=0}^{b+c-1}\frac{(i(m+1))!}{(e+i(m+1))!}\,\frac{\prod_{i=0}^{b-1}\prod_{j=0}^{a-1}(1+d+e+(c+i+j)(m+1))}{\prod_{i=0}^{b+c-1}\prod_{j=0}^{a+c-1}(1+d+e+(i+j)(m+1))}\,.$$

Proof of Corollaries 1.1, 1.2 and 1.3. Use Theorem 4.1 for shapes $\Lambda(a,b,c,e,d,0)$, $\Lambda(a,b,c,0,0,1)$, and $\Lambda(a,b,c,d,e,1)$, respectively. $\square$

The rest of the section is devoted to the proof of Theorem 4.1.

4.2 Proof of the product formulas for skew SYT

Proof of Theorem 4.1. The starting point is showing that the skew shape $\lambda/b^a = \Lambda(a,b,c,d,e,m)$ and the thick reverse hook $(b+c)^{a+c}/b^a = \Lambda(a,b,c,0,0,0)$ have the same excited diagrams. To simplify the notation, let $R = (b+c)^{a+c}$ be the rectangle $\big[(a+c)\times(b+c)\big]$.

Lemma 4.2. *The skew shapes $\Lambda(a,b,c,d,e,m)$ and $\Lambda(a,b,c,0,0,0) = R/b^a$ have the same excited diagrams.*

Proof. From the description of excited diagrams: by (P1), the cell (b,a) of $[\mu]$ cannot go past the cell $(b+c,a+c)$ so the rest of $[\mu]$ is confined in the rectangle $(b+c)^{a+c}$. $\square$

By (NHLF) and Lemma 4.2 we have:

$$\frac{f^{\lambda/b^a}}{n!} = \left[\prod_{u\in[\lambda]\setminus R}\frac{1}{h_\lambda(i,j)}\right]\sum_{D\in\mathcal{E}(R/b^a)}\prod_{(i,j)\in R\setminus D}\frac{1}{h_\lambda(i,j)}\,. \tag{4.2}$$

The sum over excited diagrams of R/b^a with hook-lengths in λ on the right hand side above evaluates to a product.

Lemma 4.3. *For λ and R as above we have:*

$$\sum_{D\in\mathcal{E}(R/b^a)}\prod_{(i,j)\in R\setminus D}\frac{1}{h_\lambda(i,j)} = \frac{\Phi(a+b+c)\,\Phi(a)\,\Phi(b)\,\Phi(c)}{\Phi(a+b)\,\Phi(b+c)\,\Phi(a+c)}\prod_{(i,j)\in R/0^cb^a}\frac{1}{h_\lambda(i,j)}\,. \tag{4.3}$$

Proof. We write the sum of excited diagrams as an evaluation of $F_{(b+c)^{a+c}/b^a}(\mathbf{x}\,|\,\mathbf{y})$.

$$\sum_{D\in\mathcal{E}(R/b^a)}\prod_{(i,j)\in R\setminus D}\frac{1}{h_\lambda(i,j)} = (-1)^m F_{(b+c)^{a+c}/b^a}(\mathbf{x}\,|\,\mathbf{y})\Big|_{\substack{x_i=\lambda_i-i+1\\ y_j=j-\lambda'_j}} \tag{4.4}$$

where $m=(b+c)(a+c)-ba$. Using Theorem 3.7 to obtain the symmetry of the series $F_{(b+c)^{a+c}/b^a}(\mathbf{x}\,|\,\mathbf{y})$ in $\mathbf{x}$:

$$(-1)^m F_{(b+c)^{a+c}/b^a}(\mathbf{x}\,|\,\mathbf{y})\Big|_{\substack{x_i=\lambda_i-i+1\\ y_j=j-\lambda'_j}} = \sum_{\Theta}\prod_{(i,j)\in\Theta}\frac{1}{h_\lambda(i,j)}\,, \tag{4.5}$$

where the sum is over tuples $\Theta := (\theta_1,\ldots,\theta_c)$ of nonintersecting paths inside $(b+c)^{a+c}$ with endpoints $\theta_p : (p,1)\to(a+p,b+c)$. Note that each tuple Θ has the same number of cells in each diagonal $i + j = k$. Also, by property (P2) of λ, the sum $(\lambda_i + \lambda'_j)$ is constant when $(i + j)$ is constant. Thus each tuple Θ will have the same contribution to the sum on the right hand side of (4.5), namely

$$\prod_{(i,j)\in\Theta} \frac{1}{h_\lambda(i,j)} = \prod_{(i,j)\in R/0^c b^a} \frac{1}{h_\lambda(i,j)}.$$

Lastly, the number of tuples Θ in (4.5) equals the number of excited diagrams $(b + c)^{a+c}/b^a$, given by (2.1). $\square$

By Lemma 4.3, (4.2) becomes the following product formula for f^{λ/b^a} :

$$\frac{f^{\lambda/b^a}}{n!} = \frac{\Phi(a+b+c)\,\Phi(a)\,\Phi(b)\,\Phi(c)}{\Phi(a+b)\,\Phi(b+c)\,\Phi(a+c)} \prod_{(i,j)\in\lambda/0^c b^a} \frac{1}{h_\lambda(i,j)}. \tag{4.6}$$

Finally, we carefully rewrite this product in terms of $\Psi(\cdot)$ and $\Psi^{(m)}(\cdot)$. $\square$

5 Final remark and a conjecture

In the full version [12] of this paper we give q-analogues of our product formulas.

It is natural to study shifted analogues of our product formulas. For nonnegative integers $a \le c, d$ and m, let $\lambda/\mu = \Lambda^\triangledown(a,c,d,m)$ be the following shifted skew partition,

$$\lambda = (c+d, c+d-1, \ldots, c+d-a+1) + \nu,$$

where $\nu = (d + (a+c-1)m, d + (a+c-2)m, \ldots, d)$ and $\mu = \delta_{a+1}$. See Figure 3 (iii).

Conjecture 5.1. *In the notation above, the number of SYT of shape $\pi = \Lambda^\triangledown(a,c,d,m)$ equals*

$$g^\pi = \frac{n!}{2^a} \cdot \frac{\Phi(2a+c)\,\Phi(a)}{\Phi(2a)\,\Phi(a+c)} \cdot \frac{\daleth(2a)\,\daleth(c)}{\daleth(2a+c)} \cdot \prod_{u\in\lambda\setminus(\delta_{a+c+1}/c^a\delta_{c+1})} \frac{1}{h_B(u)}.$$

A special case $\pi = \Lambda^\triangledown(a,c,d,0)$ is the (conjugated) *truncated rectangle shape*, and was established by the third author using a different technique [14].

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